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NEW OSCILLATION CRITERIA FOR THIRD ORDER NONLINEAR MIXED NEUTRAL DYNAMIC EQUATIONS

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Abstract. In this work, some new oscillation criteria are established for a third order nonlinear mixed neutral dynamic equation. Our results improve and extend some known results in the literature. Several examples are given to illustrate the importance of the results.

1. Intrduction.

First, we give a short review of the time scales calculus. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. For any $t \in \mathbb{T}$, we define the forward and backward jump operators by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}, \qquad \rho(t) := \sup\{s \in \mathbb{T} : s < t\},$$

respectively. The graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined by $\mu(t) := \sigma(t) - t$. A point $t \in \mathbb{T}$ is said to be right dense if

$$t < \sup \mathbb{T} \quad \text{and} \quad \sigma(t) = t,$$

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and a point $t \in \mathbb{T}$ is said to be left dense if

$$t > \inf \mathbb{T} \text{ and } \rho(t) = t.$$

Also, t is said to be right scattered if $\sigma(t) > t$, left scattered if $t > \rho(t)$. A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous if it is continuous at right dense points in $\mathbb{T}$ and its left-sided limit exists (finite) at left dense points in $\mathbb{T}$.

For a function $f : \mathbb{T} \to \mathbb{R}$, if there exists a number $\alpha \in \mathbb{R}$ such that for all $\varepsilon > 0$ there exists a neighborhood U of t with

$$|f(\sigma(t)) - f(s) - \alpha(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s|, \qquad \text{for all } s \in U,$$

then f is Δ-differentiable at t, and we call α the derivative of f at t and denote it by $f^{\Delta}(t)$,

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t}$$

if t is right scattered. When t is a right dense point, then the derivative is defined by

$$f^{\Delta}(t) = \lim_{t \to \infty} \frac{f(t) - f(s)}{t - s}$$

provided this limit exists.

If $f : \mathbb{T} \to \mathbb{R}$ is Δ-differentiable at $t \in \mathbb{T}$, then f is continuous at t. Furthermore, we assume that $g : \mathbb{T} \to \mathbb{R}$ is Δ-differentiable. The following formulas are useful:

$$\begin{aligned} f(\sigma(t)) &= f(t) + \mu(t) f^{\Delta}(t), \\ (fg)^{\Delta} &= f^{\Delta} g + f^{\sigma} g^{\Delta}, \\ \left(\frac{f}{g}\right)^{\Delta} &= \frac{f^{\Delta} g + f g^{\Delta}}{g g^{\sigma}}. \end{aligned}$$

Note that if $\mathbb{T} = \mathbb{R}$, we have

$$\sigma t) = t, \quad \mu(t) = 0, \quad f^{\Delta}(t) = f'(t), \quad \int_a^b f(t)\Delta t = \int_a^b f(t) dt.$$

When $\mathbb{T} = \mathbb{Z}$, we have

$$\sigma(t) = t + 1, \quad \mu(t) = 1, \quad f^{\Delta}(t) = \Delta f(t), \quad \int_a^b f(t)\Delta t = \sum_{t=a}^{b-1} f(t).$$

To more details about theory of time scales we refer to the books [3, 4] and the references cited therein.

By a Riccati transformation technique, we present some new oscillation criteria for the nonlinear dynamic equation of the form

$$\begin{aligned}(1.1)\quad &\Big(a(t)\left(b(t)(x(t)+p_1(t)x(t-\tau_1)+p_2(t)x(t+\tau_2))^{\Delta}\right)^{\Delta}\Big)^{\Delta}\\ &\qquad + q_1(t)x(t-\tau_3)+q_2(t)x(t+\tau_4)=0, \qquad t\geq t_0.\end{aligned}$$

Throughout this paper, we will assume the following hypotheses:

(A_1) a and b are rd-continuous positive functions on $\mathbb{T}$.

(A_2) $p_i(t)$ is rd-continuous positive functions on $\mathbb{T}$, such that, $0 \leq p_i(t) \leq p_i$, where p_i are constants for $i = 1, 2$.

(A_3) $q_i \in C_{rd}([0,\infty)_{\mathbb{T}}, [0,\infty))$ for $i = 1, 2$.

(A_4) $\tau_i \geq 0$ are constants for $i = 1, 2, 3, 4$.

If $\mathbb{T} = \mathbb{N}$, then (1.1) becomes the third order nonlinear mixed neutral difference equation

$$\begin{aligned}(1.2)\quad &\Delta\left(a(n)\Delta\left(b(n)\Delta\left(x(n)+p_1(n)x(n-\tau_1)+p_2(n)x(n+\tau_2)\right)\right)\right)\\ &\qquad + q_1(n)x(n-\tau_3)+q_2(n)x(n+\tau_4)=0, \qquad n\geq n_0.\end{aligned}$$

If $\mathbb{T} = \mathbb{R}$, then (1.1) becomes the third order nonlinear mixed neutral differential equation

$$\begin{aligned}(1.3)\quad &\Big(a(t)\left(b(t)\left(x(t)+p_1(t)x(t-\tau_1)+p_2(t)x(t+\tau_2)\right)'\right)'\Big)'\\ &\qquad + q_1(t)x(t-\tau_3)+q_2(t)x(t+\tau_4)=0, \qquad t\geq t_0.\end{aligned}$$

We set $z(t) := x(t)+p_1(t)x(t-\tau_1)+p_2(t)x(t+\tau_2)$. By a solution of (1.1) we mean a nontrivial real-valued function $x \in C^1_{rd}[\mathbb{T}_x,\infty)$, $\mathbb{T}_x \geq t_0$ which satisfies equation (1.1) on $[\mathbb{T}_x,\infty)$, where C_{rd} is the space of rd-continuous functions. A solution x of equation (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative and non-oscillatory otherwise. Equation (1.1) is called oscillatory if all its solutions are oscillatory. In recent years, there has been an increasing interest in the study of the oscillatory behavior of solutions of dynamic equations we refer to the book [1] and the papers [2, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19] and the references cited therein.

In this paper, the details of the proofs of results for nonoscillatory solutions will be carried out only for eventually positive solutions, since the arguments are similar for eventually negative solutions. The paper is organized as follows. In Section 2, we will state and prove the main oscillation theorems and we provide some examples to illustrate the main results.

2. Main results. In this section, we establish some new oscillation criteria for the equation (1.1) under the following condition

$$\int_{t_0}^{\infty} a^{-1}(s)\Delta s = \infty, \qquad \int_{t_0}^{\infty} b^{-1}(s)\Delta s = \infty. \tag{2.1}$$

In the following results, we shall use the following notations:

$$Q_1(t) := \min\{q_1(t), q_1(t-\tau_1), q_1(t+\tau_2)\},$$

$$Q_2(t) := \min\{q_2(t), q_2(t-\tau_1), q_2(t+\tau_2)\},$$

$$Q(t) = Q_1(t) + Q_2(t), \quad \delta(t) := \int_t^{\infty} a^{-\frac{1}{\alpha_2}}(v)\Delta v,$$

$$\varphi_1(t) := \frac{\beta(t)}{(\beta^\sigma(t))^2 b(t-\tau_3)} \int_{t_2}^{t-\tau_3} a^{-1}(v)\Delta v,$$

$$\varphi_2(t) := \frac{\beta(t)}{(\beta^\sigma(t))^2 b(t-\tau_1)} \int_{t_2}^{t-\tau_1} a^{-1}(v)\Delta v,$$

$$\vartheta(t,s) := \left(\frac{\beta^\Delta(t)}{\beta^\sigma(t)} - \frac{h(t,s)}{\sqrt{H(t,s)}} \right).$$

We assume that there exist functions $H, h \in C_{rd}(\mathbb{D}, \mathbb{R})$, where $\mathbb{D} = \{(t,s) \mid t \geq s \geq 0\}$ such that

(i) $H(t,t) = 0$ for $t \geq 0$,

(ii) $H(t,s) > 0$ for $t > s > 0$,

(iii) H has a nonpositive continuous Δ-partial derivative $H^{\Delta_s}(t,s)$ with respect to the second variable, and satisfies

$$h(t,s) = -\frac{H^{\Delta_s}(t,s)}{\sqrt{H(t,s)}}.$$

We begin with these Lemmas, which will be used in obtaining our main results.

Lemma 2.1. *Let $x(t)$ be an eventually positive solution of (1.1) and suppose that $z(t)$ satisfies $z^\Delta(t) > 0$, $\left(b(t)z^\Delta(t)\right)^\Delta > 0$, $\left(a(t)\left(b(t)z^\Delta(t)\right)^\Delta\right)^\Delta \leq 0$, for all $t \geq t_1$. Then there exists $t \geq t_1 \geq t_2$ such that*

$$z^\Delta(t) \geq b^{-1}(t)\left(a(t)\left(b(t)z^\Delta(t)\right)^\Delta\right) \int_{t_2}^{t} a^{-\frac{1}{\alpha_2}}(s)\Delta s. \tag{2.2}$$

Proof. Since $\left(a(t)\left(b(t)z^{\Delta}(t)\right)^{\Delta}\right)^{\Delta} \leq 0$, we have $a(t)\left(b(t)z^{\Delta}(t)\right)^{\Delta}$ is non-increasing. Then we obtain,

$$\begin{aligned} b(t)z^{\Delta}(t) &= b(t_1)\,\Delta z(t_1) + \int_{t_1}^{t} a^{-1}(s)a(s)\left(b(s)z^{\Delta}(s)\right)^{\Delta}\Delta s \\ &\geq a(t)\left(b(t)z^{\Delta}(t)\right)^{\Delta}\int_{t_1}^{t} a^{-1}(s)\Delta s. \end{aligned}$$

It follows that

$$z^{\Delta}(t) \geq b^{-1}(t)a(t)\left(b(t)z^{\Delta}(t)\right)^{\Delta}\int_{t_1}^{t} a^{-1}(s)\Delta s.$$

The proof is complete. □

Lemma 2.2. *Assume that* (2.1) *holds. Let* $x(t)$ *be an eventually positive solution of equation* (1.1)*. Then for sufficiently large* t*, there are only two possible cases:*

(I): $z^{\Delta}(t) > 0,\ \left(b(t)\left(z^{\Delta}(t)\right)\right)^{\Delta} > 0,$

or

(II): $z^{\Delta}(t) < 0,\ \left(b(t)\left(z^{\Delta}(t)\right)\right)^{\Delta} > 0.$

Proof. Let $x(t)$ be an eventually positive solution of equation (1.1). From equation (1.1) it follows that $\left(a(t)\left(b(t)z^{\Delta}(t)\right)^{\Delta}\right)^{\Delta} \leq 0$, for all $t \geq t_1$. Then, $a(t)\left(b(t)z^{\Delta}(t)\right)^{\Delta}$ is non-increasing function and thus $z^{\Delta}(t)$ and $\left(b(t)z^{\Delta}(t)\right)^{\Delta}$ are eventually of one sign. There are the following four possibilities to consider

Case (I): $z^{\Delta}(t) > 0$, $\left(b(t)z^{\Delta}(t)\right)^{\Delta} > 0$ for all large t,

Case (II): $z^{\Delta}(t) < 0$, $\left(b(t)z^{\Delta}(t)\right)^{\Delta} > 0$ for all large t,

Case (III): $z^{\Delta}(t) > 0$, $\left(b(t)z^{\Delta}(t)\right)^{\Delta} < 0$ for all large t, and

Case (IV): $z^{\Delta}(t) < 0$, $\left(b(t)z^{\Delta}(t)\right)^{\Delta} < 0$ for all large t.

We claim that $\left(b(t)z^{\Delta}(t)\right)^{\Delta} > 0$. If not, then, we have two cases: Case (III) and Case (IV).

Assume that Case (III) holds. We have $b(t)\left(z^{\Delta}(t)\right)$ is strictly decreasing and there exists a negative constant M, such that

$$a(t)\left(b(t)z^{\Delta}(t)\right)^{\Delta} < M \text{ for all } t \geq t_2.$$

Dividing by $a(t)$ and integrating the above inequality from t_2 to t, we obtain

$$b(t)\left(z^{\Delta}(t)\right) \leq b(t_2)\left(z^{\Delta}(t_2)\right) + M\int_{t_2}^{t} a^{-1}(s)\Delta s.$$

Letting $t \to \infty$, and using (2.1) then $b(t)\left(z^{\Delta}(t)\right) \to -\infty$, which contradicts that $z^{\Delta}(t) > 0$.

Assume that Case (IV) holds. Then

$$b(t)z^{\Delta}(t) \le b(t_2)z^{\Delta}(t_2) = K < 0.$$

Dividing by $b(t)$ and integrating the above inequality from t_2 to t, we obtain

$$z(t) \le z(t_2) + K\int_{t_2}^{t} b^{-1}(s)\Delta s.$$

Letting $t \to \infty$, and using (2.1), then $z(t) \to -\infty$, which contradicts the fact that $z(t) > 0$. The proof is complete. □

Lemma 2.3. *Assume that* (2.1) *holds. Let $x(t)$ be an eventually positive solution of equation* (1.1) *and suppose that* (II) *of Lemma* 2.2 *holds. If*

$$\int_{t=t_0}^{\infty}\left(b^{-1}(v)\left(\int_{u=t_0}^{v} a^{-1}(u)\left(\int_{s=t_2}^{u}(q_1(s)+q_2(s))\Delta s\right)\Delta u\right)\right)\Delta v = \infty, \tag{2.3}$$

then $x(t) \to 0$ as $t \to \infty$.

Proof. Pick $t_1 \ge t_0$ such that $x(t) > 0$, $x(t-\tau) > 0$, for $t \ge t_1$. Since $x(t)$ is a positive decreasing solution of equation (1.1), then $\lim_{t\to\infty} x(t) = A \ge 0$. Now we claim that $A = 0$. If $A > 0$, then $x(t-\tau_3) \ge A$, $x(t+\tau_4) \ge A$ for $t \ge t_2 \ge t_1$. Therefore from (A_1), (2.3) and (1.1), we have

$$\left(a(t)\left(b(t)z^{\Delta}(t)\right)^{\Delta}\right)^{\Delta} + A(q_1(t)+q_2(t)) \le 0, \quad t \ge t_2.$$

Define the function $u(t) = a(t)\left(b(t)z^{\Delta}(t)\right)^{\Delta}$ for $t \ge t_2$. Then $u^{\Delta}(t) \le -A\left(q_1(t)+q_2(t)\right)$. Integrating the above inequality from t_2 to t, we obtain

$$u(t) \le u(t_2) - A\int_{t_2}^{t}(q_1(s)+q_2(s))\Delta s.$$

From equation (2.3), it is possible to choose an integer t_3 sufficiently large such that

$$u(t) \le -\frac{A}{2}\int_{t_2}^{t}(q_1(s)+q_2(s))\Delta s,$$

for all $t \ge t_3$. Hence

$$\left(b(t)z^{\Delta}(t)\right)^{\Delta} \le -\frac{A}{2a(t)}\int_{t_2}^{t}(q_1(s)+q_2(s))\Delta s.$$

Integrating the above inequality from t_3 to t, we find

$$b(t)z^{\Delta}(t) \leq b(t_3)z^{\Delta}(t_3) - \frac{A}{2}\left(\int_{t_3}^{t} a^{-1}(u)\left(\int_{t_2}^{u}(q_1(s)+q_2(s))\Delta s\right)\Delta u\right).$$

Since $z^{\Delta}(t) < 0$ for $t \geq t_0$, the last inequality implies that

$$z^{\Delta}(t) \leq -\frac{A}{2b(t)}\left(\int_{t_3}^{t} a^{-1}(u)\left(\int_{t_2}^{u}(q_1(s)+q_2(s))\Delta s\right)\Delta u\right).$$

Integrating from t_4 to t, we find

$$z(t) \leq z(t_4) - \frac{A}{2}\int_{t_4}^{t}\left(b^{-1}(l)\left(\int_{t_3}^{l} a^{-1}(u)\left(\int_{t_2}^{u}(q_1(s)+q_2(s))\Delta s\right)\Delta u\right)\right)\Delta l.$$

Condition (2.2) implies that $z(t) \to -\infty$ as $t \to \infty$ which is contradiction with the fact that $z(t) > 0$. Then $A = 0$. i.e. $\lim_{t\to\infty} z(t) = 0$. Since $0 < x(t) \leq z(t)$ then $\lim_{t\to\infty} x(t) = 0$. The proof is complete. □

Next, we state and prove the main theorems.

Theorem 2.1. *Assume that* (2.3) *holds. Further, assume that* $\tau_1 \leq \tau_3$ *and there exists positive rd-continuous* Δ*-differentiable function* $\beta(t)$*, such that*

$$\int_{t_0}^{t}\left(\rho(s)Q(s) - \frac{(1+p_1+p_2)}{4}\frac{\left(\beta^{\Delta}(s)\right)^2 b(s-\tau_3)}{\beta(s)\int_{t_2}^{s-\tau_3} a^{-1}(u)\Delta u}\right)\Delta s = \infty. \tag{2.4}$$

Then every solution of equation (1.1) *is oscillatory or* $\lim_{t\to\infty} x(t) = 0$.

Proof. Assume that equation (1.1) has a non-oscillatory solution, say $x(t) > 0$, $x(t-\tau_1) > 0$, $x(t+\tau_2) > 0$, $x(t-\tau_3) > 0$ and $x(t+\tau_4) > 0$ for all $t \geq t_0$. From equation (1.1), we see that $z(t) > x(t) > 0$ and

$$\left(a(t)\left(b(t)z^{\Delta}(t)\right)^{\Delta}\right)^{\Delta} = -q_1(t)x(t-\tau_3) - q_2(t)x(t+\tau_4) \leq 0. \tag{2.5}$$

Then, $a(t)\left(b(t)z^{\Delta}(t)\right)^{\Delta}$ is non-increasing function and thus $z^{\Delta}(t)$ and $\left(b(t)z^{\Delta}(t)\right)^{\Delta}$ are eventually of one sign. By Lemma 2.2, there exist two possible cases (I) and (II). Assume that (I) holds. From equation (1.1), and the definition of $z(t)$, we have

$$\begin{aligned}
&\left(a(t)\left(b(t)z^{\Delta}(t)\right)^{\Delta}\right)^{\Delta}+q_1(t)x(t-t_3)+q_2(t)x(t+t_4)\\
&+p_1\left(a(t-\tau_1)\left(b(t-\tau_1)z^{\Delta}(t-\tau_1)\right)^{\Delta}\right)^{\Delta}+p_1q_1(t-\tau_1)x(t-\tau_1-\tau_3)\\
&+p_1q_2(t-\tau_1)x(t+\tau_4-\tau_1)+p_2\left(a(t+\tau_2)\left(b(t+\tau_2)z^{\Delta}(t+\tau_2)\right)^{\Delta}\right)^{\Delta}\\
&+p_2q_1(t+\tau_2)x(t+\tau_2-\tau_3)+p_2q_2(t+\tau_2)x(t+\tau_2+\tau_4)=0.
\end{aligned}$$

Thus

$$\begin{aligned}
&\left(a(t)\left(b(t)z^{\Delta}(t)\right)^{\Delta}\right)^{\Delta}+p_1\left(a(t-\tau_1)\left(b(t-\tau_1)z^{\Delta}(t-\tau_1)\right)^{\Delta}\right)^{\Delta}\\
&+p_2\left(a(t+\tau_2)\left(b(t+\tau_2)z^{\Delta}(t+\tau_2)\right)^{\Delta}\right)^{\Delta}+Q_1(t)z(t-\tau_3)+Q_2(t)z(t+\tau_4)\le 0.
\end{aligned}\tag{2.6}$$

It follows from $z^{\Delta}(t)>0$ that $z(t+\tau_4)\ge z(t-\tau_3)$. Thus, by (2.6), we obtain

$$\begin{aligned}
&\left(a(t)\left(b(t)z^{\Delta}(t)\right)^{\Delta}\right)^{\Delta}+p_1\left(a(t-\tau_1)\left(b(t-\tau_1)z^{\Delta}(t-\tau_1)\right)^{\Delta}\right)^{\Delta}\\
&\qquad+p_2\left(a(t+\tau_2)\left(b(t+\tau_2)z^{\Delta}(t+\tau_2)\right)^{\Delta}\right)^{\Delta}+Q(t)z(t-\tau_3)\le 0.
\end{aligned}\tag{2.7}$$

Define a Riccati substitution

$$\omega_1(t):=\beta(t)\frac{a(t)\left(b(t)z^{\Delta}(t)\right)^{\Delta}}{z(t-\tau_3)}.\tag{2.8}$$

Then $\omega_1(t)>0$. From (2.8), we have

$$\begin{aligned}
\omega_1^{\Delta}(t)=\beta^{\Delta}(t)&\frac{a^{\sigma}(t)\left(b^{\sigma}(t)\left(z^{\Delta}(t)\right)^{\sigma}\right)^{\Delta}}{z^{\sigma}(t-\tau_3)}+\beta(t)\frac{\left(a(t)\left(b(t)z^{\Delta}(t)\right)^{\Delta}\right)^{\Delta}}{z(t-\tau_3)}\\
&-\beta(t)\frac{a^{\sigma}(t)\left(b^{\sigma}(t)\left(z^{\Delta}(t)\right)^{\sigma}\right)^{\Delta}(z(t-\tau_3))^{\Delta}}{z(t-\tau_3)z^{\sigma}(t-\tau_3)}.
\end{aligned}\tag{2.9}$$

From Lemma 2.1, $\left(a(t)\left(b(t)z^{\Delta}(t)\right)^{\Delta}\right)^{\Delta}\le 0$ and $t-\tau_3<t$, we get

$$(z(t-\tau_3))^{\Delta}\ge b^{-1}(t-\tau_3)\left(a(t-\tau_3)\left(b(t-\tau_3)z^{\Delta}(t-\tau_3)\right)^{\Delta}\right)\int_{t_2}^{t-\tau_3}a^{-\frac{1}{\alpha_2}}(s)\Delta s$$

$$\geq b^{-1}(t-\tau_3)\left(a(t)\left(b(t)z^{\Delta}(t)\right)^{\Delta}\right)\int_{t_2}^{t-\tau_3} a^{-1}(s)\Delta s$$

$$\geq b^{-1}(t-\tau_3)\left(a^{\sigma}(t)\left(b^{\sigma}(t)(z^{\Delta}(t))^{\sigma}\right)^{\Delta}\right)\int_{t_2}^{t-\tau_3} a^{-1}(s)\Delta s. \tag{2.10}$$

From (2.9) and (2.10), we obtain

$$\omega_1^{\Delta}(t) \leq \frac{\beta^{\Delta}(t)}{\beta^{\sigma}(t)}\omega_1^{\sigma}(t) + \beta(t)\frac{\left(a(t)\left(b(t)z^{\Delta}(t)\right)^{\Delta}\right)^{\Delta}}{z(t-\tau_3)} - \frac{\beta(t)}{\left(\beta^{\sigma}(t)\right)^2 b(t-\tau_3)}\left(\omega_1^{\sigma}(t)\right)^2\int_{t_2}^{t-\tau_3} a^{-1}(s)\Delta s. \tag{2.11}$$

Next, define another function $\omega_2(t)$ by

$$\omega_2(t) := \beta(t)\frac{a(t-\tau_1)\left(b(t-\tau_1)z^{\Delta}\left(t-\tau_1\right)\right)^{\Delta}}{z(t-\tau_3)}. \tag{2.12}$$

Then $\omega_2(t) > 0$. From (2.12), we have

$$\left(\omega_2(t)\right)^{\Delta} = \frac{\beta^{\Delta}(t)}{\beta^{\sigma}(t)}\omega_2^{\sigma}(t) + \beta(t)\frac{\left(a(t-\tau_1)\left(b(t-\tau_1)z^{\Delta}(t-\tau_1)\right)^{\Delta}\right)^{\Delta}}{z(t-\tau_3)} - \beta(t)\frac{a^{\sigma}(t-\tau_1)\left(b^{\sigma}(t-\tau_1)\left(\left(z^{\Delta}(t)\right)^{\sigma}-\tau_1\right)\right)^{\Delta}\left(z(t-\tau_3)\right)^{\Delta}}{z(t-\tau_3)z^{\sigma}(t-\tau_3)}. \tag{2.13}$$

From Lemma 2.1, and $\tau_3 \geq \tau_1$, we get

$$\begin{aligned}
&(z(t-\tau_3))^{\Delta} \\
&\quad \geq b^{-1}(t-\tau_3)\left(a(t-\tau_3)\left(b(t-\tau_3)z^{\Delta}(t-\tau_3)\right)^{\Delta}\right)\int_{t_2}^{t-\tau_3} a^{-\frac{1}{\alpha_2}}(s)\Delta s \\
&\quad \geq \left(a(t-\tau_1)\left(\Delta\left(b(t-\tau_1)\Delta z(t-\tau_1)\right)\right)\right)b^{-1}(t-\tau_3)\int_{t_2}^{t-\tau_3} a^{-1}(s)\Delta s \\
&\quad \geq \left(a\left(\sigma(t)-\tau_1\right)\left(b\left(\sigma(t)-\tau_1\right)z^{\Delta}\left(\sigma(t)-\tau_1\right)\right)^{\Delta}\right)b^{-1}(t-\tau_3)\int_{t_2}^{t-\tau_3} a^{-1}(s)\Delta s.
\end{aligned}$$

Then from (2.12), (2.13) and the above inequality, we have

$$(\omega_2(t))^{\Delta} \leq \beta(t)\frac{\left(a(t-\tau_1)\left(b(t-\tau_1)z^{\Delta}(t-\tau_1)\right)^{\Delta}\right)^{\Delta}}{z(t-\tau_3)} + \frac{\beta^{\Delta}(t)}{\beta^{\sigma}(t)}\omega_2^{\sigma}(t) - \frac{\beta(t)}{\left(\beta^{\sigma}(t)\right)^2 b(t-\tau_3)}\left(\omega_2^{\sigma}(t)\right)^2 \int_{t_2}^{t-\tau_3} a^{-1}(s)\Delta s. \tag{2.14}$$

Similarly, we define another function $\omega_3(t)$ by

$$\omega_3(t) := \beta(t)\frac{a(t+\tau_2)\left(b(t+\tau_2)z^{\Delta}\left(t+\tau_2\right)\right)^{\Delta}}{z(t-\tau_3)}. \tag{2.15}$$

Then $\omega_3(t) > 0$. From (2.15), we have

$$(\omega_3(t))^{\Delta} = \frac{\beta^{\Delta}(t)}{\beta^{\sigma}(t)}\omega_3^{\sigma}(t) + \beta(t)\frac{\left(a(t+\tau_2)\left(b(t+\tau_2)z^{\Delta}(t+\tau_2)\right)^{\Delta}\right)^{\Delta}}{z(t-\tau_3)} - \beta(t)\frac{a\left(\sigma(t)+\tau_2\right)\left(b\left(\sigma(t)+\tau_2\right)z^{\Delta}\left(\sigma(t)+\tau_2\right)\right)^{\Delta}\left(z(t-\tau_3)\right)^{\Delta}}{z(t-\tau_3)z\left(\sigma(t)-\tau_3\right)}. \tag{2.16}$$

From Lemma 2.1 and $t-\tau_3 < t+\tau_2$, we get

$$\begin{aligned}
&(z(t-\tau_3))^{\Delta} \\
&\quad\geq b^{-1}(t-\tau_3)\left(a(t-\tau_3)\left(b(t-\tau_3)z^{\Delta}(t-\tau_3)\right)^{\Delta}\right)\int_{t_2}^{t-\tau_3} a^{-\frac{1}{\alpha_2}}(s)\Delta s \\
&\quad\geq \left(a(t+\tau_2)\left(b(t+\tau_2)z^{\Delta}(t+\tau_2)\right)^{\Delta}\right)b^{-1}(t-\tau_3)\int_{t_2}^{t-\tau_3} a^{-1}(s)\Delta s \\
&\quad\geq \left(a\left(\sigma(t)+\tau_2\right)\left(b\left(\sigma(t)+\tau_2\right)z^{\Delta}\left(\sigma(t)+\tau_2\right)\right)^{\Delta}\right)b^{-1}(t-\tau_3)\int_{t_2}^{t-\tau_3} a^{-1}(s)\Delta s.
\end{aligned}$$

Then from (2.15), (2.16) and the above inequality, we have

$$(\omega_3(t))^{\Delta} \leq \frac{\beta^{\Delta}(t)}{\beta^{\sigma}(t)}\omega_3^{\sigma}(t) + \beta(t)\frac{\left(a(t+\tau_2)\left(b(t+\tau_2)z^{\Delta}(t+\tau_2)\right)^{\Delta}\right)^{\Delta}}{z(t-\tau_3)} - \frac{\beta(t)}{\left(\beta^{\sigma}(t)\right)^2 b(t-\tau_3)}\left(\omega_3^{\sigma}(t)\right)^2 \int_{t_2}^{t-\tau_3} a^{-1}(s)\Delta s. \tag{2.17}$$

From (2.7), (2.11), (2.14) and (2.17), we obtain

(2.18)
$$\begin{aligned}
&(\omega_1(t))^{\Delta}+p_1\left(\omega_2(t)\right)^{\Delta}+p_2\left(\omega_3(t)\right)^{\Delta}\\
&\quad\leq -\beta(t)Q(t)+\frac{\beta^{\Delta}(t)}{\beta^{\sigma}(t)}\omega_1^{\sigma}(t)-\frac{\beta(t)}{(\beta^{\sigma}(t))^2 b(t-\tau_3)}(\omega_1^{\sigma}(t))^2\int_{t_2}^{t-\tau_3}a^{-1}(s)\Delta s\\
&\qquad+p_1\frac{\beta^{\Delta}(t)}{\beta^{\sigma}(t)}\omega_2^{\sigma}(t)-p_1\frac{\beta(t)}{\left(\beta^{\sigma}(t)\right)^2 b(t-\tau_3)}\left(\omega_2^{\sigma}(t)\right)^2\int_{t_2}^{t-\tau_3}a^{-1}(s)\Delta s\\
&\qquad+p_2\frac{\beta^{\Delta}(t)}{\beta^{\sigma}(t)}\omega_3^{\sigma}(t)-p_2\frac{\beta(t)}{\left(\beta^{\sigma}(t)\right)^2 b(t-\tau_3)}\left(\omega_3^{\sigma}(t)\right)^2\int_{t_2}^{t-\tau_3}a^{-1}(s)\Delta s.
\end{aligned}$$

Using (2.18) and the inequality

$$Bu-Au^2\leq\frac{B^2}{4A},A>0, \tag{2.19}$$

we have

$$\begin{aligned}
(\omega_1(t))^{\Delta}+p_1\left(\omega_2(t)\right)^{\Delta}+p_2\left(\omega_3(t)\right)^{\Delta}\leq -\beta(t)Q(t)+\frac{1}{4}\frac{\left(\beta^{\Delta}(t)\right)^2 b(t-\tau_3)}{\beta(t)\int_{t_2}^{t-\tau_3}a^{-1}(s)\Delta s}\\
+\frac{p_1}{4}\frac{\left(\beta^{\Delta}(t)\right)^2 b(t-\tau_3)}{\beta(t)\int_{t_2}^{t-\tau_3}a^{-1}(s)\Delta s}+\frac{p_2}{4}\frac{\left(\beta^{\Delta}(t)\right)^2 b(t-\tau_3)}{\beta(t)\int_{t_2}^{t-\tau_3}a^{-1}(s)\Delta s}.
\end{aligned}$$

Integrating the last inequality from t_3 to t, we obtain

$$\begin{aligned}
\int_{t_3}^{t}\left(\beta(s)Q(s)-\frac{(1+p_1+p_2)}{4}\frac{\left(\beta^{\Delta}(s)\right)^2 b\left(s-\tau_3\right)}{\beta(s)\int_{t_2}^{s-\tau_3}a^{-1}(u)\Delta u}\right)\Delta s\\
\leq \omega_1(t_3)+p_1\omega_2(t_3)+p_2\omega_3(t_3),
\end{aligned}$$

which yields

$$\int_{t_3}^{t}\left(\beta(s)Q(s)-\frac{(1+p_1+p_2)}{4}\frac{\left(\beta^{\Delta}(s)\right)^2 b\left(s-\tau_3\right)}{\beta(s)\int_{t_2}^{s-\tau_3}a^{-1}(u)\Delta u}\right)\Delta s\leq c_1,$$

where $c_1>0$ is a finite constant. But, this contradicts (2.4). Next we assume that (II) holds. We are then back to the proof of Lemma 2.3 to show that $\lim_{t\to\infty}x(t)=0$. The proof is complete. □

From Theorem 2.1 we can derive some oscillation criteria for equation (1.1) on different types of time scales.

Corollary 2.1. *If $\mathbb{T}=\mathbb{R}$, then Theorem* 2.1 *becomes: Assume that $\tau_1 \leq \tau_3$ and there exists $\beta \in C^1([t_0,\infty),(0,\infty])$, such that*

$$(\overline{2.1}) \qquad \int_{t_0}^{\infty} a^{-1}(t)dt=\infty, \qquad \int_{t_0}^{\infty} b^{-1}(t)dt=\infty,$$

$$(\overline{2.3}) \qquad \int_{t=t_0}^{\infty}\left(b^{-1}(v)\left(\int_{u=t_0}^{v} a^{-1}(u)\left(\int_{s=t_2}^{u}(q_1(s)+q_2(s))ds\right)du\right)\right)dv=\infty,$$

$$(\overline{2.4}) \qquad \int_{t_0}^{t}\left(\rho(s)Q(s)-\frac{(1+p_1+p_2)}{4}\frac{(\beta')^2 b(s-\tau_3)}{\beta(s)\int_{t_2}^{s-\tau_3} a^{-1}(u)du}\right)ds=\infty.$$

Hold for all sufficiently large t_1. Then every solution of equation (1.3) *is oscillatory or $\lim\limits_{t\to\infty} x(t)=0$.*

Example 2.1. Consider the following third-order differential equation

$$\Big(a(t)\left(b(t)\left(x(t)+p_1(t)x(t-\tau_1)+p_2(t)x(t+\tau_2)\right)'\right)'\Big)' + q_1(t)x(t-\tau_3)+q_2(t)x(t+\tau_4)=0, \quad t\geq t_0.$$

Let $a(t)=b(t)=1$, $p_1(t)=p_2(t)=\dfrac{1}{3}$, $q_1(t)=e^{-2}+\dfrac{e^{-1}}{3}$, $q_2(t)=\dfrac{1}{3}$, $\tau_1=\tau_2=\tau_4=1$, $\tau_3=2$. We see that $(\overline{2.1})$ holds. Take $\beta(t)=1$. Then conditions $(\overline{2.3})$ and $(\overline{2.4})$ hold. Then by Corollary 2.1, every solution of this equation is oscillatory or tends to zero. It is easy to find that $x(t)=e^{-t}$ is a solution of this equation. **However, the results established in [10, 19] do not apply to this equation.**

Corollary 2.2. *If $\mathbb{T}=\mathbb{N}$, then* (2.4) *becomes*

$$\sum_{s=n_0}^{n-1}\left(\rho(s)Q(s)-\frac{(1+p_1+p_2)}{4}\frac{(\Delta\rho(s))^2 b(s-\sigma_3)}{\rho(s)\sum_{u=n_2}^{\sigma_3(s)-1} a^{-1}(u)}\right)=\infty.$$

Then every solution of equation (1.2) *is oscillatory or $\lim\limits_{t\to\infty} x(t)=0$.*

Example 2.2. Consider the following nonlinear neutral equation

$$\Delta\left(n\Delta\left(n\Delta\left(x(n)+p_1(n)x(n-\sigma_1)+p_2(n)x(n+\sigma_2)\right)\right)\right) + \frac{\beta}{n}x(n-\sigma_3)+\frac{\gamma}{n}x(n+\sigma_4)=0,$$

where $\sigma_3 \geq \sigma_1$, $p_1+p_2 \leq 3$, β and γ are positive constants. We see that (2.1) holds. If we take $n_0 = 1$, $\rho(n) = 1$, then by Corollary 2.2, every solution of this equation is oscillatory or tends to zero provided that $\beta + \gamma > 1$.

Corollary 2.3. *Assume that all the assumptions of Theorem* 2.1 *hold, except the condition* (2.4) *is replaced by*

$$\lim_{t\to\infty}\sup\int_{t_0}^{t}\beta(s)Q(s)\Delta s=\infty,$$

$$\lim_{t\to\infty}\sup\int_{t_0}^{t}\frac{\left(\beta^{\Delta}(s)\right)^2 b(s-\tau_3)}{\beta(s)\int_{t_2}^{s-\tau_3}a^{-1}(u)\Delta u}\Delta s<\infty.$$

Then every solution of equation (1.1) *is oscillatory or* $\lim_{t\to\infty} x(t) = 0$.

Remark 2.1. Note that from Theorem 2.1, we can obtain different conditions for oscillation of all solutions of equation (1.1) by different choices of $\beta(t)$. Let $\beta(t) = 1$, $\beta(t) = t$ and $\beta(t) = t^{\lambda}$, $t \geq t_0$ and $\lambda > 1$ is a constant. By Theorem 2.1, we have the following results.

Corollary 2.4. *Assume that all the assumptions of Theorem* 2.1 *hold, except the condition* (2.4) *is replaced by*

$$\lim_{t\to\infty}\sup\int_{t_0}^{t}Q(s)\Delta s=\infty.$$

Then every solution of equation (1.1) *is oscillatory or* $\lim_{t\to\infty} x(t) = 0$.

Corollary 2.5. *If* $\mathbb{T} = \mathbb{N}$, *then* (2.4) *becomes*

$$\lim_{n\to\infty}\sup\sum_{s=n_0}^{n-1}Q(s)=\infty.$$

Then every solution of equation (1.2) *is oscillatory or* $\lim_{t\to\infty} x(t) = 0$.

Example 2.3. Consider the following nonlinear neutral equation

$$\Delta\left(a(n)\Delta\left(b(n)\Delta(x(n)+p_1(n)x(n-1)+p_2(n)x(n+\sigma_2))\right)\right) + \frac{1}{n}x(n-\sigma_3)+\frac{1}{n}x(n+\sigma_4)=0,$$

where $\sigma_1 = 1$, $q_i(n) = \dfrac{1}{n}$ for $i = 1,2$. Assume that (2.1) holds. If we take, $n_0 = 1$, $\rho(n) = 1$, then Corollary 2.5, asserts that every solution of this equation is oscillatory or tends to zero.

Corollary 2.6. *Assume that all the assumptions of Theorem* 2.1 *hold, except the condition* (2.4) *is replaced by*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left(sQ(s)-\frac{(1+p_1+p_2)}{4}\frac{b(s-\tau_3)}{s\int_{t_2}^{s-\tau_3}a^{-1}(u)\Delta u}\right)\Delta s=\infty.$$

Then every solution of equation (1.1) *is oscillatory or* $\lim_{t\to\infty} x(t) = 0$.

Corollary 2.7. *Assume that all the assumptions of Theorem* 2.1 *hold, except the condition* (2.4) *is replaced by*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left(l^{\lambda}Q(l)-\frac{(1+p_1+p_2)}{4}\frac{((\sigma(l))^{\lambda}-l^{\lambda})^2 b(l-\tau_3)}{(\mu(l))^{2\lambda}l^{\lambda}\int_{t_2}^{l-\tau_3}a^{-1}(u)\Delta u}\right)\Delta l=\infty.$$

Then every solution of equation (1.1) *is oscillatory or* $\lim_{t\to\infty} x(t) = 0$.

Theorem 2.2. *Assume that* (2.3) *holds. Let* $\tau_1 \le \tau_3$ *and* $\beta(t)$ *be a positive function. Furthermore, we assume that there exists a function* $\{H(t,s) \mid t \ge s \ge 0\}$. *If*

$$\limsup_{t\to\infty}\frac{1}{H(t,0)}\int_0^t\left[H(t,s)\beta(s)Q(s)-(1+p_1+p_2)\frac{H(t,s)\vartheta^2(t,s)}{4\varphi_1(s)}\right]\Delta s = \infty \tag{2.20}$$

$$\vartheta(t,s) := \left(H^{\Delta_s}(t,s)+H(t,s)\frac{(\beta^{\Delta}(s))_+}{\beta^{\sigma}(s)}\right)$$

then every solution of equation (1.1) *is oscillatory or* $\lim_{t\to\infty} x(t) = 0$.

Proof. Proceeding as in the proof of Theorem 2.1, we assume that equation (1.1) has a non- oscillatory solution, say $x(t) > 0$, $x(t-\tau_1) > 0$, $x(t+\tau_2) > 0$, $x(t-\tau_3) > 0$ and $x(t+\tau_4) > 0$ for all $t \geq t_0$. By Lemma 2.2, there are two possible cases. If (I) holds, from the proof of Theorem 2.1, we find that (2.18) holds for all $t \geq t_1$. From (2.18), we have

$$\begin{aligned} \beta(t)Q(t) \leq& -(\omega_1(t))^\Delta - p_1(\omega_2(t))^\Delta - p_2(\omega_3(t))^\Delta + \frac{\beta^\Delta(t)}{\beta^\sigma(t)}\omega_1^\sigma(t) \\ &-\varphi_1(t)(\omega_1^\sigma(t))^2 + p_1\frac{\beta^\Delta(t)}{\beta^\sigma(t)}\omega_2^\sigma(t) - p_1\varphi_1(t)(\omega_2^\sigma(t))^2 \\ &+p_2\frac{\beta^\Delta(t)}{\beta^\sigma(t)}\omega_3^\sigma(t) - p_2\varphi_1(t)(\omega_3^\sigma(t))^2. \end{aligned} \tag{2.21}$$

Therefore, we have

$$\begin{aligned} \int_{t_2}^t H(t,s)\beta(s)Q(s)\Delta s \leq& -\int_{t_2}^t H(t,s)(\omega_1(s))^\Delta\Delta s \\ &-p_1\int_{t_2}^t H(t,s)(\omega_2(s))^\Delta\Delta s - p_2\int_{t_2}^t H(t,s)(\omega_3(s))^\Delta\Delta s \\ &+\int_{t_2}^t H(t,s)\frac{\left(\beta^\Delta(s)\right)_+}{\beta^\sigma(s)}\omega_1^\sigma(s)\Delta s - \int_{t_2}^t H(t,s)\varphi_1(s)\left(\omega_1^\sigma(s)\right)^2\Delta s \\ &+p_1\int_{t_2}^t H(t,s)\frac{\left(\beta^\Delta(s)\right)_+}{\beta^\sigma(s)}\omega_2^\sigma(s)\Delta s - p_1\int_{t_2}^t H(t,s)\varphi_1(s)\left(\omega_2^\sigma(s)\right)^2\Delta s \\ &+p_2\int_{t_2}^t H(t,s)\frac{\left(\beta^\Delta(s)\right)_+}{\beta^\sigma(s)}\omega_3^\sigma(s)\Delta s - p_2\int_{t_2}^t H(t,s)\varphi_1(s)\left(\omega_3^\sigma(s)\right)^2\Delta s, \end{aligned}$$

Integrating by parts and using $H(t,t) = 0$, we have

$$\begin{aligned} &\int_{t_2}^t H(t,s)\beta(s)Q(s)\Delta s \\ &\quad\leq H(t,t_2)\,\omega_1(t_2) + \int_{t_2}^t H^{\Delta_s}(t,s)\omega_1^\sigma(s)\Delta s + \int_{t_2}^t H(t,s)\frac{\left(\beta^\Delta(s)\right)_+}{\beta^\sigma(s)}\omega_1^\sigma(s)\Delta s \\ &\qquad-\int_{t_2}^t H(t,s)\varphi_1(s)\left(\omega_1^\sigma(s)\right)^2\Delta s + p_1H(t,t_2)\,\omega_2(t_2) \\ &\qquad+p_1\int_{t_2}^t H^{\Delta_s}(t,s)\omega_2^\sigma(s)\Delta s + p_1\int_{t_2}^t H(t,s)\frac{\left(\beta^\Delta(s)\right)_+}{\beta^\sigma(s)}\omega_2^\sigma(s)\Delta s \end{aligned}$$

$$
\begin{aligned}
&-p_1\int_{t_2}^{t} H(t,s)\varphi_1(s)\left(\omega_2^\sigma(s)\right)^2\Delta s+p_2H\left(t,t_2\right)\omega_3(t_2)\\
&+p_2\int_{t_2}^{t} H^{\Delta_s}(t,s)\omega_3^\sigma(s)\Delta s+p_2\int_{t_2}^{t} H(t,s)\frac{\left(\beta^\Delta(s)\right)_+}{\beta^\sigma(s)}\omega_3^\sigma(s)\Delta s\\
&-p_2\int_{t_2}^{t} H(t,s)\varphi_1(s)\left(\omega_3^\sigma(s)\right)^2\Delta s\\
=\;& H\left(t,t_2\right)\omega_1(t_2)+\int_{t_2}^{t}\left(H^{\Delta_s}(t,s)+H(t,s)\frac{\left(\beta^\Delta(s)\right)_+}{\beta^\sigma(s)}\right)\omega_1^\sigma(s)\Delta s\\
&-\int_{t_2}^{t} H(t,s)\varphi_1(s)\left(\omega_1^\sigma(s)\right)^2\Delta s+p_1H\left(t,t_2\right)\omega_2(t_2)\\
&+p_1\int_{t_2}^{t}\left(H^{\Delta_s}(t,s)+H(t,s)\frac{\left(\beta^\Delta(s)\right)_+}{\beta^\sigma(s)}\right)\omega_2^\sigma(s)\Delta s\\
&-p_1\int_{t_2}^{t} H(t,s)\varphi_1(s)\left(\omega_2^\sigma(s)\right)^2\Delta s+p_2H\left(t,t_2\right)\omega_3(t_2)\\
&+p_2\int_{t_2}^{t}\left(H^{\Delta_s}(t,s)+H(t,s)\frac{\left(\beta^\Delta(s)\right)_+}{\beta^\sigma(s)}\right)\omega_3^\sigma(s)\Delta s\\
&-p_2\int_{t_2}^{t} H(t,s)\varphi_1(s)\left(\omega_3^\sigma(s)\right)^2\Delta s\\
=\;& H\left(t,t_2\right)\omega_1(t_2)+\int_{t_2}^{t} H(t,s)v(t,s)\omega_1^\sigma(s)\Delta s-\int_{t_2}^{t} H(t,s)\varphi_1(s)\left(\omega_1^\sigma(s)\right)^2\Delta s\\
&+p_1H\left(t,t_2\right)\omega_2(t_2)+p_1\int_{t_2}^{t} H(t,s)v(t,s)\omega_2^\sigma(s)\Delta s\\
&-p_1\int_{t_2}^{t} H(t,s)\varphi_1(s)\left(\omega_2^\sigma(s)\right)^2\Delta s+p_2H\left(t,t_2\right)\omega_3(t_2)\\
&+p_2\int_{t_2}^{t} H(t,s)v(t,s)\omega_3^\sigma(s)\Delta s-p_2\int_{t_2}^{t} H(t,s)\varphi_1(s)\left(\omega_3^\sigma(s)\right)^2\Delta s.
\end{aligned}
$$

By completing the square, we have

$$
\begin{aligned}
\text{(2.22)}\qquad &\int_{t_2}^{t} H(t,s)\beta(s)Q(s)\Delta s\\
&\quad\le H\left(t,t_2\right)\omega_1(t_2)+\int_{t_2}^{t}\frac{H(t,s)\vartheta^2(t,s)}{4\varphi_1(s)}\Delta s
\end{aligned}
$$

$$
\begin{aligned}
&-\int_{t_2}^{t}\left[\sqrt{H(t,s)\varphi_1(s)}\omega_1^{\sigma}(s)-\frac{\vartheta(t,s)}{2}\sqrt{\frac{H(t,s)}{\varphi_1(s)}}\right]^2\Delta s\\
&+p_1H\left(t,t_2\right)\omega_2(t_2)+p_1\int_{t_2}^{t}\frac{H(t,s)\vartheta^2(t,s)}{4\varphi_1(s)}\Delta s\\
&-p_1\int_{t_2}^{t}\left[\sqrt{H(t,s)\varphi_1(s)}\omega_2^{\sigma}(s)-\frac{\vartheta(t,s)}{2}\sqrt{\frac{H(t,s)}{\varphi_1(s)}}\right]^2\Delta s\\
&+p_2H\left(t,t_2\right)\omega_3(t_2)+p_2\int_{t_2}^{t}\frac{H(t,s)\vartheta^2(t,s)}{4\varphi_1(s)}\Delta s\\
&-p_2\int_{t_2}^{t}\left[\sqrt{H(t,s)\varphi_1(s)}\omega_2^{\sigma}(s)-\frac{\vartheta(t,s)}{2}\sqrt{\frac{H(t,s)}{\varphi_1(s)}}\right]^2\Delta s.
\end{aligned}
$$

Then,

$$
\begin{aligned}
&\int_{t_2}^{t}\left[H(t,s)\beta(s)Q(s)-(1+p_1+p_2)\frac{H(t,s)\vartheta^2(t,s)}{4\varphi_1(s)}\right]\Delta s\\
&\qquad\le H\left(t,t_2\right)\omega(t_2)+p_1H\left(t,t_2\right)\omega_2(t_2)+p_2H\left(t,t_2\right)\omega_3(t_2),
\end{aligned}
$$

which implies

$$
\begin{aligned}
&\int_{t_2}^{t}\left[H(t,s)\beta(s)Q(s)-(1+p_1+p_2)\frac{H(t,s)\vartheta^2(t,s)}{4\varphi_1(s)}\right]\Delta s\\
&\qquad\le H\left(t,0\right)\left|\omega_1(t_2)\right|+p_1H\left(t,0\right)\left|\omega_2(t_2)\right|+p_2H\left(t,0\right)\left|\omega_3(t_2)\right|.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\int_{0}^{t}\left[H(t,s)\beta(s)Q(s)-(1+p_1+p_2)\frac{H(t,s)\vartheta^2(t,s)}{4\varphi_1(s)}\right]\Delta s\\
&\qquad\le H\left(t,0\right)\left\{\int_0^t\left|\beta(s)Q(s)\right|\Delta s+\left|\omega_1(t_2)\right|+p_1\left|\omega_2(t_2)\right|+p_2\left|\omega_3(t_2)\right|\right\}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\limsup_{t\to\infty}\frac{1}{H\left(t,0\right)}\int_{0}^{t}\left[H(t,s)\beta(s)Q(s)-(1+p_1+p_2)\frac{H(t,s)\vartheta^2(t,s)}{4\varphi_1(s)}\right]\Delta s\\
&\qquad\le\int_0^t\left|\beta(s)Q(s)\right|\Delta s+\left|\omega_1(t_2)\right|+p_1\left|\omega_2(t_2)\right|+p_2\left|\omega_3(t_2)\right|<\infty,
\end{aligned}
$$

which is contrary to (2.20). If (II) holds, then we are back to the proof of Lemma 2.3 to show that $\lim_{t\to\infty} x(t) = 0$. This completes the proof of Theorem 2.2. □

Corollary 2.8. *Assume that all the assumptions of Theorem 2.2 hold, except the condition* (2.20) *is replaced by*

$$\limsup_{t\to\infty} \frac{1}{H(t,0)} \int_0^t H(t,s)\beta(s)Q(s)\Delta s,$$
$$\limsup_{t\to\infty} \frac{1}{H(t,0)} \int_0^t \frac{H(t,s)\vartheta^2(t,s)}{4\varphi_1(s)}\Delta s.$$

Then every solution of equation (1.1) *is oscillatory or* $\lim_{t\to\infty} x(t) = 0$.

Corollary 2.9. *If* $\mathbb{T} = \mathbb{N}$, *then* (2.20) *becomes*

$$\overline{(2.20)} \quad \limsup_{m\to\infty} \frac{1}{H(m,0)} \sum_{n=0}^{m-1} \left(H(m,n)\,\beta(n)Q(n) - (1+p_1+p_2)\frac{\vartheta^2(m,n)\,H(m,n)}{4\varphi_1(n)} \right) = \infty.$$

Then every solution of equation (1.2) *is oscillatory or* $\lim_{t\to\infty} x(t) = 0$.

Theorem 2.3. *Assume that* (2.3) *holds. Further, assume that* $\tau_3 \le \tau_1$ *and there exists a positive non decreasing function* $\beta(t)$, *such that*

$$\limsup_{t\to\infty} \int_{t_0}^t \left[\beta(s)Q(s) - \frac{(1+p_1+p_2)}{4} \frac{\left(\beta^\Delta(t)\right)^2 b(t-\tau_1)}{\beta(t)\int_{t_2}^{t-\tau_1} a^{-1}(s)\Delta s} \right] \Delta s = \infty. \tag{2.23}$$

Then every solution of equation (1.1) *is oscillatory or* $\lim_{t\to\infty} x(t) = 0$.

Proof. Assume that equation (1.1) has a non- oscillatory solution, say $x(t) > 0$, $x(t-\tau_1) > 0$, $x(t+\tau_2) > 0$, $x(t-\tau_3) > 0$ and $x(t+\tau_4) > 0$ for all $t \ge t_0$. Proceeding as in the proof of Theorem 2.1, we get (2.6). By Lemma 2.2, there exist two possible cases (I) and (II). Assume that (I) holds. Then, we obtain (2.7).

From $z^\Delta(t) > 0$, $\tau_3 \le \tau_1$, we obtain

$$\left(a(t)\left(b(t)z^\Delta(t)\right)^\Delta\right)^\Delta + p_1\left(a(t-\tau_1)\left(b(t-\tau_1)z^\Delta(t-\tau_1)\right)^\Delta\right)^\Delta + p_2\left(a(t+\tau_2)\left(b(t+\tau_2)z^\Delta(t+\tau_2)\right)^\Delta\right)^\Delta + Q(t)z(t-\tau_1) \le 0. \tag{2.24}$$

Using the Riccati transformation

$$\omega_1(t) := \beta(t)\frac{a(t)\left(b(t)z^{\Delta}(t)\right)^{\Delta}}{z\left(t-\tau_1\right)}. \tag{2.25}$$

Then $\omega_1(t) > 0$. From (2.25), we have

$$\begin{aligned}(\omega_1(t))^{\Delta} = \beta^{\Delta}(t)&\frac{\left(a^{\sigma}(t)\left(b^{\sigma}(t)\left(z^{\Delta}(t)\right)^{\sigma}\right)^{\Delta}\right)^{\sigma}}{z\left(\sigma(t)-\tau_1\right)} + \beta(t)\frac{\left(a(t)\left(b(t)z^{\Delta}(t)\right)^{\Delta}\right)^{\Delta}}{z(t-\tau_1)} \\ &- \beta(t)\frac{a^{\sigma}(t)\left(b^{\sigma}(t)\left(z^{\Delta}(t)\right)^{\sigma}\right)^{\Delta}\left(z(t-\tau_1)\right)^{\Delta}}{z(t-\tau_1)z\left(\sigma(t)-\tau_1\right)}.\end{aligned} \tag{2.26}$$

From Lemma 2.1, $\left(a(t)\left(b(t)z^{\Delta}(t)\right)^{\Delta}\right)^{\Delta} \le 0$ and $t-\tau_1 < t$, we get

$$(z(t-\tau_1))^{\Delta} \ge b^{-1}(t-\tau_1)\left(a^{\sigma}(t)\left(b^{\sigma}(t)\left(z^{\Delta}(t)\right)^{\sigma}\right)\right)\int_{t_2}^{t-\tau_1} a^{-1}(s)\Delta s. \tag{2.27}$$

From (2.26) and (2.27), we obtain

$$\begin{aligned}(\omega_1(t))^{\Delta} \le \frac{\beta^{\Delta}(t)}{\beta^{\sigma}(t)}\omega_1^{\sigma}(t) + \beta(t)&\frac{\left(a(t)\left(b(t)z^{\Delta}(t)\right)^{\Delta}\right)^{\Delta}}{z(t-\tau_1)} \\ &- \frac{\beta(t)\left(\omega_1^{\sigma}(t)\right)^2}{\left(\beta^{\sigma}(t)\right)^2 b(t-\tau_1)}\int_{t_2}^{t-\tau_1} a^{-1}(s)\Delta s.\end{aligned} \tag{2.28}$$

Next, define another function $\omega_2(t)$ by

$$\omega_2(t) := \beta(t)\frac{a(t-\tau_1)\left(b(t-\tau_1)z^{\Delta}\left(t-\tau_1\right)\right)^{\Delta}}{z(t-\tau_1)}. \tag{2.29}$$

Then $\omega_2(t) > 0$. From (2.29), we have

$$\begin{aligned}(\omega_2(t))^{\Delta} = \frac{\beta^{\Delta}(t)}{\beta^{\sigma}(t)}\omega_2^{\sigma}(t) + \beta(t)&\frac{\left(a(t-\tau_1)\left(b(t-\tau_1)z^{\Delta}(t-\tau_1)\right)^{\Delta}\right)^{\Delta}}{z(t-\tau_1)} \\ - \beta(t)&\frac{a\left(\sigma(t)-\tau_1\right)\left(b\left(\sigma(t)-\tau_1\right)z^{\Delta}\left(\sigma(t)-\tau_1\right)\right)^{\Delta}\left(z(t-\tau_1)\right)^{\Delta}}{z(t-\tau_1)z\left(\sigma(t)-\tau_1\right)}.\end{aligned} \tag{2.30}$$

From Lemma 2.1, we get

$$\begin{aligned}&(z(t-\tau_1))^{\Delta}\\&\geq \left(a\left(\sigma(t)-\tau_1\right)\left(b\left(\sigma(t)-\tau_1\right)z^{\Delta}\left(\sigma(t)-\tau_1\right)\right)^{\Delta}\right)b^{-1}(t-\tau_1)\int_{t_2}^{t-\tau_1}a^{-1}(s)\Delta s.\end{aligned}$$

Then from (2.29), (2.30) and the above inequality, we have

$$\begin{aligned}(\omega_2(t))^{\Delta} \leq{}& \beta(t)\frac{\left(a(t-\tau_1)\left(b(t-\tau_1)z^{\Delta}(t-\tau_1)\right)^{\Delta}\right)^{\Delta}}{z(t-\tau_1)}+\frac{\beta^{\Delta}(t)}{\beta^{\sigma}(t)}\omega_2^{\sigma}(t)\\&-\frac{\beta(t)}{\left(\beta^{\sigma}(t)\right)^2 b(t-\tau_1)}\left(\omega_2^{\sigma}(t)\right)^2\int_{t_2}^{t-\tau_1}a^{-1}(s)\Delta s.\end{aligned}\tag{2.31}$$

Similarly, we define another function $\omega_3(t)$ by

$$\omega_3(t):=\beta(t)\frac{a(t+\tau_2)\left(b(t+\tau_2)z^{\Delta}\left(t+\tau_2\right)\right)^{\Delta}}{z(t-\tau_1)}.\tag{2.32}$$

Then $\omega_3(t)>0$. From (2.32), we have

$$\begin{aligned}(\omega_3(t))^{\Delta}={}&\frac{\beta^{\Delta}(t)}{\beta^{\sigma}(t)}\omega_3^{\sigma}(t)+\beta(t)\frac{\left(a(t+\tau_2)\left(b(t+\tau_2)z^{\Delta}(t+\tau_2)\right)^{\Delta}\right)^{\Delta}}{z(t-\tau_1)}\\&-\beta(t)\frac{a\left(\sigma(t)+\tau_2\right)\left(b\left(\sigma(t)+\tau_2\right)z^{\Delta}\left(\sigma(t)+\tau_2\right)\right)^{\Delta}\left(z(t-\tau_1)\right)^{\Delta}}{z(t-\tau_1)z\left(\sigma(t)-\tau_1\right)}.\end{aligned}\tag{2.33}$$

From Lemma 2.1, and $t-\tau_1<t+\tau_2$, we get

$$\begin{aligned}&(z(t-\tau_1))^{\Delta}\\&\geq \left(a\left(\sigma(t)+\tau_2\right)\left(b\left(\sigma(t)+\tau_2\right)z^{\Delta}\left(\sigma(t)+\tau_2\right)\right)^{\Delta}\right)b^{-1}(t-\tau_1)\int_{t_2}^{t-\tau_1}a^{-1}(s)\Delta s.\end{aligned}$$

Then from (2.32), (2.33) and the above inequality, we have

$$\begin{aligned}(\omega_3(t))^{\Delta}\leq{}&\frac{\beta^{\Delta}(t)}{\beta^{\sigma}(t)}\omega_3^{\sigma}(t)+\beta(t)\frac{\left(a(t+\tau_2)\left(b(t+\tau_2)z^{\Delta}(t+\tau_2)\right)^{\Delta}\right)^{\Delta}}{z(t-\tau_1)}\\&-\frac{\beta(t)}{\left(\beta^{\sigma}(t)\right)^2 b(t-\tau_1)}\left(\omega_3^{\sigma}(t)\right)^2\int_{t_2}^{t-\tau_1}a^{-1}(s)\Delta s.\end{aligned}\tag{2.34}$$

From (2.24), (2.28), (2.31) and (2.34), we obtain

$$
\begin{aligned}
(2.35)\ (\omega_1(t))^{\Delta} + p_1\,(\omega_2(t))^{\Delta} + p_2\,(\omega_3(t))^{\Delta} \\
\le -\beta(t)Q(t) + \frac{\beta^{\Delta}(t)}{\beta^{\sigma}(t)}\omega_1^{\sigma}(t) - \frac{\beta(t)}{(\beta^{\sigma}(t))^2\, b(t-\tau_1)}\,(\omega_1^{\sigma}(t))^2 \int_{t_2}^{t-\tau_1} a^{-1}(s)\Delta s \\
+p_1\frac{\beta^{\Delta}(t)}{\beta^{\sigma}(t)}\omega_2^{\sigma}(t) - p_1\frac{\beta(t)}{(\beta^{\sigma}(t))^2\, b(t-\tau_1)}\,(\omega_2^{\sigma}(t))^2 \int_{t_2}^{t-\tau_1} a^{-1}(s)\Delta s \\
+p_2\frac{\beta^{\Delta}(t)}{\beta^{\sigma}(t)}\omega_3^{\sigma}(t) - p_2\frac{\beta(t)}{(\beta^{\sigma}(t))^2\, b(t-\tau_1)}\,(\omega_3^{\sigma}(t))^2 \int_{t_2}^{t-\tau_1} a^{-1}(s)\Delta s.
\end{aligned}
$$

Using (2.19) and (2.35), we have

$$
\begin{aligned}
(\omega_1(t))^{\Delta} + p_1\,(\omega_2(t))^{\Delta} + p_2\,(\omega_3(t))^{\Delta} \le -\beta(t)Q(t) + \frac{1}{4}\frac{\left(\beta^{\Delta}(t)\right)^2 b(t-\tau_1)}{\beta(t)\int_{t_2}^{t-\tau_1} a^{-1}(s)\Delta s} \\
+\frac{p_1}{4}\frac{\left(\beta^{\Delta}(t)\right)^2 b(t-\tau_1)}{\beta(t)\int_{t_2}^{t-\tau_1} a^{-1}(s)\Delta s} + \frac{p_2}{4}\frac{\left(\beta^{\Delta}(t)\right)^2 b(t-\tau_1)}{\beta(t)\int_{t_2}^{t-\tau_1} a^{-1}(s)\Delta s}.
\end{aligned}
$$

Integrating the last inequality from t_3 to t, we obtain

$$
\begin{aligned}
\int_{t_3}^{t}\left[\beta(s)Q(s) - \frac{(1+p_1+p_2)}{4}\frac{\left(\beta^{\Delta}(t)\right)^2 b(t-\tau_1)}{\beta(t)\int_{t_2}^{s-\tau_1} a^{-1}(s)\Delta s}\right]\Delta s \\
\le \omega_1(t_3) + p_1\omega_2(t_3) + p_2\omega_3(t_3),
\end{aligned}
$$

which yields

$$
\int_{t_3}^{t}\left[\beta(s)Q(s) - \frac{(1+p_1+p_2)}{4}\frac{\left(\beta^{\Delta}(t)\right)^2 b(t-\tau_1)}{\beta(t)\int_{t_2}^{s-\tau_1} a^{-1}(s)\Delta s}\right]\Delta s \le c_1,
$$

where $c_1 > 0$ is a finite constant. But, this contradicts (2.23). Next we assume that (II) holds. We are then back to the proof of Lemma 2.3 to show that $\lim_{t\to\infty} x(t) = 0$. The proof is complete. □

Corollary 2.10. *If* $\mathbb{T} = \mathbb{N}$, *then* (2.23) *becomes*

$$
(\overline{2.23})\qquad \lim_{n\to\infty}\sup\sum_{s=n_0}^{n-1}\left(\beta(s)Q(s) - \frac{(1+p_1+p_2)}{4}\frac{(\Delta\beta(s))^2\, b\,(s-\sigma_1)}{\beta(s)\sum_{u=n_2}^{\sigma_1(s)-1} a^{-1}(u)}\right) = \infty.
$$

Then every solution of equation (1.2) *is oscillatory or* $\lim_{t\to\infty} x(t) = 0$.

Example 2.4. Consider the following nonlinear neutral equation

$$\Delta\left(\frac{1}{n}\Delta\left(\frac{1}{n}\Delta\left(x(n)+p_1(n)x\left(n-2\right)+p_2(n)x\left(n+1\right)\right)\right)\right)$$
$$+\frac{\beta}{n^2}x\left(n-\tau_3\right)+\frac{\gamma}{n^2}x\left(n+\tau_4\right)=0,$$

where $\tau_1 \geq \tau_3$, $p_1+p_2 \leq 3$, β and γ are positive constants. We see that (2.1) holds. If we take $n_0=1$, $\beta(n)=1$, then by Corollary 2.10, every solution of this equation is oscillatory or tends to zero provided that $\beta+\gamma>1$.

Theorem 2.4. *Assume that* (2.3) *holds. Let* $\tau_3 \leq \tau_1$ *and* $\beta(t)$ *be a positive function. Furthermore, we assume that there exists a function* $\{H(t,s) \mid t \geq s \geq 0\}$. *If*

$$\lim_{m\to\infty}\sup\frac{1}{H\left(m,0\right)}\sum_{n=0}^{m-1}\left(H\left(m,n\right)\beta(n)Q(n)\right.$$
$$\left.-\left(1+p_1+p_2\right)\frac{\vartheta^2\left(m,n\right)H\left(m,n\right)}{4\varphi_2(n)}\right)=\infty, \tag{2.36}$$

then every solution of equation (1.1) *is oscillatory or* $\lim\limits_{t\to\infty}x(t)=0$.

Proof. Proceeding as in the proof of Theorem 2.3, we assume that equation (1.1) has a non-oscillatory solution, say $x(t)>0$, $x(t-\tau_1)>0$, $x(t+\tau_2)>0$, $x(t-\tau_3)>0$ and $x(t+\tau_4)>0$ for all $t \geq t_0$. By Lemma 2.2, there are two possible cases. If (I) holds, from the proof of Theorem 2.3, we find that (2.35) holds for all $t \geq t_1$. From (2.35), we have

$$\begin{aligned}\beta(t)Q(t) \leq\; & -\left(\omega_1(t)\right)^{\Delta}-p_1\left(\omega_2(t)\right)^{\Delta}-p_2\left(\omega_3(t)\right)^{\Delta}+\frac{\beta^{\Delta}(t)}{\beta^{\sigma}(t)}\omega_1^{\sigma}(t)\\ & -\varphi_2(t)\left(\omega_1^{\sigma}(t)\right)^2+p_1\frac{\beta^{\Delta}(t)}{\beta^{\sigma}(t)}\omega_2^{\sigma}(t)-p_1\varphi_2(t)\left(\omega_2^{\sigma}(t)\right)^2\\ & +p_2\frac{\beta^{\Delta}(t)}{\beta^{\sigma}(t)}\omega_3^{\sigma}(t)-p_2\varphi_2(t)\left(\omega_3^{\sigma}(t)\right)^2.\end{aligned} \tag{2.37}$$

The remainder of the proof is similar to that of the proof of Theorem 2.2 and hence the details are omitted. □

Corollary 2.11. *If* $\mathbb{T}=\mathbb{N}$, *then* (2.36) *becomes*

$$(\overline{2.36}) \qquad \lim_{m\to\infty} \sup \frac{1}{H(m,0)} \sum_{n=0}^{m-1} \Bigg(H(m,n)\,\beta(n) Q(n) - (1+p_1+p_2) \frac{\vartheta^2(m,n)\,H(m,n)}{4\varphi_2(n)} \Bigg) = \infty.$$

then every solution of equation (1.2) *is oscillatory or* $\lim_{t\to\infty} x(t) = 0$.

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