Capacity of Synchronous CDMA Systems with Near-Far Effects and Design of Suboptimum Signature Codes

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Abstract—This paper deals with near-far effects on various aspects of Code Division Multiple Access (CDMA) systems. Initially, we propose a new class of codes for over-loaded synchronous wireless CDMA systems that are robust against near-far effects; and then we provide a low complexity decoder for a subclass of such codes. Moreover, bounds for the sum capacity of CDMA systems in the presence of near-far effects are derived. An important contribution of this paper is the development of a method that translates a near-far sum capacity problem with imperfect channel state estimation to the evaluation of the capacity for a CDMA system with perfect channel state estimation. To show the power and utility of the results, a number of sum capacity bounds for special cases are numerically evaluated.

I. INTRODUCTION

In a CDMA system, each user is assigned a signature vector to transmit its data through a common channel. Different users have different distances from the receiver; thus, the received signals do not have the same power at the receiver end. Fading due to multi-path and shadowing can also create power variations at the receiver end for different users. Near-far problem can be studied from two different aspects; firstly, the design of near-far resistant codes and practical decoding schemes; secondly, the evaluation of the sum channel capacity.

Several multiuser detectors have been developed that are resistant to near-far effects. Some of these detectors are 1) optimum multiuser detector which was studied by [1], [2], 2) deccorelating detector [3], [4] and 3) Minimum Mean Squared Error (MMSE) detector [5]. The optimum multiuser detector achieves optimum near-far resistance under perfect channel state estimation condition but is computationally complex. The deccorelating and MMSE detectors are sub-optimum but, in the absence of the user channel state estimation, these suboptimum detectors become optimum. There are other near-far resistant detectors that are discussed in [6] and [7].

Initially, we propose uniquely decodable codes such as COW codes developed in [8] as near-far resistant signature codes and then provide a low complexity decoder for a subclass of such codes. We can implement large signature matrices with low complexity decoding.

One of the main concerns in this paper is the evaluation of the channel capacity. In the absence of near-far effects, the channel capacity has been evaluated for real and complex inputs [9] and [10]. However, for the finite input alphabets, only lower and upper bounds have been evaluated [8], [11]–[13]; a recent review of these papers is published in [14]. Asymptotic results for finite input sum capacity have been derived by [15] and [16]. But these asymptotic results are based on replica theory that has not been proven rigorously for all cases [17]. The asymptotic results discussed in [16] also covered the near-far effects with perfect channel state estimation.

We derive different bounds in the absence and presence of channel state estimation. Although the asymptotic sum capacity in [16] is not rigorously derived, it falls between our bounds with the assumption of perfect channel state estimation. In the present paper, we have also derived a method that can estimate the sum capacity when perfect channel state estimation is not available. This method depends on the sum capacity evaluation in the absence of near-far effects. We have used the bounds derived from [12] as well as the asymptotic results by [15] and [16] to find new asymptotic bounds for the sum capacity where there is imperfect channel state estimation.

The rest of the paper is organized as follows: In Section II we will introduce a channel model in the presence of near-far effects. In the subsequent section, some relevant bounds will be derived for uniquely decodable codes. The same section includes numerical results related to Bit Error Rate (BER) versus $E_b/N_0$ for the proposed signature codes and decoders. In Section IV we will derive asymptotic lower and upper bounds for channel capacity for two different scenarios, namely, perfect and imperfect channel state estimation. The conclusion and future works are covered in Section V.

II. CHANNEL MODEL

In a DS-CDMA system, each user is assigned a signature vector. Each user multiplies its signature by its data and transmits it through a common channel. All vectors are added up together in the channel and the resultant vector embedded in noise is received. In such a system, without perfect power control, the assumption of receiving equal powers from all transmitters is no longer valid. Thus, in a synchronous CDMA system with $n$ users and $m$ chips in the presence of noise and
near-far effects, the channel model is

$$ Y = \sum_{i=1}^{n} \frac{1}{\sqrt{m}} A_i M_i X_i + N = \frac{1}{\sqrt{m}} A M X + N, \quad (1) $$

where \( A = [A_1|\cdots|A_n] \) is the \( m \times n \) signature matrix, \( M = \text{diag}(M_1,\cdots,M_n) \), in which \( M_i \) is the channel gain. \( X = [X_1,\cdots,X_n]^T \in \mathbb{Z}^n \) is the user data vector, where \( \mathbb{Z} \) is the input alphabet and \( N \) is i.i.d. Gaussian noise vector. In a CDMA system with no near-far effects, the diagonal matrix \( M \) is the identity matrix. Also assume that \( M_i \)'s are i.i.d. random variables and \( M_i = G_i + E_i \), where \( G_i \)'s are the estimation of the amplitudes at the receiver and \( E_i \)'s are the estimation errors.

Also define the Power Control Factor (PCF) of a CDMA system as

$$ \text{PCF}_{\text{db}} = 10 \log_{10} \frac{\mathbb{E}[\text{Re}(G_i)^2]}{\text{Var}[	ext{Re}(E_i)]}, \quad (2) $$

where \( \text{Re}(\cdot) \) is the real part function. PCF is the ratio of the estimated channel power divided by the channel state estimation error.

### III. Uniquely Decodable Codes

#### A. Error-less Codes for CDMA Systems with Near-Far Effects in the Absence of Noise

In the absence of channel state estimation, we assume that \( M_i \)'s have symmetric distributions around one, \( G = I \) and \( \mathcal{I} = \{\pm 1\} \). Also we suppose a compact support distribution for \( E_i \)'s; thus \( E_i \)'s belong to the interval \([-\eta,\eta]\). We rewrite (1) as

$$ Y = \frac{1}{\sqrt{m}} A X + \frac{1}{\sqrt{m}} A Z + N, \quad (3) $$

where \( Z_i = E_i X_i \). Based on the assumption of binary bipolar input and symmetric distribution of \( Z \), since the conditional probability \( \mathbb{P}(Z|X) \) is always equal to \( \mathbb{P}(Z) \) we conclude that \( Z \) is independent of \( X \).

The first question that we would like to address is to find the maximum value of \( \eta \) such that the mapping from \( X \) to \( Y \) in (3) is uniquely decodable in the absence of noise. This is possible if the \( 2^n \cdot m \)-dimensional shapes \( \{X + [-\eta,\eta] \times \mathcal{I}\}^n \) are mutually disjoint for different values of \( X \)- see Fig. 1.

Define \( \eta_{\text{sup}}(A) \) to be the supremum value of \( \eta \) for which these shapes are disjoint. Thus, for a uniformly distributed \( E_i \) on \([-\eta,\eta]\), we have

$$ \text{PCF}_{\text{inf}}(A) = 10 \log_{10} \frac{1}{\eta_{\text{sup}}^2 / 3}. \quad (4) $$

In the following subsection, we find lower and upper bounds for \( \eta_{\text{sup}}(A) \).

#### 1) Lower and Upper Bounds for \( \eta_{\text{sup}} \):

**Theorem 1:** Lower Bound for \( \eta_{\text{sup}} \) in Binary Input and Arbitrary Signature Matrix CDMA Systems

For any norm \( \| \cdot \| \) on \( \mathbb{R}^m \), we get

$$ \eta_{\text{sup}}(A) \geq \frac{1}{2} \min_{X \in \{0,\pm 2\}^n} \| \frac{1}{\sqrt{m}} A X \|. \quad (5) $$

The proof is given in Appendix A.

**Theorem 2:** Upper Bound for \( \eta_{\text{sup}} \) in Binary Input and Real Signature Matrix CDMA Systems

For any \( m \times n \) signature matrix \( A \), we have the following inequality:

$$ \eta_{\text{sup}}(A) \leq \frac{1}{2 \pi - 1}. \quad (6) $$

The proof is given in Appendix B.

**Example 1:**

For the matrix

$$ A_{2 \times 4} = \begin{bmatrix} 1 & 0 & 0.5 & -0.5 \\ 0 & 1 & 0.5 & 0.5 \sqrt{3} \end{bmatrix}, $$

the upper bound of Theorem 2 states that \( \eta_{\text{sup}}(A_{2 \times 4}) \leq 0.33 \). We have numerically evaluated the lower bound given in (5) for 100 random norms on \( \mathbb{R}^2 \) and have found that \( \eta_{\text{sup}}(A_{2 \times 4}) \geq 0.18 \). For generating random norm, we have chosen a random matrix \( A \) and a random number \( p > 1 \). Define \( \| X \|_p^A = \| A X \|_p \) where \( \| \cdot \|_p \) denotes the \( \ell_p \) norm. Computer simulations have shown that \( \eta_{\text{sup}}(A_{2 \times 4}) \approx 0.21 (\text{PCF}_{\text{inf}} \approx 18.32 \text{dB}) \).

**Example 2:**

We have evaluated the lower and upper bounds of \( \eta_{\text{sup}}(A_{8 \times 13}) \) for the \( 8 \times 13 \) COW signature matrix which is the uniquely decodable binary matrix as suggested in B. The upper bound given in Theorem 2 is 0.48 and the lower bound of (5) obtained by evaluating over 100 random norms similar to Example 1 is 0.13. The simulation results show that \( \eta_{\text{sup}}(A_{8 \times 13}) \approx 0.23 (\text{PCF}_{\text{inf}} \approx 17.53 \text{dB}) \).
Now, we would like to construct large signature matrices that are robust against near-far effects:

2) Constructing Large Signature Matrices from Small Ones:

The evaluation of lower bound (26) for large size matrices needs huge amount of computations and is not practical. For this reason, in this section, we propose a method for constructing large robust matrices from small ones.

**Theorem 3: Constructing Large Matrices**
Assume $P$ is an invertible $k \times k$ matrix, then

$$\eta_{sup}(P \otimes A) = \eta_{sup}(A),$$  \hspace{1cm} (7)

where $\otimes$ is the Kronecker product.

The proof is given in Appendix C.

By using the lower bound (26) we can find small size near-far resistant signature matrices and by using Theorem 3 these matrices can be enlarged without changing $\eta_{sup}$.

From the above theorem, we derive the following corollary:

**Corollary 1:**
Using Corollary 1 and Theorem 3 for any invertible matrix $P$, we can derive another lower bound

$$\eta_{sup}(A) \geq \min_{X \in \{0, \pm 1\}^{kn}} \| (P \otimes A) X \| / \max_{X \in \{\pm 1\}^{kn}} \| (P \otimes A) X \|,$$  \hspace{1cm} (8)

In the next subsection, we propose a very low complexity decoder for a subclass of these codes.

**B. A Decoding Method for a Class of Near-Far Resistant Codes**

For highly over-loaded systems, conventional methods for estimating the user powers do not work. However, the decoding method presented in this section can give impressive results in the absence of channel state estimation for noisy channels and near-far effects.

In [19], a very low complexity method for decoding COW signature codes is proposed. Here, we use those ideas to decode near-far resistant matrices. For overloaded systems, the generalized central limit theorem suggests that from (3), we can approximate $\frac{1}{\sqrt{m}} A Z + N$ with a Gaussian vector ($W$) with zero mean and auto-covariance matrix $\frac{n^2}{3m} A A^T + \sigma^2 I$. This approximation becomes better as the loading factor grows [20]. Thus, from now on we consider the channel model as

$$Y = \frac{1}{\sqrt{m}} A X + W,$$  \hspace{1cm} (9)

where $W$ is a zero mean Gaussian random vector with the covariance matrix $\frac{3}{2m} A A^T + \sigma^2 I$. Similar to [21] and [8], we prove a lemma that significantly decreases the complexity of the decoding problem for a signature matrix that is obtained by Kronecker product similar to Theorem 3.

**Lemma 1:**
Assume $P$ is an invertible matrix and $D_{km \times kn} = P_{k \times k} \otimes \frac{1}{\sqrt{m}} A_{m \times n}$. The decoding problem of a system with the signature matrix $D$ can be decoupled to $k$ decoding of a system with the signature matrix $\frac{1}{\sqrt{m}} A$.

The proof is given in Appendix D. The following lemma reduces the decoder complexity even further.

**Lemma 2:**
Suppose $\frac{1}{\sqrt{m}} A_{m \times n}$ is full rank. The decoding problem for a system with the signature matrix $\frac{1}{\sqrt{m}} A$ can be performed by $2^n - m$ Euclidean distance calculations instead of $2^n$.

Please refer to Appendix C for the proof.

**Example 3:**
From Theorem 3 and Example 1 $D_{64 \times 128} = I_{32} \otimes A_{2 \times 4}$ ($A_{2 \times 4}$ is the matrix in Example 1 and $\eta_{sup}(D_{64 \times 128}) \approx 0.21$, i.e., $PCF_{inf}(D_{64 \times 128}) \approx 18.32 dB$. Since $I_{32}$ is a unitary matrix, the first two columns of $A$ is also a unitary matrix and the rows of $A$ are orthogonal to each other, Lemmas 1 and 2 result in a decoder for a system with signature matrix $D_{64 \times 128}$. The proposed method of decoding requires $32 \times 2^2$ Euclidean norm calculations instead of $2^{128}$ such calculations in direct implementation. The performance of this code in an AWGN channel is simulated in the next section.

**Example 4:**
Similar to Example 3 for $D_{64 \times 104} = H_8 \otimes \frac{1}{\sqrt{8}} A_{8 \times 13}$, where $H_8$ is an $8 \times 8$ Hadamard matrix and $\frac{1}{\sqrt{8}} A_{8 \times 13}$ is the matrix in Example 2 $\eta_{sup}(D_{64 \times 104}) \approx 0.23$, i.e., $PCF_{inf}(D_{64 \times 104}) = 17.53 dB$. The advantage of this matrix is that its entries are $\pm 1$ (it is in fact a COW matrix). Since $\frac{1}{\sqrt{8}} H_8$ is a unitary matrix, according to Lemma 1 the decoder of $D_{64 \times 104}$ can be implemented by 8 decoders of $\frac{1}{\sqrt{8}} A_{8 \times 13}$. This implies significant reduction in the complexity of the decoder, i.e., $8 \times 2^{13}$ Euclidean norm calculations instead of $2^{104}$ such calculations. However, using Lemma 2 we obtain a sub-optimum decoder with $8 \times 2^5$ Euclidean norm calculations. This decoder is not ML because the rows of $\frac{1}{\sqrt{8}} A_{8 \times 13}$ are not orthogonal but its performance is good.

**C. Simulation Results**

We have simulated two overloaded binary (64, 128) and (64, 104) CDMA systems. The code matrices used for these simulations are $D_{64 \times 104}$ and $D_{64 \times 128}$, which are introduced in Examples 3 and 4. The advantage of the system with 104 users is that its signature matrix is binary antipodal which is practically favorable. In our simulations, we have assumed that the near-far effects for each user is a white random process, i.e., there is no correlation between its time samples. Obviously, this scenario is much worse than what occurs in practical situations. The assumption of correlation of Markov models and Viterbi algorithm and decoding is a part of our future activities [22], [23].

The advantage of the decoding method presented in subsection III-B is that we have assumed that the receiver has no knowledge about the received user powers.
A. Perfect Channel State Estimation

For perfect channel state estimation, $\rho = 0$ and hence the user amplitudes are known without any ambiguity at the receiver. The following two theorems are related to lower and upper bounds for the sum capacity of the CDMA systems with near-far effects.

**Theorem 4: Lower Bound for the Sum Capacity of CDMA Systems with Perfect Channel State Estimation**

In a CDMA system with perfect channel state estimation, we have the following lower bound for the average sum capacity

$$c(\beta, I, S_\pi, \eta, g, 0) \geq \sup \sup_{\gamma} \left\{ -\frac{1}{2\beta} (\gamma \log e - \log (1 + \gamma)) \right\} - \log e \times \sup_{\theta \in \mathbb{R}} \left\{ -\frac{1}{2\beta} \ln \left( 1 + \frac{2\eta_\gamma\beta\theta}{\sigma_p^2(1 + \gamma)(\sigma_q^2 + \mu_s^2)} \right) - I(\theta) \right\}, \quad (10)$$

where $I(\theta) = \sup_{x \in \mathbb{R}} \{ \theta x - \ln E(e^{(XG_1)^2}) \}$ is the Legendre transform of $(XG_1)^2$, in which $G_1$ is as defined in Section \ref{appendix} and $X$ is the difference random variable as defined in \ref{26}. For the proof, please refer to Appendix \ref{app}

**Theorem 5: Upper Bound for the Sum Capacity of CDMA Systems with Perfect Channel State Estimation**

If the input alphabets come from a finite set, we have the following upper bound for the average sum capacity

$$c(\beta, I, S_\pi, \eta, g, 0) \leq \min \left( \log |I|, \frac{1}{2\beta} \max_{p(\cdot)} \log (1 + \beta \frac{\text{Var}[A_1G_1X_1]}{\sigma^2}) \right), \quad (11)$$

where $A_1, G_1, X_1$ and $N_1$ are independent random variables with distributions $\pi(\cdot), g(\cdot), p(\cdot)$ and $\mathcal{N}(0, \sigma^2)$, respectively. For the proof, please refer to Appendix \ref{app}

B. Imperfect Channel State Estimation

We will use the bounds derived in the previous subsection as well as the asymptotic derivation for sum channel capacity from the CDMA literature to obtain lower and upper bounds.

**Theorem 6: Lower and Upper Bounds for the Sum Capacity of CDMA Systems with Imperfect Channel State Estimation**

Suppose that $\rho$ is not zero, which implies that we have an imperfect estimation of user powers.

$$c(\beta, \{\pm 1\}, S_\pi, \eta, g, \rho) \geq c(\beta, \{\pm 1\}, S_\pi, \eta_\pi, g, 0), \quad (12)$$

$$c(\beta, \{\pm 1\}, S_\pi, \eta, g, \rho) \leq c(\beta, \{\pm 1\}, S_\pi, \eta_\rho, g, 0), \quad (13)$$

Simulations are plotted for various values of PCF. Figs. \ref{fig2} and \ref{fig3} show that for PCF values greater than PCF$_{\text{inf}}$, the BER tends to zero as $E_b/N_0$ grows, and for PCF values less than PCF$_{\text{inf}}$, the BER saturates and error-less transmission is not possible. The simulation results show the robustness of the proposed codes against noise and near-far effects. Both systems employ the proposed decoder and thus have very low complexity.

IV. CHANNEL CAPACITY

In this section, we will derive lower and upper bounds for capacity of CDMA systems with near-far effects. Initially, we need some definitions and assumptions. Suppose $E$ is a diagonal matrix with i.i.d Gaussian random variables with variance $\rho^2$ and $G$ is a diagonal matrix with i.i.d random variables with distribution $g(\cdot)$ as defined in Section \ref{appendix}. For a fixed $g(\cdot)$ and $\rho$, define $c(\beta, I, S_\pi, \eta, g, \rho)$ to be the per user capacity averaged over all random matrices $A$ with i.i.d components of distribution $\pi(\cdot)$ with average $E_b/N_0$ of $\eta$. Here, $\beta$ is the loading factor, $I$ and $S$ are the input and the signature alphabets, respectively. Also, we have an additional assumption which is $\mu_s = 0$ where $p(\cdot)$ is the probability distribution function on $I$. This assumption is practically favorable since we prefer to have transmitters with zero transmitting mean.

Below, we have four subsections. Subsection \ref{IV-A} is related to real systems with perfect channel state estimation, subsection \ref{IV-B} is related to real systems with imperfect channel state estimation, subsection \ref{IV-C} is related to complex systems with perfect and imperfect channel state estimations, and subsection \ref{IV-D} is on simulation results.

![Fig. 2. BER versus $E_b/N_0$ for binary CDMA system with 64 chips and 104 users (binary inputs and signature).](image1)

![Fig. 3. BER versus $E_b/N_0$ for binary CDMA system with 64 chips and 128 users (binary inputs).](image2)
where

\[ \eta_t = \frac{\eta}{1 + \frac{\mu^2}{\sigma^2 + \mu^2} (1 + \eta (1 + \sqrt{\beta})^2)} \]

\[ \eta_u = \frac{\eta}{1 + \frac{\mu^2}{\sigma^2 + \mu^2} (1 + \eta (1 - \sqrt{\beta})^2)} \]  \hspace{1cm} (14)

For the proof please refer to Appendix [I].

**Example 5:**
For the binary input vectors and signature matrices, from Theorem 6 and [15], we get

\[ \frac{1}{2\beta} \log (1 + 2\eta_t \beta (1 - \theta)) + q(\lambda, \theta) \log (e) \geq c(\beta, \{\pm 1\}, \{\pm 1\}, \eta, \delta (-1), \rho) \geq \frac{1}{2\beta} \log (1 + 2\eta_u \beta (1 - \theta)) + q(\lambda, \theta) \log (e) \]  \hspace{1cm} (15)

where \( \delta (-1) \) is the unit delayed Dirac delta function representing the pdf of a point process and \( q(\lambda, \theta) \) is defined by

\[ q(\lambda, \theta) = \frac{1}{\sigma^2 + \lambda (1 - \theta)} - \int \ln (\cosh (\sqrt{\lambda} Z + \lambda)) D_Z \]  \hspace{1cm} (16)

in which \( D_Z \) is the standard normal measure,

\[ \lambda = \frac{1}{\sigma^2 + \beta (1 - \theta)}, \quad \theta = \int \tanh (\sqrt{\lambda} Z + \lambda) D_Z \]  \hspace{1cm} (17)

This result is based on replica theory which is a non-rigorous mathematical analysis. A rigorous proof of [13] is given in [17] for \( \beta \leq \alpha_s \approx 1.49 \).

**Example 6:**
For the binary input vectors and signature matrices, we can also use the lower bound derived in [12] to get

\[ c(\beta, \{\pm 1\}, \{\pm 1\}, \eta, \delta (-1), \rho) \geq 1 - \inf_{\gamma \in [0, 1]} \sup_{t \in [0, 1]} \left[ h(t) + \frac{1}{2\beta} \left( \gamma \log e - \log (1 + \gamma (1 + 8t \eta)) \right) \right] \]  \hspace{1cm} (18)

where \( h(t) = -t \log t - (1 - t) \log (1 - t) \).

**Example 7:**
Again for the binary CDMA system with perfect power control we have an upper bound derived in [12]

\[ c(\beta, \{\pm 1\}, \{\pm 1\}, \eta, \delta (-1), \rho) \leq \min \left( 1, \frac{1}{2\beta} \log (1 + 2\beta \eta_u) \right) \]  \hspace{1cm} (19)

In this example similar to [12], [15] we conjecture uniform input distribution maximizes the capacity.

These examples are simulated and given in subsection [V-D].

**C. Complex-Valued Channels**

By a complex valued channel, we mean that the entries of \( E \) are i.i.d. complex Gaussian random variables with independent real and imaginary parts of variance \( \rho^2 \). Similarly, the entries of \( N \) are i.i.d. complex Gaussian random variables with independent real and imaginary parts of variance \( \sigma^2 \).

By perfect/imperfect channel state estimation, we mean that the receiver has an accurate/inaccurate estimation of both amplitude and phase of the complex matrix \( M \).

Theorems [7] and [8] are related to the lower and upper bounds for the sum capacity with perfect channel state estimation. Theorem [9] is, on the other hand, related to the bounds with imperfect channel state estimation.

**Theorem 7: Lower Bound for the Sum Capacity of CDMA Systems with Perfect Channel State Estimation**
Assume \( S \subset \mathbb{R} \). The lower bound for the sum capacity is given by

\[ c(\beta, I, S, \eta, g, 0) \geq \sup_{\gamma} \left\{ -\frac{1}{\beta} (\gamma \log e - \log (1 + \gamma)) - \log e \times \right. \]

\[ \left. \sup_{\theta \in \mathbb{R}} \left\{ -\frac{1}{2\beta} \ln \left( 1 + \frac{2 \eta \beta^2}{\sigma^2 (1 + \gamma)(\sigma^2 + \mu^2)} \right) (\theta_1 + \theta_2) \right. \right. \]

\[ + \left. \left. \left( \frac{2 \eta \beta}{\sigma^2 (1 + \gamma)(\sigma^2 + \mu^2)^2} (\theta_1 \theta_2 - \theta_3^2) - I(\theta) \right) \right\} \right. \]  \hspace{1cm} (20)

where \( I(\theta) \) is the Legendre transform of \( \text{Re}(G_1 X_1)^2, \text{Im}(G_1 X_1)^2, \text{Re}(G_1 X_1) \text{Im}(G_1 X_1) \).

The proof is given in Appendix [I].

**Corollary 2:** The extension of above theorem to the complex valued \( S \) is

\[ c(\beta, I, S, \eta, g, 0) \geq \sup_{\gamma} \left\{ -\frac{1}{\beta} (\gamma \log e - \log (1 + \gamma)) - \log e \times \right. \]

\[ \sup_{\theta \in \mathbb{R}} \left\{ -\frac{1}{2\beta} \ln \left( 1 + \frac{2 \eta \beta^2}{\sigma^2 (1 + \gamma)(\sigma^2 + \mu^2)} \right) (\theta_1 + \theta_2) \right. \right. \]

\[ + \left. \left. \left( \frac{2 \eta \beta}{\sigma^2 (1 + \gamma)(\sigma^2 + \mu^2)^2} (\theta_1 \theta_2 - \theta_3^2) - I(\theta) \right) \right\} \right. \]  \hspace{1cm} (21)

where \( I(\theta) \) is the Legendre transform of \( \text{Re}(b_1 G_1 X_1)^2, \text{Im}(b_1 G_1 X_1)^2, \text{Re}(b_1 G_1 X_1) \text{Im}(b_1 G_1 X_1) \), where \( b_1 \) is the first entry of \( b \).

**Theorem 8: Upper Bound for the Sum Capacity of CDMA Systems with Perfect Channel State Estimation**
In such a system the sum capacity is upper bounded by

\[ c(\beta, I, S, \eta, id, 0) \leq \min \left( \log |I|, \frac{1}{2\beta} \max_{\theta} \log \left( \frac{\det G}{\rho^2} \right) \right) \]  \hspace{1cm} (22)
where $\Sigma$ is the covariance matrix of real and imaginary parts of $\sqrt{\beta}a_1X_1 + N_1$, in which $a_1$ and $X_1$ are two independent random variables with corresponding distributions $\pi(\cdot)$ and $p(\cdot)$, and $N_1$ is a complex Gaussian random variable with independent real and imaginary with variance of $\sigma^2$.

The proof is similar to Appendix C.

**Theorem 9:** Lower and Upper Bounds for the Sum Capacity of CDMA Systems with Imperfect Channel State Estimation

In the absence of perfect channel state information, we can derive the following bounds

\[
c(\beta, I, S, \eta, g, \rho) \geq c(\beta, I, S, \eta, g, 0),
\]

\[
c(\beta, I, S, \eta, g, \rho) \leq c(\beta, I, S, \eta, g, 0).
\]

where,

\[
\eta_l = \frac{\eta}{1 + \frac{2\rho^2}{\sigma^2} \left(1 + \eta \left(1 + \sqrt{\beta}\right)^2\right)},
\]

\[
\eta_u = \frac{\eta}{1 + \frac{2\rho^2}{\sigma^2} \left(1 + \eta \left(1 - \sqrt{\beta}\right)^2\right)}.
\]

The proof is similar to the proof of Theorem 6.

**D. Numerical Results**

Examples 5, 6 and 7 have been numerically evaluated. Figure 4 shows a comparison between two lower and two upper bounds obtained from (15), (18) and (19). As $\eta$ increases, Tanaka’s formula approaches the upper bound proposed in (19) (shown under perfect channel state estimation assumption in [12]). Hence, the gap between the upper bounds is omitted for larger values of $\beta$. Figure 5 shows bounds for three different values of PCF, namely, 15 dB, 18 dB, and the case of perfect power control. It is interesting to note that since $\beta$ is small, the upper bounds obtained for different values of PCF coincide with the capacity. Figure 6 shows the dependence of bounds on $\beta$. When $\beta$ increases, a greater power control is necessary to achieve the same capacity.

Except for the one curve which is identified by perfect power control in Fig. 7, we have assumed that user amplitudes have distribution $N(1, \rho^2)$ (which is equivalent to Rician power distribution). It interesting to note the gap when perfect channel state estimation is available.

Figure 8 shows different curves for the flat fading channel which distribution of power is Rayleigh.

Except Fig. 9 other figures are about real systems. Figure 9 makes a comparison between real and complex systems; as expected, the lower bound for the complex system falls below that of the real system.

**V. CONCLUSION AND FUTURE WORKS**

We first studied uniquely decodable codes that were near-far resistant. For every matrix, we proposed lower and upper bounds for the maximum near-far effects ($\eta_{\text{HFU}}$). One topic of interest is to find matrices that tolerate wide near-far effects. Simple sub-optimum decoders were also discussed. Also we derived asymptotic bounds for the sum capacity with the assumption of perfect channel state estimation. One of the contributions of this paper is the development of a method that translates a near-far sum capacity problem with imperfect channel state estimation to the evaluation of the capacity for a CDMA system with perfect channel state estimation.

For future work, we suggest to use a Markov chain for the power model which can improve the bounds. We also suggest to find the sum capacity for finite dimensional CDMA systems with near-far effects.

**APPENDIX**

**A. Proof of Theorem 1** Lower Bound for $\eta_{\text{HFU}}$ in Binary Input and Arbitrary Signature Matrix CDMA Systems

In this subsection a generalized version of the Theorem 1 is developed and proved.

**Theorem 10:** Lower Bound for $\eta_{\text{HFU}}$ for Arbitrary Input and Arbitrary Signature Matrix CDMA Systems

The alphabets of the data input and signature matrix can take complex values; for any norm $\| \cdot \|$ on $\mathbb{R}^{2m}$, we have

\[
\eta_{\text{HFU}}(A) \geq \frac{1}{2} \min_{X \in \mathcal{I}^n} \max_{\hat{X} \in \mathcal{X}^n} \| F(\frac{\hat{X}}{\sqrt{m}} A \hat{X}) \|,
\]

where $\mathcal{I} = \{0, 1\}^n$ and $\mathcal{F} : \mathbb{C}^m \rightarrow \mathbb{R}^{2m}$ is a function such that $\mathcal{F}(z_1, \ldots, z_m) = (\text{Re}(z_1), \text{Im}(z_1), \ldots, \text{Re}(z_m), \text{Im}(z_m))$.

**Proof:** If there exists $X_1, X_2 \in \mathcal{I}^n$ and $E_1, E_2$ such that $\frac{1}{\sqrt{m}} A(I +$
\[ E_1 X_1 = \frac{1}{\sqrt{m}} A (I + E_2) X_2, \] there exists \( Z_1, Z_2 \) such that \( \frac{1}{\sqrt{m}} A (X_1 + Z_1) = \frac{1}{\sqrt{m}} A (X_2 + Z_2). \) Thus, \( \frac{1}{\sqrt{m}} A (X_1 - X_2) = \frac{1}{\sqrt{m}} A (Z_2 - Z_1). \) Hence, the equation is formed as \( \frac{1}{\sqrt{m}} A X^* = \frac{1}{\sqrt{m}} A Z^* \) where \( X^* \in \mathcal{I}^n. \) Suppose that \( z_1 \) is the first entry of \( Z^*, \) therefore \( z_1 = \eta_1 x_1 + \eta_2 x_2 \) for some \( \eta_1, \eta_2 \in [\eta, \eta] \) and \( x_1, x_2 \in \mathcal{I}. \) Thus we have \( z_1 = \left( \frac{1}{\sqrt{m}} \right) + (\frac{1}{\sqrt{m}}) (-x_1) + (\frac{1}{\sqrt{m}}) (x_2) + (\frac{1}{\sqrt{m}}) (x_2) \) hence \( z_1 \) can be written as a nonnegative linear combination of four elements of \( \mathcal{I}, \) which the coefficients sum up to \( 2\eta. \) Consequently there are \( \eta_1 \cdots \eta_i \in [0, \eta] \) and \( Z_1^1, \cdots, Z_i^i \in \mathcal{I}^n \) such that \( \eta_1 + \cdots + \eta_i = 2\eta \) and \( Z^i = Z_1^1 + \cdots + \eta_i Z_i^i. \) For the proof, we show that if \( \eta = \frac{1}{2} \left( \min_{X \in \mathcal{I}^n} \frac{\left\| A X \right\|}{\max_{X \in \mathcal{I}^n} \left\| A X \right\|} - \varepsilon \right), \) for any \( \varepsilon > 0, \) there are no such \( X^* \) and \( Z^*. \) Obviously, \[
\left\| \mathcal{F} \left( \frac{1}{\sqrt{m}} A X^* \right) \right\| = \left\| \mathcal{F} (\eta_1 Z_1^1 + \cdots + \eta_i Z_i^i) \right\| = \left\| \mathcal{F} (\eta_1 Z_1^1) + \cdots + \eta_i Z_i^i \right\| \leq \left\| F (Z_1^1) \right\| + \cdots + \eta_i \left\| F (Z_i^i) \right\| \leq 2\eta \max_{X \in \mathcal{I}} \left\| F \left( \frac{1}{\sqrt{m}} A X \right) \right\|. \]
Thus, \[
\left\| F \left( \frac{1}{\sqrt{m}} A X^* \right) \right\| \leq 2\eta \max_{X \in \mathcal{I}} \left\| F \left( \frac{1}{\sqrt{m}} A X \right) \right\| = \left( \min_{X \in \mathcal{I} - \{0\}^n} \left\| F \left( \frac{1}{\sqrt{m}} A X \right) \right\| / \max_{X \in \mathcal{I}^n} \left\| F \left( \frac{1}{\sqrt{m}} A X \right) \right\| - \varepsilon \right) \left\| F \left( \frac{1}{\sqrt{m}} A Z^* \right) \right\| \leq \min_{X \in \mathcal{I} - \{0\}^n} \left\| F \left( \frac{1}{\sqrt{m}} A X \right) \right\|, \]
which means that there are no \( X^* \) and \( Z^* \) such that \( \frac{1}{\sqrt{m}} A X^* = \frac{1}{\sqrt{m}} A Z^*. \) □

**B. Proof of Theorem 2** Upper Bound for \( \eta_{\text{sup}} \) in Binary Input and Real Signature Matrix CDMA Systems

Assume that the image of the shape \([-1, +1]^n\) by the linear transformation \( \frac{1}{\sqrt{m}} A X \) is \( \mathcal{F}. \) Assume that the \( m \)-dimensional volume of \( \mathcal{F} \) is \( \nu. \) Since the channel is error-less, the \( n \)-dimensional cubes around the points \( \{\pm 1\}^n \) must be mapped to non-overlapping shapes. Because these shapes are in the positions of the image of the shape \( Z \), we have

\[
\text{Volume} \left( \frac{1}{\sqrt{m}} A [-1, 1 + \eta]^n \right) \geq 2^n \nu \eta m \nu. \]

Thus, \( (1 + \eta)^m \nu \geq 2^n \eta m \nu. \) \hspace{1cm} (30)

Since this is valid for all \( \eta, \) therefore this is valid for \( \eta_{\text{sup}}. \) □

**C. Proof of Theorem 3** Constructing Large Matrices

Assume \( \eta > \eta_{\text{sup}}(A). \) According to the proof of Theorem 10 there exist \( X^* \in \mathcal{I} \) and \( Z^* \in [-\eta, +\eta]^n \) such that \( \frac{1}{\sqrt{m}} A X^* = \frac{1}{\sqrt{m}} A Z^*. \) Let \( X = [X^1 \cdots X^{k^n}]^T \in \{0, \pm 1\}^{kn} \) and \( Z = [Z^1 \cdots Z^{k^n}]^T \in [-\eta, +\eta]^{kn}, \) we have

\[
\begin{align*}
\left( P^{-1} \otimes \frac{1}{\sqrt{m}} A \right) \left( \mathbf{P} \otimes \frac{1}{\sqrt{m}} A \right) \mathbf{X} &= \left( I \otimes \frac{1}{\sqrt{m}} A \right) \mathbf{X} = \\
\left( I \otimes \frac{1}{\sqrt{m}} A \right) \mathbf{Z} &= \left( P^{-1} \otimes I \right) \left( \mathbf{P} \otimes \frac{1}{\sqrt{m}} A \right) \mathbf{Z},
\end{align*}
\]

thus \( \left( \mathbf{P} \otimes \frac{1}{\sqrt{m}} A \right) \mathbf{X} = \left( \mathbf{P} \otimes \frac{1}{\sqrt{m}} A \right) \mathbf{Z}. \) Consequently, \( \eta_{\text{sup}}(P \otimes C) \leq \eta_{\text{sup}} \left( \frac{1}{\sqrt{m}} A \right). \) Now consider
where $p(\cdot)$ and $\pi(\cdot)$ are probability distributions on $\mathcal{I}$ and $\mathcal{S}$, respectively, $r = \sqrt{\frac{\sigma_2^2}{\sigma_2^2 + \mu_2^2}}$, $b$ and $X$ are, respectively, vectors of length $n$ with i.i.d. entries of distribution $\pi(\cdot)$ and $\tilde{p}(\cdot)$, in which $\tilde{p}(\cdot)$ is defined to be the probability distribution on $\tilde{\mathcal{I}}$, which is the difference of two independent random variables of pdf $p(\cdot)$.

By changing the order of $E_G$ and $\sup$ and applying Jensen’s inequality for log function we get

\[
C(m, n, \mathcal{I}, \mathcal{S}_\pi, \eta, g) \geq \sup_{p, \gamma} \left\{ -m(\gamma \log e - \log (1 + \gamma)) - \log E_{\tilde{X},G} \left( \left( E_b \left( e^{\frac{-\gamma r^2}{(1+\gamma)m}}|b^r G \tilde{x}|^2 \right) \right)^m \right) \right\},
\]

(34)

Thus

\[
c(\beta, \mathcal{I}, \mathcal{S}_\pi, \eta, g, 0) = \lim_{n \to \infty} \frac{1}{n} C(m, n, \mathcal{I}, \mathcal{S}_\pi, \eta, g) \geq \sup_{p, \gamma} \left\{ -\frac{1}{1+2\beta}(\gamma \log e - \log (1 + \gamma)) + \lim_{n \to \infty n} \frac{1}{n} \log E_{\tilde{X},G} \left( \left( E_b \left( e^{\frac{-\gamma r^2}{(1+\gamma)m}}|b^r G \tilde{x}|^2 \right) \right)^m \right) \right\},
\]

(35)

Now using Varadhan’s lemma we compute

\[
\log e \times \lim_{n \to \infty n} \frac{1}{n} \log E_{\tilde{X},G} \left( \left( E_b \left( e^{\frac{-\gamma r^2}{(1+\gamma)m}}|b^r G \tilde{x}|^2 \right) \right)^m \right).
\]

(36)

Substitute $r$ and let $n \to \infty$

\[
E_b \left( e^{\gamma r^2/(1+\gamma)m} |b^r G \tilde{x}|^2 \right) \approx
\left( 1 + \frac{2\gamma \beta}{\sigma_2^2 (1 + \gamma) (\sigma_2^2 + \mu_2^2)} \left( \frac{1}{n} \sum_{i=1}^n (G_{i, \tilde{x}})^2 \right)^{\frac{\sigma_2^2}{\mu_2^2}} \right)^{-\frac{\mu_2^2}{\sigma_2^2}}.
\]

(37)

Letting $f(\theta) = -\frac{1}{2\theta} \log(1 + \frac{2\gamma \theta}{(1+\gamma)(\sigma_2^2 + \mu_2^2)})$ and using Varadhan’s Lemma

\[
\log e \times \lim_{n \to \infty n} \frac{1}{n} \log E_{\tilde{X},G} \left( \left( E_b \left( e^{\frac{-\gamma r^2}{(1+\gamma)m}}|b^r G \tilde{x}|^2 \right) \right)^m \right)
= \log e \times \lim_{n \to \infty n} \frac{1}{n} \log E_{\tilde{X},G} \left( E_b \left( e^{\frac{nf(\frac{\sigma_2^2}{\mu_2^2})}{\sigma_2^2 + \mu_2^2}} \sum_{i=1}^n (G_{i, \tilde{x}})^2 \right) \right)
\]

(38)

So the desired result follows.

\[\square\]
Due to symmetry, it is easy to show that $\mathbb{E}_A \{ \mathbb{H}(Y_1) \}$ is the same for all $1 \leq i \leq m$ and is equal to $\mathbb{E}_{A^1} \{ \mathbb{H}(Y_1) \}$, where $A^1$ denotes the first row of $A$; thus,

$$\mathbb{E}_A \left\{ \frac{1}{n} \max_{p(x)} I(X;Y) \right\} \leq \mathbb{E}_{A^1} \left\{ \frac{1}{\beta} \mathbb{H}(Y_1) - \mathbb{H}(N_1) \right\}.$$  

(40)

According to the central limit theorem, when $n, m \to \infty$ and $\frac{p}{m} \to \beta$, $Y_1$ will be a complex Gaussian random variable with a covariance matrix of $\Sigma$. Hence, for a fixed distribution $p(\cdot)$,

$$c(\beta, I, S_x, \eta, id, 0) \leq \frac{1}{\beta} \log \left( \frac{\det \Sigma}{\sigma^4} \right) = \frac{1}{2\beta} \log \frac{\det \Sigma}{\sigma^4}. $$

(41)

Maximizing over all distributions $p(\cdot)$, one can get the second term.

□

H. Proof of Theorem 6: Lower and Upper Bounds for the Sum Capacity of CDMA Systems with Imperfect Channel State Estimation

The system model in (1) can be written as

$$Y = \frac{1}{\sqrt{m}} A^T X \left( \frac{1}{\sqrt{m}} A^T X + N \right).$$

(42)

Assume that $I = \{ \pm 1 \}$, then the entries of $EX$ are i.i.d Gaussian random variables of variance $\rho^2$ independent of entries of $GX$.

Suppose that the minimum and maximum eigenvalues of $\frac{1}{\sqrt{m}} A^T A$ are $\lambda_{\min}$ and $\lambda_{\max}$, respectively. If $\frac{1}{\sqrt{m}} A^T X + N$ are substituted with $\left( \sqrt{\lambda_{\min}} \rho^2 + \sigma^2 \right) W$ and with $\left( \sqrt{\lambda_{\max}} \rho^2 + \sigma^2 \right) W$, $W$ is a standard Gaussian vector, two systems, with a capacity greater and less than the system represented by (42) are obtained.

Since entries of matrix $A$ are chosen independently at random from a set $S$ with a distribution $\pi(\cdot)$, with $\mu = 0$ and $m, n \to \infty$ such that $n/m \to \beta$. Then by using the Marcenko-Pastur theorem [24], the following equations are obtained:

$$\mathbb{P} \left( \lambda_{\min} \geq \sigma^2 \left( \sqrt{\beta} - 1 \right)^2 \right) \to 1,$$

(43)

$$\mathbb{P} \left( \lambda_{\max} \leq \sigma^2 \left( \sqrt{\beta} + 1 \right)^2 \right) \to 1.$$  

(44)

Therefore, by utilizing the proposed lower and upper bounds for CDMA systems with perfect power control, it is possible to achieve lower and upper bounds for CDMA systems with near-far effects. Note that when $\rho = 0$, these formulas yield perfect channel state estimation formulas.

□

I. Proof of Theorem 7: Lower Bound for the Sum Capacity of CDMA Systems with Perfect Channel State Estimation

The proof is very similar to the proof of theorem 6. The main difference is in computing

$$\log e \times \lim_{n \to \infty} \frac{1}{n} \ln \mathbb{E}_X \mathcal{G} \left( \left( \mathbb{E}_b \left( e^{\frac{r^2}{2}} |X|^2 \right) \right)^m \right)$$

(45)

Substituting $r$ and letting $n \to \infty$

$$\mathbb{E}_b \left( e^{\frac{r^2}{2}} |X|^2 \right) \approx \left( 1 + \frac{2\eta^2 \beta}{\sigma_p^2 (1 + \gamma)(\sigma^2 + \mu^2_n) \left( \frac{1}{n} \sum_{i=1}^n \text{Re}(G_i, X_i) \right)^2 + \left( \frac{1}{n} \sum_{i=1}^n \text{Im}(G_i, X_i) \right)^2 \right)^{-\frac{1}{2}}.$$  

(46)

Let $f(\theta_1, \theta_2, \theta_3) = -\frac{1}{2\beta} \ln \left( 1 + \frac{2\eta^2 \beta}{\sigma_p^2 (1 + \gamma)(\sigma^2 + \mu^2_n) \left( \frac{1}{n} \sum_{i=1}^n \text{Re}(G_i, X_i) \right)^2 + \left( \frac{1}{n} \sum_{i=1}^n \text{Im}(G_i, X_i) \right)^2 \right)^{-\frac{1}{2}}$ then Varadhan’s lemma implies

$$\log e \times \lim_{n \to \infty} \frac{1}{n} \ln \mathbb{E}_X \mathcal{G} \left( \left( \mathbb{E}_b \left( e^{\frac{r^2}{2}} |X|^2 \right) \right)^m \right) = \log e \times \lim_{n \to \infty} \frac{1}{n} \ln \mathbb{E}_X \mathcal{G} \left( e^{n f \left( \sum_{i=1}^n \text{Re}(G_i, X_i)^2 + \sum_{i=1}^n \text{Im}(G_i, X_i)^2 - \sum_{i=1}^n \text{Re}(G_i, X_i) \text{Im}(G_i, X_i) \right)} \right).$$

(47)

Hence the desired result follows.

□

ACKNOWLEDGMENT

We are sincerely grateful to Dr. K. Alishahi and Mr. M. Mansouri for their helpful comments.

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