Utilization of discrete transforms to conquer the problems of multi-tone systems

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Abstract

This paper presents a new implementation of discrete multi-tone (DMT) systems based on different discrete transforms that include the discrete sine transform (DST), discrete cosine transform (DCT), and discrete wavelet transform (DWT). The implementation also considers time-domain equalization to mitigate channel distortion. Compared to the fast Fourier transform discrete multi-tone (FFT-DMT) system, the proposed implementations have an advantage in that their energy-compaction property helps in reducing the channel effects. The performance of the DST-DMT, DCT-DMT, DWT-DMT, and FFT-DMT systems, employing a time-domain equalizer (TEQ), is investigated in the paper. It has been

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demonstrated by computer simulations that the proposed implementations outperform the FFT-DMT system and that the utilization of the TEQ can lead to higher bit rates. © 2013 The Franklin Institute. Published by Elsevier Ltd. All rights reserved.

1. Introduction

Discrete multi-tone (DMT) systems have had a wide spread in the last few decades due to their ability to reduce the effect of severe channel degradations. One of the major problems in DMT systems is the interference problem, whether it is inter-symbol interference (ISI) or inter-carrier interference (ICI) [1,2]. The data is transmitted in DMT systems in the form of symbols. Pre-appending a guard period of \(v\) samples to each DMT symbol eliminates the ISI, when \(v \geq L - 1\), where \(L\) is the length of the channel impulse response (CIR) [1]. When the guard period is a cyclic prefix (CP), i.e. a copy of the last \(v\) samples of a DMT symbol, the ICI is reduced [2]. The guard period reduces the channel throughput by a factor of \(N/(N+v)\), where \(N\) is both the symbol length and the FFT length. When \(v\) becomes large relative to \(N\), this factor decreases so that the performance loss may be prohibiting. Hence, \(v\) is chosen to be relatively small compared to \(N\). The asymmetric digital subscriber line (ADSL) and the very high-bit-rate digital subscriber line (VDSL) standards set \(v\) to \(N/16\). In practice, however, ADSL and VDSL channel impulse responses may exceed \(N/16\) samples [3].

A possible remedy to the relatively long channel impulse responses is to use a channel shortening equalizer, commonly known as the TEQ [4,5]. This equalizer is designed so that it shortens the length of the effective channel, representing the cascade of the original channel and the TEQ. The TEQ is a finite impulse response (FIR) filter. Therefore, the effective channel can be modeled as a delay of \(\Delta\) samples followed by an FIR filter, whose impulse response is the target impulse response (TIR) of \(\nu+1\) samples [4]. The TIR would fit into a target window of \(\nu+1\) samples starting at sample index \(\Delta+1\) in the shortened impulse response (SIR). The rest of the SIR would ideally be zero [6].

Different TEQ design methods optimize the FIR coefficients based on a training data under different criteria. The minimum mean square error (MMSE) design minimizes the mean square error between the output of the physical path consisting of the channel and the TEQ and the output of a virtual path consisting of a transmission delay \(\Delta\) and the TIR [1,7–9]. The maximum shortening SNR (MSSNR) method attempts to minimize the ISI in the time domain [3]. This method maximizes the ratio of the energy of the effective channel impulse response inside a target window of \(\nu+1\) samples to that outside the target window. The Minimum-ISI (Min-ISI) method generalizes the MSSNR method by weighting the ISI in unused and noisy sub-channels [6]. The traditional TEQ-Frequency-Domain Equalizer (TEQ-FEQ) structure equalizes all sub-channels in a combined fashion, which may limit the bit rate performance. An alternative receiver architecture was proposed in [10]. In this receiver, the authors suggested the transfer of the TEQ operations to the frequency domain by moving the TEQ into the FEQ. The combined TEQ-FEQ would yield a multi-tap FEQ structure, in which each sub-channel (tone) is separately equalized. Another alternative structure was proposed in [11], in which the authors proposed the transfer of the FEQ into the TEQ to yield complex-valued TEQ filter banks. Combined equalization approaches yield higher data rates than decoupled approaches for the downstream ADSL case [12].

Multi-carrier modulation (MCM) techniques including the DMT and the orthogonal frequency division multiplexing (OFDM) have been used, extensively, in several communication standards, such as IEEE 802.11a, IEEE 802.16a, and wire-line digital communication systems, such as the
ADSL [13–15]. All of these systems employ complex exponential functions as orthogonal bases. Particularly, in the DMT systems, modulation with the IFFT and demodulation with the FFT create orthogonal sub-channels [1, 16]. However, the complex exponential functions are not the only orthogonal bases that can be used to construct baseband multi-carrier signals. Discrete transforms such as DST, DCT, and DWT can be used to implement MCM schemes [14]. It is known that the DWT has an excellent sub-band decomposition property and both the DCT and DST have energy compaction properties. They use only real arithmetics. This reduces the signal processing complexity required in MCM schemes, especially, for real pulse-amplitude modulation signaling, where the FFT processing uses complex arithmetics and suffers from the in-phase/in-quadrature imbalance problems, which may cause appreciable performance degradations [17]. A number of researchers have recently proposed the use of discrete transforms in MCM systems, particularly, in OFDM systems [13–15].

In this paper, we present new implementations of the DMT system based on discrete transforms with a TEQ filter bank. The performance of these implementations is tested and compared with that of the FFT-DMT system using the same TEQ filter bank given in [18] over the eight CSA loops. In [18], an FIR TEQ is designed for each tone; hence the FFT becomes a bank of Goertzel filters. In this setting, a single-tap FEQ is also used. The rest of this paper is organized as follows. The FFT-DMT system is presented in Section 2. The different implementations of the DMT system are discussed in Sections 3–5. The TEQ design algorithm is presented in Section 6. The simulation parameters are given in Section 7. In Section 8, the simulation results are presented and discussed. Finally, Section 9 gives concluding remarks.

2. The FFT-DMT system

Let \( u_i \) be the \( i \)th \( N \times 1 \) sample DMT symbol to be transmitted. The preceding and the following symbols are \( u_{i-1} \) and \( u_{i+1} \) symbols, respectively. The transmitted signal variance is \( \sigma_i^2 \), \( v \) is the length of the CP and \( h = [h_0, h_1, \ldots, h_{N-1}]^T \) is the \( N \times 1 \) channel impulse response. The vector \( w = [w_0, w_1, \ldots, w_{M-1}]^T \) is the \( M \times 1 \) TEQ. We assume that the TEQ sub-channel filters are of equal size (\( M \)) for simplicity.

Let \( \Delta \) be the transmission delay of the signal between the transmitter and the receiver and let

\[
U_{ISI}^d = U_i^d + U_{i-1}^d + U_{i+1}^d
\]  

(1)
be the convolutional matrix of the DMT symbols $i-1$, $i$, and $i+1$. Define the $N \times (N + M - 1)$ matrix $\mathbf{U}_i^A = [(\mathbf{U}_i^A)_R, (\mathbf{U}_i^A)_L]$, where $(\mathbf{U}_i^A)_R$ and $(\mathbf{U}_i^A)_L$ are, respectively, given by:

$$
(\mathbf{U}_i^A)_R = \begin{bmatrix}
0 & \cdots & \cdots & \cdots & 0 \\
\mu_i^{N-v} & \cdots & \cdots & \cdots & \mu_i^{N-v} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\mu_i^{N-2} & \cdots & \mu_i^{N-v} & 0 & \cdots & 0 \\
\mu_i^{N-1} & \cdots & \mu_i^{N-v-1} & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & \mu_i^{N-1} & \cdots & \mu_i^{A-M-1}
\end{bmatrix}
$$

$$
(\mathbf{U}_i^A)_L = \begin{bmatrix}
\mu_i^A & \cdots & \mu_i^0 & \mu_i^{N-1} & \cdots & \mu_i^{N-v} \\
\mu_i^{A+1} & \cdots & \mu_i^0 & \mu_i^{N-1} & \cdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\mu_i^{N-1} & \cdots & \cdots & \mu_i^0 & \mu_i^{N-1} & 0 \\
0 & \mu_i^{N-1} & \cdots & u_i^1 & u_i^0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & \cdots & 0 & u_i^{N-1} \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{bmatrix}
$$

Matrices $\mathbf{U}_{i+1}^A$ and $\mathbf{U}_{i-1}^A$ of size $N \times (N + M - 1)$ are, respectively, given by:

$$
\mathbf{U}_{i+1}^A = \begin{bmatrix}
\mu_i^{N-v} & 0 & \cdots & 0 & \cdots & 0 & \cdots & - & - & - \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\mu_i^0 & \mu_i^{N-1} & \cdots & \mu_i^{N-v} & 0 & \cdots & 0 & \cdots & \cdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\mu_i^{A-1} & \cdots & \mu_i^0 & \mu_i^{N-1} & \cdots & \mu_i^{N-v} & 0 & \cdots & \cdots & \cdots \\
\end{bmatrix}
$$

$$
\mathbf{U}_{i-1}^A = \begin{bmatrix}
| & \mu_i^{N-1} & \mu_i^{N-2} & \cdots & \mu_i^{A-M+1} \\
| & \vdots & \vdots & \vdots & \vdots \\
0 & \mu_i^{N-1} & \cdots & \mu_i^{N+v-M+1} \\
| & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & \mu_i^{N-1} \\
| & - & - & - & -
\end{bmatrix}
$$
Let \( H \) be the \((N + M - 1) \times M\) convolution matrix of the CIR and TEQ given by:

\[
H = \begin{bmatrix}
  h_0 & 0 & 0 & \cdots & 0 \\
  h_1 & h_0 & 0 & \cdots & 0 \\
  \vdots & \vdots & \ddots & \ddots & \vdots \\
  h_{M-1} & \cdots & \cdots & h_0 \\
  h_N & \cdots & \cdots & h_{N-1} \\
  0 & h_N & \cdots & h_{N-1} \\
  \vdots & \vdots & \ddots & \ddots & \vdots \\
  0 & 0 & \cdots & h_{N-1}
\end{bmatrix}
\]

For the conventional FFT-DMT system, the channel input sequence \( \{x_n\} \) at the output of the IFFT block, as shown in Fig. 1, can be presented as [19]:

\[
x_{\text{fft}} = \frac{1}{N} \sum_{k=0}^{N-1} X_k e^{j2\pi kn/N}, \quad n = 0, 1, \ldots, N - 1
\]

where \( X_k \) is the encoded bit stream. So, we can define the vector

\[
q_k = [1, e^{j2\pi k/N}, \ldots, e^{j2\pi (N-1)k/N}]^T
\]

such that the inner product of \( q_k^H \) with an \( N \)-point vector gives the \( k \)th FFT coefficient of that vector, where \( (\cdot)^H \) is the Hermitian conjugate transpose operator.

Let the near-end crosstalk (NEXT) or Additive White Gaussian Noise (AWGN) vector be defined as:

\[
n_{xx} = \{n_{-M+1}^{xx}, n_{-M+2}^{xx}, \ldots, n_0^{xx}, \ldots, n_{N-1}^{xx}\}^T
\]

The \((N+M-1) \times M\) AWGN or NEXT convolution matrix with the TEQ, \( G_{\text{AWGN}} \) or \( G_{\text{NEXT}} \), is given by:

\[
G_{xx} = \begin{bmatrix}
  n_0^{xx} & n_1^{xx} & \cdots & n_{-M+1}^{xx} \\
  n_1^{xx} & n_0^{xx} & \cdots & n_{-M+2}^{xx} \\
  \vdots & \vdots & \ddots & \vdots \\
  n_{M-1}^{xx} & \cdots & \cdots & n_0^{xx} \\
  n_N^{xx} & \cdots & \cdots & n_{N-1}^{xx} \\
  \vdots & \vdots & \ddots & \ddots \\
  n_{N+M}^{xx} & n_{N+M-1}^{xx} & \cdots & n_N^{xx}
\end{bmatrix}
\]

Using these definitions, we write the received data in the \( k \)th sub-channel as:

\[
Y^k_R(w) = q_k^H(U_{\text{ISI}}^d H + G_{\text{AWGN}} + G_{\text{NEXT}})w, \quad k \in \{0, \ldots, N/2 - 1\}
\]

The received data contains the noise due to the ISI, ICI, AWGN, and NEXT. It suffers from the effects of the channel. Now, we see the dependence of the received signal on the TEQ. The ideal received signal has no noise and is formatted to fit the demodulation scheme. In DMT
modulation, this means that the received symbol has minimal noise due to AWGN, NEXT, and ISI. We can design the TEQ to process the received samples to achieve this target.

We will express the desired signal as a function of the TEQ taps. The desired circular convolution of the \( k \)th symbol and the CIR in the \( k \)th sub-channel, after the TEQ and FFT, can be written as:

\[
Y^k_D(w) = q^H \left[ U^4_{i, \text{circ}} \right] H w, \quad k \in \{0, \cdots, N/2 - 1\}
\]

The \( N \times (N + M - 1) \) circulant matrix \( \left[ U^4_{i, \text{circ}} \right] \) is given by:

\[
\left[ U^4_{i, \text{circ}} \right] = \begin{bmatrix}
  u^4_i & \cdots & u^0_i & u^{N-1}_i & \cdots & u^{-M+2}_i \\
  \vdots & & \vdots & \vdots & & \vdots \\
  u^{N-1}_i & \cdots & u^4_i & u^{N-3}_i & \cdots & u^{-M-1}_i \\
  u^0_i & \cdots & u^{N-1}_i & u^4_i & \cdots & u^{N-M+2}_i \\
  \vdots & & \vdots & \vdots & & \vdots \\
  u^{-M-1}_i & \cdots & u^0_i & u^{N-M-1}_i & \cdots & u^4_i \\
\end{bmatrix}
\]

So, the received data \( Y^k_R(w) \) can be rewritten as:

\[
Y^k_R(w) = Y^k_D(w) + \left( Y^k_R(w) - Y^k_D(w) \right)
\]

We can write SNR\(_{k}^{\text{Model}}(w)\) for all \( k \) as

\[
\frac{E\left[ Y^k_R(w)Y^H_k D(w) \right]}{E\left[ Y^k_R(w) - Y^k_D(w) \right] Y^H_k R(w) - Y^k_D(w)]}
\]

where \( E[\cdot] \) is the statistical expectation operator and \( [\cdot]^{\text{Model}} \) stands for model. The proposed SNR model is the ratio of the desired data, which excludes the effects of the noise including the ISI and ICI, to the difference between the received data and the desired data.

Derive \( \hat{A}_k \) as

\[
\hat{A}_k = \sigma^2_s H^T \mathbf{Q}^q \left[ \mathbf{Q}^q \right] H
\]

where \( \mathbf{Q}^q_k \) is an \( (N + M - 1 \times N) \) matrix given by:

\[
\mathbf{Q}^q_k = \begin{bmatrix}
  q^{N_D+1}_k & \cdots & q^1_k & q^0_k & \cdots & q^{-M_D-2}_k \\
  q^{N_D+2}_k & \cdots & q^1_k & q^0_k & \cdots & q^{-M_D-2}_k \\
  \vdots & & \vdots & \vdots & & \vdots \\
  q^0_k & q^1_k & \cdots & \cdots & \cdots & q^{N_D+1}_k \\
  \vdots & & \vdots & \vdots & & \vdots \\
  q^{M_D-3}_k & q^{M_D-2}_k & \cdots & q^0_k & \cdots & q^{N_D-2}_k \\
\end{bmatrix}
\]

where \( N_D = N - \Delta, M_D = M + \Delta \) and \( N_1 = N - 1 \).

Similarly, we can derive \( \mathbf{B}_k \) as,

\[
\mathbf{B}_k = 2\sigma^2_s \left[ H^T_k V_k V^H_k H_a + H^T_k W_k W^H_k H_b \right] + \mathbf{Q}^q_k \left[ \sigma^2_{\text{AWGN}} \mathbf{I} + \sum \text{NEXT} \right] \left[ \mathbf{Q}^q_k \right]^H
\]

where \( \sigma^2_{\text{AWGN}} \) is the noise variance, \( \sum \text{NEXT} \) is the toeplitz variance matrix of the NEXT, and \( \mathbf{I} \) is the \( M \times M \) identity matrix. Without loss of generality, we can define a constraint set \( \mathbf{J} = \{ w : w^T w = ||w||^2 = 1 \} \), so that \( \mathbf{B}_k \) becomes independent of \( w \) over this constraint set.
Matrix $Q_{\text{noise}}^k$ of size $M \times (N + M - 1)$ is defined as:

$$Q_{\text{noise}}^k = \begin{bmatrix}
0 & 0 & \cdots & 0 & q_k^0 & \cdots & q_k^{N-1} \\
0 & 0 & \cdots & q_k^0 & \cdots & q_k^{N-1} & 0 \\
\vdots & & \ddots & \vdots & \vdots & \vdots & \vdots \\
q_k^0 & \cdots & q_k^{N-1} & 0 & \cdots & 0 & 0
\end{bmatrix}$$  \hspace{1cm} (19)

where $q_k^{(\cdot)}$ are members of the vector $q_k$ defined by Eq. (8). $V_k$ is a $\Delta \times \Delta$ upper diagonal matrix defined as:

$$V_k = \begin{bmatrix}
q_k^{N-\Delta} & q_k^{N-\Delta+1} & \cdots & q_k^{N-2} & q_k^{N-1} \\
q_k^{N-\Delta+1} & q_k^{N-\Delta} & \cdots & q_k^{N-1} & 0 \\
\vdots & & \ddots & \vdots & \vdots \\
q_k^{N-1} & 0 & \cdots & 0 & 0
\end{bmatrix}$$  \hspace{1cm} (20)

$W_k$ is a lower diagonal $(N - \nu - \Delta + M - 1) \times (N - \nu - \Delta + M - 1)$ matrix defined as:

$$W_k = \begin{bmatrix}
0 & 0 & \cdots & 0 & q_k^0 \\
0 & 0 & \cdots & q_k^0 & q_k^1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
q_k^0 & q_k^1 & \cdots & q_k^{N-\nu-\Delta+M-2}
\end{bmatrix}$$  \hspace{1cm} (21)

$\tilde{A}_k$ and $\tilde{B}_k$ are Hermitian symmetric matrices. Now Eq. (15) becomes:

$$\text{SNR}_{\text{Model}}^k(w) = \frac{w^T \tilde{A}_k w}{w^T \tilde{B}_k w}$$  \hspace{1cm} (22)

The $\text{SNR}_{\text{Model}}^k(w)$ is a ratio of quadratic functions of $w$. The SNR model becomes equivalent to the SNR that could be measured at the output of the FFT in an ADSL system, when the ISI and ICI have been removed from the received signal.

Using the SNR model, the number of bits per symbol that can be supported is:

$$b_{\text{DMT}}(w) = \sum_{k \in \mathbf{s}} \log_2 \left( 1 + \frac{\text{SNR}_{\text{Model}}^k(w)}{\Gamma} \right) = \sum_{k \in \mathbf{s}} \log_2 \left( \frac{w^T \tilde{A}_k w}{w^T \tilde{B}_k w} \right) \text{ Bits/symbol}$$  \hspace{1cm} (23)

where $A_k = \Gamma \tilde{B}_k + \tilde{A}_k$ and $B_k = \Gamma \tilde{B}_k$. $k$ is the sub-channel index, $\mathbf{s}$ is the set of the indices of the used $\tilde{N}$ sub-channels out of $N/2 + 1$ sub-channels, and $b_k(w)$ is the number of bits per data symbol in the sub-channel $k$. $\Gamma$ is the SNR gap, and it is a function of several factors including modulation method, allowable probability of error, gain of any applied coding, and desired system margin.

Maximizing the number of bits allocated to a single channel, $b_k(w)$ involves maximizing the argument of the log function. Since the log function is a monotonically increasing function of a non-negative argument, maximizing its non-negative argument will also maximize the function. The mathematical notation for this statement is:

$$b_k^{\text{opt}} = \max_{w_k: ||w_k||^2 = 1} \left[ \log_2 \left( \frac{w_k^T A_k w_k}{w_k^T B_k w_k} \right) \right] = \log_2 \left[ \max_{w_k: ||w_k||^2 = 1} \left( \frac{w_k^T A_k w_k}{w_k^T B_k w_k} \right) \right]$$  \hspace{1cm} (24)
This is the well-known generalized eigenvalue problem \[18\] and the solution is the generalized eigenvector \( \mathbf{w}_{\text{opt}}^k \) corresponding to the largest generalized eigenvalue \( \lambda_{\text{opt}}^k \) of \( (A_r^k, B_r^k) \)

\[
\lambda_{\text{opt}}^k = \left( \mathbf{w}_{\text{opt}}^k \right)^T A_r^k \mathbf{w}_{\text{opt}}^k = \left( \mathbf{w}_{\text{opt}}^k \right)^T B_r^k \mathbf{w}_{\text{opt}}^k \quad (25)
\]

hence,

\[
b_{\text{opt}}^k = \log_2 (\lambda_{\text{opt}}^k)
\]

where \((\cdot)^r\) denotes the real part. The number of bits per symbol that can be supported in the case of the conventional FFT-DMT system with the TEQ filter bank is:

\[
b_{\text{opt}}^{\text{DFT-DMT}} = \sum_{k \in s} \log_2 \left[ \frac{\left( \mathbf{w}_{\text{opt}}^k \right)^T A_r^k \mathbf{w}_{\text{opt}}^k}{\left( \mathbf{w}_{\text{opt}}^k \right)^T B_r^k \mathbf{w}_{\text{opt}}^k} \right] \quad \text{Bits/symbol} \quad (26)
\]

3. The DST-DMT system

For the DST-DMT system with TEQ filter bank, the channel input sequence \( \{x_n\} \) at the output of the IDST block, as shown in Fig. 2, can be represented as \[18\]:

\[
x_{n,\text{DST}} = \left( \frac{2}{N} \right)^{\frac{1}{2}} \cdot \sum_{k=0}^{N-1} A_k \cdot X_k \cdot \sin \left[ \frac{\pi k}{2N} (2n + 1) \right], \quad n = 0, 1, \ldots, N-1 \quad (27)
\]

where

\[
A_k = \begin{cases} 
\frac{1}{\sqrt{2}} & \text{for } k = 0 \\
1 & \text{otherwise} 
\end{cases}
\quad (28)
\]

and \( X_k \) is the encoded bit stream. So, the defined vector in Eq. (8) will be:

\[
\mathbf{q}_{k,\text{DST}} = \left[ \sqrt{\frac{2}{N}} \sin \left( \frac{\pi k}{2N} \right), \sqrt{\frac{2}{N}} \sin \left( \frac{3\pi k}{2N} \right), \ldots, \sqrt{\frac{2}{N}} \sin \left( \frac{\pi k}{2N} (2(N-1) + 1) \right) \right]^T \quad (29)
\]
such that the inner product of $\mathbf{q}^H_{k,DST}$ with an $N$-point vector gives the $k$th DST coefficient of that vector. Note that in the above equation, $\sqrt{2/N}$ will be $\sqrt{1/N}$ for $k = 0$.

Depending on the defined vector in Eq. (9) and using the above-mentioned algorithm, we can derive a new $\mathbf{w}^\text{opt}_{k,DST}$ corresponding to the largest generalized eigenvalue $\lambda^\text{opt}_{k,DCT}$ of $(\mathbf{A}'_{k,DST}, \mathbf{B}'_{k,DST})$.

$$\lambda^\text{opt}_{k,DST} = \frac{\mathbf{w}^\text{opt}_{k,DST}^T \mathbf{A}'_{k,DST} \mathbf{w}^\text{opt}_{k,DST}}{(\mathbf{w}^\text{opt}_{k,DST})^T \mathbf{B}'_{k,DST} (\mathbf{w}^\text{opt}_{k,DST})}$$

hence,

$$b^\text{opt}_{k,DST} = \log_2 (\lambda^\text{opt}_{k,DST})$$

where $b^\text{opt}_{k,DST}$ is the number of bits per data symbol in the sub-channel $k$. The number of bits per symbol that can be supported with the proposed DST-DMT system implementing a TEQ filter bank is:

$$b^\text{opt}_{DST-DMT} = \sum_{k \in s} \log_2 \left[ \frac{(\mathbf{w}^\text{opt}_{k,DST})^T \mathbf{A}'_{k,DST} (\mathbf{w}^\text{opt}_{k,DST})}{(\mathbf{w}^\text{opt}_{k,DST})^T \mathbf{B}'_{k,DST} (\mathbf{w}^\text{opt}_{k,DST})} \right] \text{ Bits/symbol}$$

4. The DCT-DMT system

For the DCT-DMT system with TEQ filter bank, the channel input sequence $\{x_n\}$ at the output of the IDCT block, as shown in Fig. 3, can be presented as [18]:

$$x_{n,dct} = \left( \frac{2}{N} \right)^{1/2} \sum_{k=0}^{N-1} A_k \cdot X_k \cdot \cos \left[ \frac{\pi k}{2N} (2n + 1) \right], \quad n = 0, 1, \cdots, N - 1$$

where

$$A_k = \begin{cases} \frac{1}{\sqrt{2}} & \text{for } k = 0 \\ 1 & \text{otherwise} \end{cases}$$

and $X_k$ is the encoded bit stream.

So, the defined vector in Eq. (8) will be:

$$\mathbf{q}_{k,DCT} = \left[ \sqrt{\frac{2}{N}} \cos \left( \frac{\pi k}{2N} \right), \sqrt{\frac{2}{N}} \cos \left( \frac{3\pi k}{2N} \right), \cdots, \sqrt{\frac{2}{N}} \cos \left( \frac{\pi k}{2N} (2N - 1 + 1) \right) \right]^T$$

such that the inner product of $\mathbf{q}^H_{k,DCT}$ with an $N$-point vector gives the $k$th DCT coefficient of that vector. Note that in Eq. (35), $\sqrt{2/N}$ will be $\sqrt{1/N}$ for $k = 0$.

Depending on the defined vector in Eq. (35) and using the above-mentioned algorithm, we can derive a new $\mathbf{w}^\text{opt}_{k,DCT}$ corresponding to the largest generalized eigenvalue $\lambda^\text{opt}_{k,DCT}$ of $(\mathbf{A}'_{k,DCT}, \mathbf{B}'_{k,DCT})$

$$\lambda^\text{opt}_{k,DCT} = \frac{\mathbf{w}^\text{opt}_{k,DCT}^T \mathbf{A}'_{k,DCT} (\mathbf{w}^\text{opt}_{k,DCT})}{(\mathbf{w}^\text{opt}_{k,DCT})^T \mathbf{B}'_{k,DCT} (\mathbf{w}^\text{opt}_{k,DCT})}$$

hence,

$$b^\text{opt}_{k,DCT} = \log_2 (\lambda^\text{opt}_{k,DCT})$$
where $b_{k,\text{DCT}}^{\text{opt}}$ is the number of bits per data symbol in the sub-channel $k$. The number of bits per symbol that can be supported with the DCT-DMT system implementing a TEQ filter bank is:

$$b_{\text{DCT-DMT}}^{\text{opt}} = \sum_{k \in s} \log_2 \left( \frac{(w_{k,\text{DCT}})^{\text{opt}}}{(w_{k,\text{DCT}}^{\text{opt}})^{\text{T}} A_{k,\text{DCT}} (w_{k,\text{DCT}})^{\text{opt}}} \right)$$ Bits/symbol \hspace{1cm} (38)

5. The DWT-DMT system

The proposed DWT-DMT system model is shown in Fig. 4. The channel input is given by:

$$x_{n,\text{DWT}} = \sum_{k=0}^{N-1} X_k \phi(k) \hspace{1cm} k = 0, 1, \ldots, N-1$$ \hspace{1cm} (39)

where $\phi(k)$ is the wavelet basis function.

The conventional DWT may be regarded as equivalent to filtering the input signal with a bank of bandpass filters, whose impulse responses are all approximately given by scaled versions of a
mother wavelet. The scaling factor between adjacent filters is usually 2:1 leading to octave bandwidths and center frequencies that are one octave apart [21–23]. The outputs of the filters are usually maximally decimated so that the number of DWT output samples equals the number of input samples and that the transform is invertible as shown in Fig. 5.

The DWT is normally implemented by a binary tree of filters as shown for the 1-D case in Fig. 5. The art of finding a good wavelet lies in the design of the set of filters, $H_0$, $H_1$, $G_0$ and $G_1$ to achieve various trade-offs between spatial- and frequency-domain characteristics while satisfying the Perfect Reconstruction (PR) condition [23]. Now we are going to discuss the PR condition. In Fig. 5-a, the process of decimation and interpolation by 2 at the output of $H_0$ and $H_1$ effectively sets all odd samples of these signals to zero.
For the low-pass branch, this is equivalent to multiplying \( x_0(n) \) by \((1/2)(1 + (-1)^n)\). Hence \( X_0(z) \) is converted to \( \{X_0(z) + X_0(-z)\} \). Similarly, \( X_1(z) \) is converted to \((1/2)\{X_1(z) + X_1(-z)\} \).

Thus, the expression for \( Y(z) \) is given by [23]:

\[
Y(z) = \frac{1}{2} \{X_0(z) + X_0(-z)\} G_0(z) + \frac{1}{2} \{X_1(z) + X_1(-z)\} G_1(z)
\]

\[
= \frac{1}{2} X(z) \{H_0(z)G_0(z) + H_1(z)G_1(z)\} + \frac{1}{2} X(-z) \{H_0(-z)G_0(z) + H_1(-z)G_1(z)\}
\]

(40)

The first PR condition requires aliasing cancellation and forces the above term in \( X(-z) \) to be zero. Hence \( \{H_0(-z)G_0(z) + H_1(-z)G_1(z)\} = 0 \), which can be achieved if [23]:

\[
H_1(z) = z^{-k}G_0(-z) \quad \text{and} \quad G_1(z) = z^kH_0(-z)
\]

(41)

where \( k \) must be odd (usually \( k = \pm 1 \)).

The second PR condition is that the transfer function from \( X(z) \) to \( Y(z) \) should be unity [23]:

\[
\{H_0(z)G_0(z) + H_1(z)G_1(z)\} = 2
\]

(42)

If we define a product filter \( P(z) = H_0(z)G_0(z) \) and substitute from Eq. (29) into Eq. (30), then the PR condition becomes [22]:

\[
H_0(z)G_0(z) + H_1(z)G_1(z) = P(z) + P(-z) = 2
\]

(43)

This needs to be true for all \( z \), and since the odd powers of \( z \) in \( P(z) \) cancel with those in \( P(-z) \), it is required that \( p_0 = 1 \) and \( p_n = 0 \) for all \( n \) even and non-zero. The polynomial \( P(z) \) should be a zero-phase polynomial to minimize distortion. In general, \( P(z) \) is of the following form [24]:

\[
P(z) = \cdots + p_5z^5 + p_3z^3 + p_1z + 1 + p_1z^{-1} + p_3z^{-3} + p_5z^{-5} + \cdots
\]

(44)

The design method for the PR filters can be summarized in the following steps [24]:

1. Choose \( p_1, p_3, p_5, \cdots \) to give a zero-phase polynomial \( P(z) \) with good characteristics.
2. Factorize \( P(z) \) into \( H_0(z) \) and \( G_0(z) \) with similar low-pass frequency responses.
3. Calculate \( H_1(z) \) and \( G_1(z) \) from \( H_0(z) \) and \( G_0(z) \).

To simplify this procedure, we can use the following relation:

\[
P(z) = P_1(Z) = 1 + p_{t,1}Z + p_{t,3}Z^3 + p_{t,5}Z^5 + \cdots
\]

(45)

where

\[
Z = \frac{1}{2}(z + z^{-1})
\]

(46)

The Haar wavelet is the simplest type of wavelets. In discrete form, Haar wavelets are related to a mathematical operation called the Haar transform. The Haar transform serves as a prototype for all other wavelet transforms [23]. Like all wavelet transforms, the Haar transform decomposes a discrete signal into two sub-signals of half its length. One sub-signal is a running average or trend; the other sub-signal is a running difference or fluctuation. This uses the simplest possible \( P_1(Z) \) with a single zero at \( Z = -1 \). It is represented as follows [24]:

\[
P_1(Z) = 1 + Z \quad \text{and} \quad Z = \frac{1}{2}(z + z^{-1})
\]

(47)
thus
\[ P(z) = \frac{1}{2}(z + 2 + z^{-1}) = \frac{1}{2}(z + 1)(1 + z^{-1}) = G_0(z)H_0(z) \] (48)

We can find \( H_0(z) \) and \( G_0(z) \) as follows:
\[ H_0(z) = \frac{1}{2}(1 + z^{-1}) \] (49)
\[ G_0(z) = (z + 1) \] (50)

Using Eq. (41) with \( k = 1 \):
\[ G_1(z) = zH_0(-z) = \frac{1}{2}z(1 - z^{-1}) = \frac{1}{2}(z - 1) \] (51)
\[ H_1(z) = z^{-1}G_0(-z) = z^{-1}(-z + 1) = (z^{-1} - 1) \] (52)

For the Le-Gall wavelet transform, we have [24]:
\[ P_i(Z) = (1 + Z)^2(1 + aZ) \] (53)

Going through the factorization process with \( a = 1/2 \), we get [23]:
\[ H_0(z) = \frac{1}{8}(-z^2 + 2z + 6 + 2z^{-1} - z^{-2}) \] (54)
\[ G_0(z) = \frac{1}{2}(z + 2 + z^{-1}) \] (55)

Using Eq. (41) with \( k = 1 \), we get [24]:
\[ G_1(z) = zH_0(-z) = \frac{1}{8}z(-z^2 - 2z + 6 - 2z^{-1} - z^{-2}) \] (56)
\[ H_1(z) = z^{-1}G_0(-z) = \frac{1}{2}z^{-1}(-z + 2 - z^{-1}) \] (57)

Another form of using wavelets is through the wavelet packet transform. The wavelet packet transform is a generalization of wavelet transform that offers a wide range of scales for signal analysis. In wavelet packet analysis, the details as well as the approximations are split to yield \( 2^m \) different ways to represent the signal, where \( m \) is the decomposition level. A single decomposition using wavelet packets generates a large number of bases, which offer a more complex and flexible analysis. An entropy-based criterion is used to select the most suitable decomposition level for a signal. The wavelet packet decomposition and reconstruction trees are shown in Fig. 5c.

The input signal \( x \) is split by filters \( H_0 \) and \( H_1 \) into a low-pass component \( x_0 \) and a high-pass component \( x_1 \), both of which are decimated (down-sampled) by 2. The low-pass component is then split further into \( x_{00} \) and \( x_{01} \), which are again decimated by 2 as shown in Fig. 5b. This process continues as far as required [23]. The outputs of the DWT are the band-pass coefficients \( x_1, x_{01}, x_{001}, \ldots, \) and the final low-pass coefficients \( x_{0000} \). Because of the decimation, the total output sample rate equals the input sample rate, and thus there is no redundancy in the transform. In order to reconstruct the signal, a pair of reconstruction filters \( G_0 \) and \( G_1 \) is used in the arrangement of Fig. 5c, and usually the filters are designed such that the output signal \( Y(z) \) is identical to the input signal \( X(z) \). Hence, in Fig. 5c, \( x_{000} \) may be reconstruction from \( x_{0000} \) and \( x_{0001} \); and then \( x_{00} \) from \( x_{000} \) and \( x_{001} \); and so on back to \( x \), using an inverse tree of \( G \) filters [24].
We can find a new \( w_{k,DWT}^{opt} \) corresponding to the largest generalized eigenvalue \( \lambda_{k,DWT}^{opt} \) of \( (A_{k,DWT}^{opt}, B_{k,DWT}^{opt}) \) as:

\[
\lambda_{k,DWT}^{opt} = \frac{(w_{k,DWT}^{opt})^T A_{k,DWT}^{opt} (w_{k,DWT}^{opt})}{(w_{k,DWT}^{opt})^T B_{k,DWT}^{opt} (w_{k,DWT}^{opt})} = \frac{(w_{k,DWT}^{opt})^T A_{k,DWT}^{opt} (w_{k,DWT}^{opt})}{(w_{k,DWT}^{opt})^T B_{k,DWT}^{opt} (w_{k,DWT}^{opt})}
\]

(58)

hence,

\[
b_{k,DWT}^{opt} = \log_2(\lambda_{k,DWT}^{opt})
\]

(59)

where \( b_{k,DWT}^{opt} \) is the number of bits per data symbol in the sub-channel \( k \). The number of bits per symbol that can be supported with the proposed DWT-DMT system implementing a TEQ filter bank is:

\[
b_{DWT-DMT}^{opt} = \sum_{k \in s} 2 \log_2 \left[ \frac{(w_{k,DWT}^{opt})^T A_{k,DWT}^{opt} (w_{k,DWT}^{opt})}{(w_{k,DWT}^{opt})^T B_{k,DWT}^{opt} (w_{k,DWT}^{opt})} \right] \text{ Bits/symbol}
\]

(60)

6. TEQ design algorithm

The first derivative of Eq. (23) is:

\[
db_{DMT}(w) = \frac{2}{\ln 2} \sum_{k \in s} r_k(w)[A_k^r - \lambda_k(w)B_k^r] w
\]

(61)

where

\[
r_k(w) = \frac{1}{w^T A_k w} \quad \text{and} \quad \lambda_k(w) = \frac{w^T A_k w}{w^T B_k w}
\]

(62)

Notice that \( b_{DMT}(w) = \sum_{k \in s} \log_2[\lambda_k(w)] \). Thus, increasing \( \lambda_k(w) \) increases \( b_{DMT}(w) \). Now, we can write:

\[
C_k(w) = r_k(w)[A_k^r - \lambda_k(w)B_k^r] \quad \text{and} \quad C_k(w, s) = \sum_{k \in s} C_k(w).
\]

(63)

let

\[
F(\lambda) = \max_{w \in J} w^T C(w, s) w = \max_{w \in J} \sum_{k \in s} r_k[w^T A_k^r w - \lambda_k w^T B_k^r w]
\]

(64)

where

\[
\lambda = [\lambda_1, \ldots, \lambda_k, \ldots]^T, k \in s.
\]

Set an iteration counter \( i = 0 \), a smoothing factor \( \alpha = 0 \), and the values of \( r_k \) and \( \lambda_k \) to zero for all \( k \). The algorithm procedure can be written as follows:

1. \( r_k = \alpha r_k + (1 - \alpha) \frac{1}{w^T A_k w}, \forall k \in s \)
2. \( \lambda_k = \alpha \lambda_k + (1 - \alpha) \frac{w^T A_k w}{w^T B_k w}, \forall k \in s \)
3. Compute \( C_k(w, s) = \sum_{k \in s} r_k[w^T A_k^r w - \lambda_k w^T B_k^r w] \)
4. \( w_{\text{new}} = \arg \max_{v \in J} \{v^T C(w, s) v, v \in J\} \)
5. If \( ||w_{\text{new}} - w|| < \epsilon \) or \( i > i_{\text{max}} \) then return \( w_{\text{opt}} \).
6. If \( b_{DMT}(w_{\text{new}}) < b_{DMT}(w) \), set \( \alpha = (1 + \alpha)/2 \), else \( w_{\text{opt}} = w_{\text{new}} \).
Fig. 6. SNR achieved using the FFT-DMT system (a), the DCT-DMT system (b), and the DWT-DMT system (c) in the presence of the TEQ filter bank for CSA loop 1.
7. 

8. 

9. Go back to step 1 and repeat.

7. Simulation parameters

We use the eight CSA loops as our test channels. The CSA is an identifiable subset of the current subscriber loop population in the US. It consists mainly of 24- and 26-gauge twisted pairs whose lengths could reach 12 kft and 9 kft, respectively. The parameters of these channels are well-known in the literature [4]. All channel impulse responses consist of 512 samples sampled at a rate of 2.208 MHz. We add a fifth-order Chebyshev high-pass filter with cut-off frequency of 5.4 kHz and pass-band ripples of 0.5 dB to each CSA loop to take into account the effect of the splitter at the transmitter. The DC channel (channel 0), channels 1 to 5, and the Nyquist channel are not used. We model the channel noise as $-140$ dBm AWGN distributed over the bandwidth of 1.104 MHz plus NEXT. The NEXT noise consists of 8 ADSL disturbers as described in the ANSI T1.413-1995 standard [4]. The input signal power is 23 dBm distributed equally over all used sub-channels, and the discrete transforms length is set to $N=512$. Also, we set $M=17$, and $v=32$. The coefficients of the FIR filters (TEQs) are obtained from the MATLAB discrete...
Fig. 8. SNR achieved using the FFT-DMT system (a), the DCT-DMT system, and the DWT-DMT system (c) in the presence of the TEQ filter bank for CSA loop 8.
Fig. 9. Bit allocation to each sub-channel using the FFT-DMT system (a), the DCT-DMT system (b), and the DWT-DMT system (c) in the presence of the TEQ filter bank for CSA loop 1.
Fig. 10. Bit allocation to each sub-channel using the FFT-DMT system (a), the DCT-DMT system (b), and the DWT-DMT system (c) in the presence of the TEQ filter bank for CSA loop 4.
Fig. 11. Bit allocation to each sub-channel using the FFT-DMT system (a), the DCT-DMT system (b), and the DWT-DMT system (c) in the presence of the TEQ filter bank for CSA loop 8.
Bandwidth optimization is applied by shutting down (not assigning any transmit power to) the sub-channels with initial SNR lower than the SNR required to transmit two bits with a given SNR gap of $9.8 + 6 - 4.2 = 11.6$ dB. This corresponds to a system margin of 6 dB and a coding gain of 4.2 dB. No bit loading algorithms have been used. So, all bit rate results are calculated from the SNR distribution after the TEQ filter bank is placed into the system. We assume that the power allocation is constant over all used sub-channels and that it is not changed after the TEQ filter bank is placed in the system.

8. Simulation results

Simulation results are presented to analyze and compare the performance of the FFT-DMT, DST-DMT, DCT-DMT, and DWT-DMT systems in the presence of the TEQ filter bank. Fig. 6 shows the SNR achieved by the FFT-DMT system, the DCT DMT system, and the DWT-DMT system for CSA loop 1. This figure reveals that the performance of the proposed DWT-DMT system with the TEQ filter bank outperforms those of the other systems in the uniformity of SNR for different subcarriers. The DWT-DMT system has a semi-flat magnitude response over most of the spectrum except at the positions of high ISI, while for the other systems, the SNR decreases as the frequency increases, as displayed in Figs. 6–8 for CSA loops 1, 4 and 8, respectively.

Fig. 9 shows the bit allocation to each sub-channel for the FFT-DMT system, the DCT-DMT system, and the DWT-DMT system when a TEQ filter bank is used for CSA loop 1. It is clear that the DWT-DMT system achieves a high bit allocation for each sub-channel over most of the spectrum except at the positions of high ISI, because each sub-channel carries a different number of bits depending on its SNR. The number of bits assigned to each sub-channel in the FFT-DMT and DCT-DMT systems decreases as the frequency increases, as displayed in Figs. 9–11 for CSA loops 1, 4 and 8, respectively.

Table 1 shows the achievable bit rates in Mbps for the conventional FFT-DMT system and the DWT-DMT system when the TEQ filter bank is employed. The results are displayed for the CSA loops 1 to 8. It is clear that the DWT-DMT system achieves higher data rates for each loop than those of the conventional FFT-DMT system in the range of (2.899–5.369) Mbps. In the

<table>
<thead>
<tr>
<th>Loop</th>
<th>Bit rate of the conventional FFT-DMT system in Mbps</th>
<th>Bit rate of the proposed DWT-DMT system with TEQ filter bank in Mbps</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>MSSN</td>
<td>Min-ISI</td>
</tr>
<tr>
<td>1</td>
<td>5.48</td>
<td>8.592</td>
</tr>
<tr>
<td>4</td>
<td>5.17</td>
<td>8.2</td>
</tr>
<tr>
<td>5</td>
<td>6.514</td>
<td>8.64</td>
</tr>
</tbody>
</table>

multi-tone time-domain equalizer (DMTTEQ) Toolbox that was implemented by the Embedded Signal Processing Lab. at the University of Texas [20].
FFT-DMT system, three approaches for equalizer design have been investigated, but the DWT-DMT system has superior performance over all these approaches. This superiority of the DWT-DMT system is mainly attributed to the energy compaction property of the DWT that forces the ISI to be in small-valued or negligible coefficients, leading to an enhancement of the SNR per subcarrier, and hence an enhancement of the bit rate.

9. Conclusions

In this paper, a new DWT-DMT system employing a TEQ filter bank has been proposed to achieve higher bit rates. Simulation experiments have shown that the proposed DWT-DMT system employing TEQ filter bank provides a better performance when compared with the conventional system. Specifically, it achieves higher SNR values in each sub-channel, over most of the spectrum except at the positions of high ISI, than those of the conventional systems for the eight CSA loops. The proposed DWT-DMT system with the TEQ filter bank also achieves higher bit rates than those obtained by the conventional systems. This is intuitively not surprising, because the sub-band decomposition and energy compaction properties of the DWT leave most of the samples at the end of each symbol close to zero, which in turn reduces the ISI, dramatically, leading to a great performance enhancement.

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References


