RESEARCH ARTICLE | SEPTEMBER 13 2023

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AIP Conf. Proc. 2845, 060006 (2023)
https://doi.org/10.1063/5.0157108

An Investigation into the Module Theory of Multiplicative Subsets

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Abstract. If R is a commutative ring containing an identity element. Multiplicative subset Sin R (Mult-($S^{-1}R$)), means if $1\in S$ and $ab\in S$ for all a, b $\in$ S. In this article, we characterize some particular cases of multiplicative subset in general and we studied the multiplicative subset by using two types of module as projective and flat modules. The main results about multiplicative subsets in module theory have relations with projective and flat module. We proved that if S is a Mult-($S^{-1}R$), so $S^{-1}R$ is a flat R-module. Also we obtained if S is a Mult-($S^{-1}R$) and P denotes a projective-module, so $S^{-1}P$ is a projective $S^{-1}R$-module. The other result if S is a Mult-($S^{-1}R$), and M a module, then $M=S^{-1}M$ iff M is an $S^{-1}R$-module. Eventually, we obtained the following result; if S is a multiplicative-subset of a ring R, then $S^{-1}R=0$ iff S contains a nilpotent element.

Keyword: Flat Module, Integral Domain, Multiplicative Subset, Projective Module, Prime Ideal.

INTRODUCTION

In recent studies, researchers have introduced many extensions on module theory such as [1-8]. In this work, we necessity to demonstrate several important facts, firstly S is called Mult-($S^{-1}R$) if $1\in S$ and if $a,b\in S$ indicates $ab\in S$. "If S is a multiplicative-subset of a ring R, and M is an R-module we describe a new equivalence relation on M × S by (m; s)$\sim$ (n; t)$\Leftrightarrow$ u(tm-sn)=0 for some u $\in$ S and it is with ease checked these do not be contingent on the choices of representatives for the equivalence classes, and that we gain in this method an $S^{-1}R$-module such that $S^{-1}M=\{m/s:m \in M; s \in S\}$" (see [9,10], It is necessary to note that $S^{-1}R \cong R/I$ for some ideal $I \subset R$ such that the image of the map $Spec(S^{-1}R) \rightarrow Spec(R)$ is closed. Also, it should be mentioned that by [2,3], "It is well-known that if R is reduced, then $S^{-1}R$ is reduced also for every multiplicatively closed sets S and if R is a ring, S a multiplicative-subset, M and N modules, then

$$S^{-1}(M\otimes_R N) = S^{-1}M\otimes_R N = S^{-1}M\otimes_{S^{-1}R}S^{-1}N = S^{-1}M\otimes_R S^{-1}N\text{''}.$$

For S=R\{0} and P is a maximal ideal of integral domain R; $S\cap P=\emptyset$, we get P is prime. In [11], we see "a subset S of a ring R is multiplicatively closed if the element $1\in S$, and product of multiplication a.b $\in$ S for all a; b $\in$S". "Also, $S^{-1}R$ is ring of the field of fractions F of R, so any multiplicative subset T of R not containing the zero, in this case $T^{-1}R$ is a subring of F consisting of the fractions a/t with $a\in R$ and $t\in T$" (see [12]).

Here the definition of flat module is adopted in the usual sense such as in [18] and the definition of a projective module in [18]. Nielsen also looked at the relationship between the multiplicative subset and the fractions module. $U^{-1}M$ classes of equivalence (see [13-19]).

The National University of Science and Technology International Conference for Pure and Applied Sciences
AIP Conf. Proc. 2845, 060006-1–060006-7; https://doi.org/10.1063/5.0157108
Published by AIP Publishing. 978-0-7354-4652-6/$30.00

In this paper, we will use two modules in module theory to find connection between these modules and Mult-($S^{-1}R$). The main result is over commutative ring R if S is a Mult-($S^{-1}R$), then $S^{-1}R$ is a flat. Also we proved that if M is a flat and S denotes multiplicative(Mult) subset of R, then $S^{-1}M$ is a flat and $S^{-1}P$ is a projective ($S^{-1}R$)-module.

MAIN RESULTS

Definition 1. Let S be a subset of commutative ring R. Then S is called Mult-($S^{-1}R$) if

1. The number 1 belong to S.
2. ab belong to S for all a, b in S.

Example 1. We have *R*‚- {0} is integral domain. Then *R*‚- {0} is Mult-($S^{-1}R$).
Example 2. The R-*P*‚ is a Mult-($S^{-1}R$) of R and *P*‚ is prime ideal. Let x, y ∈ R-*P*‚. Then x, y ∉P. Hence xy ∉P. Thus xy ∈ R-P. Let a ∈ P. Then a.1 ∈ P. So P is active prime and a ∈ P ant this means 1 on belong P. Thus 1 ∈ R-P.
Example 3. If a belong to R, then $\{a^n$: n is non-negative integer$\}=\{1, a, a^2, \ldots\}$ is a Mult-($S^{-1}R$), because if $1 \in\{1, a, a^2, \ldots\}$ and x, y $\in\{1, a, a^2, \ldots\}$. Then $x=a^n$, $y=a^n$. So $xy= a^n b^n=a^{n+m} \in\{1, a, a^2, \ldots\}$.
Definition 2. If S is a Mult-($S^{-1}R$) of comm. ring R, then R×S={(r, s): r ∈R and s ∈ S} the set of all pair (r, s) and we define ∼ on R×S by:

$$(r, s)\sim (r^1, s^1) \Leftrightarrow \exists\ t \in S \ni t(rs^1 - r^1s)=0.$$

Also $\frac{r}{s}$ denote the equivalence class containing (r, s). We define:

$$S^{-1}R=\{\frac{r}{s} : r \in R \text{ and } s \in S\}$$

The set of all equivalence classes of ∼.
Remark 1.

1. If ($S^{-1}R$, +, .) is a comm. ring with identity, then it is called the fractions ring R by S and the zero of $S^{-1}R$ is $\frac{0}{1}=\frac{0}{s}$ ∀ s ∈ S, the identity of $S^{-1}R$ is $\frac{1}{1}=\frac{s}{s}$ ∀ s ∈ S.
2. Let R and S be a Mult-($S^{-1}R$) of R and I ideal in R:

$$S^{-1}I = \{\frac{a}{s}: a \in I \text{ and } s \in S\}$$

is an ideal of $S^{-1}R$.

3. For two ideals I and J of R and S is Mult-($S^{-1}R$), then
   - (*) $S^{-1}(I+J)=S^{-1}(I)+S^{-1}(J)$.
   - (**) $S^{-1}(I\cap J)=S^{-1}(I)\cap S^{-1}(J)$.

Assume that *M*‚ S are a module and Mult-($S^{-1}R$) and:

$$M'\times S=\{(m, s): m \in M' \text{and } s \in S\}.$$

Also we define ∼ on M×S by:

$$(m, s) \sim (m_1, s_1) \Leftrightarrow \exists\ t \in S \ni t(s_1m-sm_1)=0.$$

Let $\frac{m}{s}$ denote to equivalent class containing and let:

$$S^{-1}M=\{\frac{m}{s}: m \in M \text{ and } s \in S\}.$$

We define addition and multiplication on $S^{-1}M$ as:

$$(m/s)+ (m_1/s_1) = (s_1m+sm_1)/ss_1 \text{ and } r.(m/s) = (rm/s)\ \forall\ m/s\ ,\ m_1/s_1 \in S^{-1}M \text{ and } r \in R.$$

Then $S^{-1}M$ is an R-module. It is called the module of fraction of M by S.

Lemma 1. LetM' be an R-module. If S is a multiplicative subset of R, then $M\otimes S^{-1}R=S^{-1}M$.

Proof. We define:

$$g: M\otimes S^{-1}R\rightarrow S^{-1}M \text{ by } g(m, r/s)=(rm)/s\ \forall\ m \in M \text{ and } (r/s) \in S^{-1}R.$$

Now to prove that g is bilinear mapping.

$$g(r_1m_1 + r_2m_2), r/s)=r(r_1m + r_2m_2)/s$$

$$= (rr_1m_1+ rr_2m_2)/s$$

$$= (r_1rm_1/s) + (r_2rm_2/s)$$

$$= r_1(rm_1/s) + r_2(rm_2/s)$$

$$= r_1g(m_1, r/s) + r_2g(m_2, r/s)$$

Also

$$g(m, a_1(r_1/s_1) + a_2(r_2/s_2))=(ma_1r_1s_2+ma_2r_2s_1)/s_1s_2$$

$$= a_1(mr_1s_2)/s_1s_2 + a_2(mr_2s_1)/s_1s_2$$

$$= a_1(mr_1)/s_1 + a_2(mr_2)/s_2.$$

$$= a_1g(m, r_1/s_1) + a_2g(m, r_2/s_2)$$

Thus there exists a unique home:

$$g: M\otimes S^{-1}R\rightarrow S^{-1}M\ \ni h o \beta=g.$$

We define:

f: $S^{-1}M\rightarrow MS^{-1}R\ \ni f(m/s)=m\otimes(1/s)\ \forall\ m \in M'$ and $s \in S$

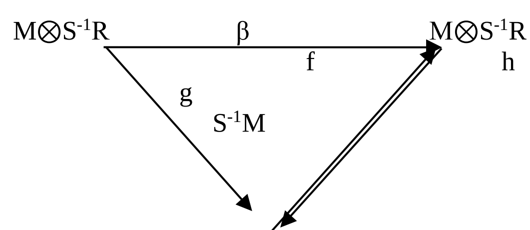

FIGURE 1. $M\otimes S^{-1}R$ unique home

Note that f is well-define and homomorphism because

$$f((m_1/s_1)+(m_2/s_2))=f(m_1s_2+m_2s_1)/s_1s_2$$

$$=(m_1s_2+m_2s_1) \otimes (1/s_1s_2)$$

$$=(m_1s_2\otimes(1/s_1s_2))+(m_2s_1\otimes(1/s_1s_2))$$

$$=(m_1 s_2 \otimes (1/s_1)) + ((1/s_1) m_2 s_1 \otimes (1/s_2))$$

$$=(m_1 \otimes (1/s_1)) + (m_2 \otimes (1/s_2))$$

$$=f(m_1/s_1) + f(m_2/s_2).$$

So

$$(foh)(m \otimes (r/s)) = f(h(\beta(m, r/s)$$

$$=f(g(m, r/s)$$

$$=f(rm/s)$$

$$=rm \otimes (1/s)$$

$$=m \otimes (r/s).$$

Thus

$$foh = Im \otimes S^{-1}R \ni m = Im \text{ and } r/s = S^{-1}R$$

and hence

$$(hof)(m/s) = h(f(m/s))$$

$$=h(m \otimes (1/s))$$

$$=h(\beta(m, 1/s)$$

$$=g(m, 1/s)$$

$$=m/s.$$

So of=IS^{-1}M and M⊗S^{-1}R≈S^{-1}M
In the following theorem, we show a best relation between Mult-(S^{-1}R) and flat R-module.
Theorem 1. If R be a comm. ring with unity and if S is a Mult-(S^{-1}R), then S^{-1}R is a flat R-module.
Proof. Let A and B be any two R-modules and let μ:A→B be a monomorphism mapping. Now we take μ⊗I=A⊗S^{-1}R→B⊗ S^{-1}R is a monomorphism. From Lemma 2.7, we have
A⊗S^{-1}R≈S^{-1}A and B⊗S^{-1}R≈S^{-1}B. Consider the following diagram:

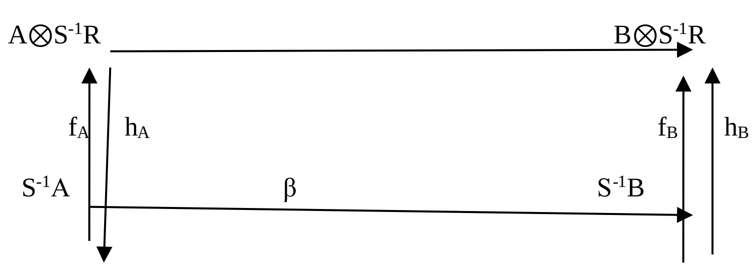

FIGURE 2. Monomorphism mapping

Now we prove this diagram is commutative.

$$f_B o\beta = (\mu \otimes I) o f_A$$

and

$$f_B o\beta(a/S)=f_B(\beta(a/S))$$

$$=f_B(\mu(a)/S)$$

$$=\mu(a)\otimes\frac{1}{s}.$$

Then

$$((\mu\otimes I)\ o\ f_A)\ (a/S)=\mu\otimes I(f_A(a/S))=\mu\otimes I(a\otimes\frac{1}{s})=\mu(a)\otimes I(\frac{1}{s})=\mu(a)\otimes\frac{1}{s}=f_B o\beta=(\mu\otimes I)of_A.$$

We define β: $S^{-1}A \rightarrow S^{-1}B$ by

$$\beta=h_B o(\mu\otimes I)of_A.\ \textit{So } \beta \textit{ is an R-homomorphism.}$$

$$\forall\ \frac{a}{s}\in S^{-1}A;\ \textit{we have}$$

$$\beta(\frac{a}{s})=h_B(\mu\otimes I)\ (f_A(\frac{a}{s}))$$

$$=h_B(\mu\otimes I)a\otimes\frac{1}{s}$$

$$=h_B\ (\mu(a)\otimes\frac{1}{s})$$

$$=h_B\beta(\mu(a),\frac{1}{s})$$

$$=g(\mu(a),\frac{1}{s})$$

$$=\mu(a)/S \qquad \text{………. (1)}$$

Let $w\in\ker(\mu\otimes I)$. Then there exists $\frac{a}{s}\in S^{-1}A$ such that $w=f_A(\frac{a}{s})$, because $f_A(\frac{a}{s})$, because f_A is onto. Then

$$0=(\mu\otimes I)(w)$$

$$=(\mu\otimes I)(f_A(\frac{a}{s}))$$

$$=f_B(\beta(\frac{a}{s}))$$

because the diagram is commutative and so $0=f_B(\mu(a)/S)$ by (1).

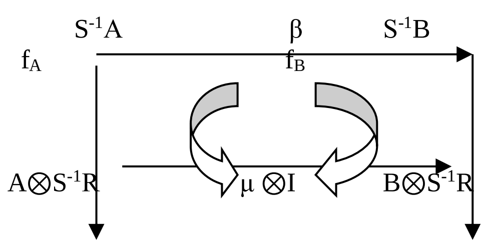

FIGURE 3. Diagram is commutative

So

$\mu(a)/S = \frac{0}{1}$, because f_B is a monomorphism. Hence

$$\exists t\in S \ni t(\mu(a) - 0.S) = 0 \text{ and } t\,\mu(a) = 0.$$

So $\mu(ta) = 0$

and then $ta = 0$, because μ is a monomorphism.

Now $w= f_A(\frac{a}{s})=f_A(\frac{at}{st}) =f_A(\frac{0}{st}) = 0$. Therefore $w= 0$ and then $\mu\otimes I$ is a monomorphism. Thus $S^{-1}R$ is a flat.

Next we need to understand how projective modules behave under multiplicative subset.

Theorem 2. Let S be a Mult-($S^{-1}R$) and P be a projctive-module. Then $S^{-1}P$ is a projctive $S^{-1}R$-module.

Proof. Let K be a module, suchthat $F: = K \oplus P$ is free. So that $S^{-1}F = S^{-1}P \oplus S^{-1}L$ and that $S^{-1}F$ is free over $S^{-1}R$. Then, $S^{-1}P$ is a projective $S^{-1}R$-module.

Corollary 1. If M is a flat R-module and S is the multiplicative-subset of R, then $S^{-1}M$ is a flat R-module.

Proof. Since M is an R-module and S is Mult-($S^{-1}R$), then $S^{-1}R$ is knowing a flat. Now since M is a flat and $S^{-1}R$ is flat, then $M\otimes S^{-1}R\approx S^{-1}M$. So $S^{-1}M$ is a flat.

Corollary 2. The set E is a Mult-($S^{-1}Z$) such that $E^{-1}Z$ is the area that deals with rational numbers.

Theorem 3. Let S be a Mult-($S^{-1}R$) such that R is a commutative ring. For an ideal I; $S^{-1}(\text{Rad } I) = \text{Rad }(S^{-1}(I))$.

Proof: First we need to prove that $S^{-1}(\text{Rad } I) \subseteq \text{Rad }(S^{-1}I)$., If $x/y \in S^{-1}(\text{Rad } I)$, then $x \in \text{Rad}(I)$ and $y \in S$. For $n > 0$, $x^n\in I$. As S is Mult-($S^{-1}R$), $y^n \in S$. Then $(x/y)^n = x^n y^n \in S^{-1}I$, and so $S^{-1}(\text{Rad } I) \subseteq \text{Rad }(S^{-1}I)$.

Now, let $x/y \in\text{Rad}(S^{-1}I)$. So if $m > 0$, $x^m/y^m =(x/y)^m \in S^{-1}I$. So, $\forall z \in I$, $w \in S$ we get $x^m/y^m=zw$. Thus $\exists s \in S$, such that $s(x^m w - y^m z) = 0$, or, $x^m sw =y^m sz$. Since $z \in I$, then $x^m sw = y^m sz \in I$. But S is a Mult-($S^{-1}R$) with y, s, $w \in S$; $ysw \in S$ and $(xsw)^m = x^m sw(sw)^{m-1} \in I$, then $(ysw)^m \in S$. Hence $(xsw/ysw)^m \in S^{-1}I$. Thus $xsw/ysw \in \text{Rad }(S^{-1}I)$. Any $\bar{s} \in S$, $\bar{s}\,(xysw - xysw) =0$, we get $x/y= xsw/ysw \in \text{Rad}(S^{-1}I)$. So $\text{Rad }(S^{-1}(I)) \subseteq S^{-1}(\text{Rad}(I))$.

Theorem 4. Let S be a Mult-($S^{-1}R$) and $\beta:R \mapsto R/I$ be the canonical projection such that I is denotes an ideal of R. Then $\beta S = \beta(S)$ is a Mult-($S^{-1}R/I$).

Proof. $\forall s, \bar{s} \in S$, as S is a Mult-($S^{-1}R$), we get $s\bar{s} \in S$. Thus $\beta(S)\beta(\bar{s})=\beta(s\bar{s})\in\beta(S)=\beta S$, and so βS is a Mult-($S^{-1}R$),.

As a result from Theorem 2, we introduce the following:

Corollary 3. A mapping $\beta: S^{-1}R \mapsto (\beta S)^{-1}(R/I)$ given by r/s $\beta(r)/\beta(s)$ knowed a well-defined of function.

Proof. If $r/s = \bar{r}/\bar{s} \in S^{-1}R$, then $r\bar{s} = \bar{r}s$. Since $\beta: R \mapsto R/I$ is knowing a hom.,

$$\beta(r)\beta(\bar{s})= \beta\,(r\bar{s})$$

$$= \beta\,(\bar{r}s)$$

$$= \beta(r)\beta(\bar{s}).$$

Finally

$$\beta\,(r/s) = \beta\,(r)/\beta\,(s)$$

$$= \beta\,(\bar{r})/\beta\,(\bar{s})$$

$$= \beta(\bar{r}/\bar{s}) \in (\beta S)^{-1}(R/I).$$

Theorem 5. Let S be a Mult-($S^{-1}R$),. Then, the proper ideals of a ring $S^{-1}R$ are form $IS^{-1}R = S^{-1}I = ({}^{i}/_{s} : i \in I, s \in S)$ with I ideal in R and $I \cap S = \emptyset$.

Proof. If J be a proper ideal of $S^{-1}A$. Let $I = J\cap A$. Then I is an ideal of A, and assume $I\cap S \neq\emptyset$. Take $s \in S\cap I$, then $s/1 \in J$ and hence $J = A$, a contradiction, so $I\cap S=\emptyset$.

Theorem 6. Let S be a Mult-($S^{-1}R$), and M'= a module. Then that $M=S^{-1}M'$= iff M'=is an $S^{-1}R$-module.

Proof. Let $M = S^{-1}M$, then very clear M is an $S^{-1}R$-module. As opposed to that, if M is an $S^{-1}R$-module, then from M and identity map we can characterize $S^{-1}M$; whence, $M=S^{-1}M$.

Theorem 7. If M be an R-module and If S is a Mult-($S^{-1}R$), then $S^{-1}M= 0$ if $\text{Ann}(M)\cap S \neq\emptyset$.

Proof. Let $f \in \text{Ann}(M)\cap S$ and let $\frac{m}{t} \in S^{-1}M$. Then $(\frac{f}{1})(\frac{m}{t}) =(\frac{fm}{t})$. Hence $\frac{m}{t}$=0. Thus $S^{-1}M= 0$.

Theorem 8. If S be a multiplicative subset of a ringR. Then $S^{-1}R=0$ iff S contains a nilpotent element.

Proof. From ([5], 1.1), $S^{-1}R = 0 \Leftrightarrow \frac{1}{1} = 0/1$. But, $\frac{1}{1} = 0/1 \Leftrightarrow 0 \in S$. So S is a multiplicative, $0 \in S \Leftrightarrow$ S contains a nilpotent element.

Corollary 4. All the rings of fractions of Noetherian rings are Noetherian.

Proof. Let S be a closed Mult-($S^{-1}R$) ∋ R is Noetherian Let $J \lhd S^{-1}R$, so $J = S^{-1}I$ ($\exists I \lhd R$), because all ideals of $S^{-1}R$ are extended. But A is Noetherian, so $I = < x_1 , \dots , x_n >$ for some $x_1, \dots , x_n \in R$, whence $J = < x_1/1 , \dots , x_n/1 >$. Thus all ideals of $S^{-1}R$ are f.g, which shows $S^{-1}R$ is Noetherian.

CONCLUSION

Multiplicative model is one of important branch of model theory. in this work, we Investigated a comprehensive study on this important branch. Several results were presented followed by proofs that confirm the importance of this branch. We proved that if S is a Mult-($S^{-1}R$), so $S^{-1}R$ is a flat R-module. Also we obtained if S is a Mult-($S^{-1}R$) and P denotes a projective-module, so $S^{-1}P$ is a projective $S^{-1}R$-module. The other result if S is a Mult-($S^{-1}R$), and M a module, then $M' = S^{-1} M'$ iff M' is an $S^{-1}R$-module. Finally, we hope that we have succeeded in demonstrating the importance of this concept.

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