Rapid 3-D forward model of potential fields with application to the Palinuro Seamount magnetic anomaly (southern Tyrrenian Sea, Italy)

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Received 2 July 2008; revised 18 September 2008; accepted 20 October 2008; published 13 February 2009.

[1] We show a set of forward model equations in the Fourier domain for calculating the 3-D gravity and magnetic anomalies of a given 3-D distribution of density or magnetization. One property of the potential field equations is that they are given by convolution products, providing a very simple analytic expression in the Fourier domain. Under this assumption, the domain of the density or magnetization parameters is connected by a biunivoc relationship with the data space, and potential field anomalies can be seen as filtered versions of the corresponding density or magnetization distributions. A very fine spatial discretization can be obtained by using a large number of points within a unique 3-D grid, where both the source distributions and field data are defined. The main advantage of this formulation is that it dramatically reduces execution times, providing a very fast forward model tool useful for modeling anomalies at different altitudes. We use this method to evaluate an average magnetization of $8 A/m$ for the Palinuro Seamount in the Tyrrenian Sea (southern Italy), thus performing a joint interpretation of morphological and newly acquired magnetic data.

Citation: Caratori Tontini, F., L. Cocchi, and C. Carmisciano (2009), Rapid 3-D forward model of potential fields with application to the Palinuro Seamount magnetic anomaly (southern Tyrrenian Sea, Italy), J. Geophys. Res., 114, B02103, doi:10.1029/2008JB005907.

1. Introduction

[2] Forward calculation of potential field anomalies is essential for quantitative interpretation. A model’s anomaly, based on geologic or any other external “a priori” information, may be calculated and then compared with an observed anomaly to infer some parameters of the generating source [Blakely, 1995]. Moreover, forward models are also essential for inversion methods, because they provide the basic equations for automatically determining some source parameters, depending on the particular adopted model used to describe the source properties. Many papers deal with this kind of calculation; providing a complete description of available models is beyond the scope of this introduction. For a recent review, see Nabighian [2005a, 2005b].

[3] Closed-form equations were first developed for simple sources. For example, Singh and Sabina [1978] calculated the magnetic anomaly due to a vertical right cylinder with arbitrary polarization. Bhattacharyya [1964] calculated the magnetic anomaly of a prismatic body, while Nagy [1966] developed similar calculations for a gravity anomaly. Later, Plouff [1976] developed a closed-form equation for the gravity anomaly of a finite-thickness horizontal plate, which is useful for terrain corrections. Prismatic bodies are very important as they provide a practical method for approximating a more complex source by the principle of superposition, saturating the source volume without “holes”. In a similar way, Talwani and Ewing [1960] (for gravity) and then Talwani [1965] (for magnetic data) approximated real sources by sets of stacked laminae.

[4] For 3-D sources, polyhedral bodies provide a very useful geometry to represent arbitrary shapes [Okabe, 1979; Pohanka, 1988; Furness, 1994; Singh and Guptaarma, 2001; Holstein, 2002a, 2002b]. Particularly in the case of magnetic data, if the magnetization is uniform, the magnetization distribution is equivalent to a magnetic charge on the facets of the polyhedron [Bott, 1963; Barnett, 1976; Hansen and Wang, 1988].

[5] Another commonly adopted model is based on approximating the real source, in the case of a strike extension, by a 2-D body with its strike direction perpendicular to a polygonal cross section [Talwani et al., 1959], and its subsequent refinements that were adopted for end corrections [Shuey and Pasquale, 1973; Rasmussen and Pedersen, 1979; Cady, 1980] to consider the real case where geological bodies have finite lengths.

[6] Modeling of potential field data has also been widely developed in the Fourier domain, where many geometric characteristics of the source are expressed as simple multiplicative factors within the anomaly spectrum, and transformations are based on simple linear relations [Bhattacharyya, 1967]. In this sense, Bhattacharyya [1966] evaluated the spectrum of the magnetic anomaly of a right-rectangular prism. Pedersen [1985] calculated the spectral
expressions for magnetic and gravity anomalies of ellipsoidal bodies. Parker [1972] provided a very interesting method for calculating potential field anomalies of 2-D sources confined within a layer enclosed between uneven top and bottom surfaces. This forward method is useful for modeling potential field anomalies generated by terrain or bathymetry [Blakely, 1995]. Generally speaking, Fourier domain modeling is fast, provided some care is taken concerning errors due to aliasing or edge effects.

[7] In this paper, we show a new set of equations defining 3-D potential field anomalies of a 3-D source in the Fourier domain, using an approach similar to the gravity computations of Peng et al. [1995]. A 3-D grid of regularly spaced points provides a finite discretization of the real infinite 3-D space, where both density or magnetization distributions and field data are defined as discrete approximations of real continuous functions. Within this grid, the Fourier operator is defined by the discrete Fast Fourier Transform (FFT) method [Cooley and Tukey, 1965]. Typically, the top half-space defines the data space, while the bottom half-space of the grid is attributed to the source domain, so that density or magnetization distributions vanish in the top half-space. Our method also provides the field inside the source volume, so that in the final analysis, from a theoretical point of view, both density or magnetization distributions and field data can be considered as defined in the whole grid, at least vanishing in some parts of the space.

[8] We demonstrate the reliability of these new 3-D forward equations, thus defining the 3-D Fourier transform of the gravity and magnetic anomalies as functions of the corresponding 3-D Fourier transforms of the density or magnetization distributions. The corresponding fields in the space domain at the nodes of the grid are thus obtained by an inverse FFT transformation. The relation connecting the particular density or magnetization distribution with the observed anomaly has a very simple algebraic expression in the Fourier domain, and it shows the relevant property that a potential field anomaly can be seen as a filtered version of the corresponding density or magnetization distribution. This method is very fast, allowing for a complete description of the 3-D field by a grid of $100^3$ points within a few seconds.

[9] As an application of these forward model equations, we estimate the average magnetization of the Palinuro Seamount in the Tyrrenian Sea in southern Italy, obtained by a recent ship-borne survey performed in May 2008, and we analyze the corresponding residual magnetic anomaly coming from the terrain corrections computed with the estimated average magnetization of 8 Am.$^{-1}$.

2. Fourier Representation of Potential Field Equations

[10] We define the variables and symbols adopted for calculating the forward equations. The continuous 3-D Fourier transform is defined according to the following convention:

$$\mathcal{F}_{3-D}[\gamma(x)] \equiv \hat{\gamma}(k) \equiv \int_{\mathbb{R}^3} \gamma(x)e^{-ik \cdot x}dx,$$

(1)

where (1) $\mathcal{F}_{3-D}$ is the 3-D Fourier operator; (2) $x = (x,y,z)$ is the spatial position; (3) $k$ is the 3-D wave vector, where $k = (k_x,k_y,k_z)$; and (4) $\mathbb{R}^3$ is the infinite 3-D domain of real numbers where the function $\gamma(x)$ is defined.

[11] Figure 1 provides a sketch of the adopted geometry and the variables used for this calculation. The $z$ axis is positive downward, while the $x$ and $y$ axes are north and east oriented, respectively.

[12] In general, the function $\gamma(x)$ is characterized by good convergence properties, satisfying the condition:

$$\int_{\mathbb{R}^3} |\gamma(x)|dx < +\infty,$$

(2)

which automatically ensures the existence of the Fourier transform of equation (1). This is the case, for example, for the functions defining the 3-D source distributions, which are limited and vanish outside of a bounded volume. However, equation (2) is only a sufficient condition for the existence of the Fourier transform. By introducing weak convergence in the space of the distributions, the Fourier transform can be adapted to deal with all the functions that have asymptotic behavior which can be approximated by a polynomial function of positive degree [Churchill et al., 1974]. The corresponding Fourier transform may not be continuous, and may be defined by a distribution that is weakly approximated by a set of "classical functions". The Fourier transform is thus defined in the space of the distributions, and integrals are intended in the sense of the Cauchy principal part, but, as we shall see, the physical equations defining the field are well-behaved.

[13] The inverse Fourier transformation is given by

$$\mathcal{F}^{-1}_{3-D}[\hat{\gamma}(k)] \equiv \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \hat{\gamma}(k)e^{ik \cdot x}dk,$$

(3)

by introducing the normalization factor $1/(2\pi)^3$. An important result that will be used in the following is

$$\mathcal{F}_{3-D} \left[ \frac{1}{|x|} \right] = \int_{\mathbb{R}^3} \frac{1}{|x|} e^{-ik \cdot x}dx = \frac{4\pi}{|k|^2},$$

(4)
where $\| \|$ is the Pythagorean norm, $|x| = \sqrt{x^2 + y^2 + z^2}$. This result, which is the 3-D Green’s function for the potential, will be demonstrated in Appendix A, and it is one particular case where the extension of the Fourier operator to the space of the distribution is important, since in this case, the condition of equation (2) is not satisfied.

3. Gravity Fourier Equations

3.1. Introduction

[14] The expression of the vertical gravity anomaly in the spatial domain is

$$\Delta g_z(x_0) = G \int_{\mathbb{R}^3} \frac{\rho(x)(z - z_0)}{|x - x_0|^3} \, dx,$$  \hspace{1cm} (5)

where (1) $\Delta g_z$ is the gravity anomaly, in $m \cdot s^{-2}$; (2) $G = 6.67 \times 10^{-11} \text{N} \cdot m^2/\text{kg}^2$ is the gravitational constant; (3) $\rho$ is the density distribution in $\text{kg} \cdot m^{-3}$; (4) $x = (x,y,z)$ is the vector position of the generic point inside of the source volume, over which the integration is performed; and (5) $x_0 = (x_0,y_0,z_0)$ is the vector position of the measurement.

[15] If the density distribution vanishes outside of a bounded volume, we can extend the integral from the finite source volume to all of $\mathbb{R}^3$, as implicitly assumed in equation (5). The corresponding expression of the gravity anomaly in the Fourier domain is given by

$$\mathcal{F}_{3-D}[\Delta g_z] = \int_{\mathbb{R}^3} \Delta g_z(x_0) e^{-ik \cdot x} \, dx_0 = G \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\rho(x)(z - z_0)}{|x - x_0|^3} e^{-ik \cdot x} \, dx \, dx_0.$$  \hspace{1cm} (6)

[16] In the case of a bounded source distribution, this Fourier transform should exist. This is because, if the density or magnetization distributions vanish outside of a finite volume, the corresponding potential fields are similarly bounded, with power law decay with distance $d$ from the source of $d^{-2}$ and $d^{-3}$ for the gravity and magnetic cases, respectively. These kinds of power law decays should ensure that Cauchy integrals in the space of the distributions are convergent. The integral of equation (6) can thus be easily evaluated because it is a convolution product, taking the following expression:

$$\mathcal{F}_{3-D}[\Delta g_z] = -G \mathcal{F}_{3-D}[\rho] \mathcal{F}_{3-D} \left[ \frac{z_0}{|x_0|^3} \right] = -G \mathcal{F}_{3-D}[\rho] \mathcal{F}_{3-D} \left[ \frac{\partial}{\partial z_0} \left( \frac{1}{|x_0|} \right) \right] = iGk_z \mathcal{F}_{3-D}[\rho] \mathcal{F}_{3-D} \left[ \frac{1}{|x_0|} \right] = i4\pi G \frac{k_z}{|k|^2} \mathcal{F}_{3-D}[\rho],$$  \hspace{1cm} (7)

where we have used the derivative rule of the Fourier transform, which gives the factor $ik_z$, and the result of equation (4). Equation (7) provides a very simple compact expression for the Fourier transform of the gravity anomaly given the Fourier transform of the density distribution. In the Fourier domain, these quantities are related by a simple algebraic product, which practically acts as an anisotropic, non-normalized, smoothing filter, emphasizing mainly the $k_z$ component of the Fourier transformed density distribution. This expression is valid even within the source volume. The anomaly in the spatial domain can be obtained by the inverse transformation defined in equation (2). We note that equation (7) can be adapted to calculate other field components or gradients by simply multiplying for the proper combinations of the components of the vector $(ik_x,ik_y,ik_z)$, according to the Fourier derivative rule. In particular, for the gravity gradient tensor $U_{ij}$ we have

$$\mathcal{F}_{3-D}[U_{ij}] = -4\pi G \frac{k_k g_k}{|k|^2} \mathcal{F}_{3-D}[\rho],$$  \hspace{1cm} (8)

where $i,j \in \{x,y,z\}$, which is obviously symmetric and satisfies the Laplace equation in its Fourier form, as given by the following trace:

$$\mathcal{F}_{3-D}[U_{xx} + U_{yy} + U_{zz}] = 4\pi G \mathcal{F}_{3-D}[\rho].$$  \hspace{1cm} (9)

3.2. Gravity Anomaly of a Point-Like Mass

[17] To confirm the correctness of the previous equations, we discuss the case of a point-like mass $m$ centered at the origin of the Cartesian system. The density distribution in this case is given by

$$\rho(x) = m \delta(x),$$  \hspace{1cm} (10)

where $\delta(x)$ is the Dirac delta function, whose Fourier transform is $\mathcal{F}_{3-D}[\delta(x)] = 1$. The gravity anomaly at point $x_0$ on the space is thus obtained by an inverse Fourier transformation, as follows:

$$\Delta g_z(x_0) = \int_{\mathbb{R}^3} 4\pi Gm \frac{k_z}{|k|^2} \frac{dk}{(2\pi)^3} e^{ik \cdot x_0}.$$  \hspace{1cm} (11)

[18] The previous equation can be simplified using the derivative rule $ik_z \rightarrow \partial/\partial z_0$ in the following form:

$$\Delta g_z(x_0) = \frac{\partial}{\partial z_0} \int_{\mathbb{R}^3} \frac{4\pi Gm}{|k|^2} \frac{dk}{(2\pi)^3} e^{ik \cdot x_0} = \frac{Gm}{2\pi^2} \frac{\partial}{\partial z_0} \int_{\mathbb{R}^3} \frac{dk}{|k|^2} e^{ik \cdot x_0}.$$  \hspace{1cm} (12)

where the last integral can be evaluated in polar coordinates as follows:

$$\Delta g_z(x_0) = \frac{Gm}{2\pi} \frac{\partial}{\partial z_0} \int_{0}^{+\infty} k^2 dk \int_{0}^{\pi} \sin \theta \frac{\partial}{\partial k} \left( \frac{k}{|k|^2} \right) d\theta \int_{0}^{2\pi} d\phi.$$  \hspace{1cm} (13)

By integrating along $\phi$ and $\theta$, we are left with

$$\Delta g_z(x_0) = \frac{2Gm}{\pi} \frac{\partial}{\partial z_0} \left[ \frac{\sin k |x_0|}{k |x_0|} \right] dk = \frac{2Gm}{\pi} \frac{\partial}{\partial z_0} \left[ \frac{1}{|x_0|} \int_{0}^{+\infty} \frac{\sin \alpha}{\alpha} d\alpha \right].$$  \hspace{1cm} (14)
[19] Given
\[
\int_0^{+\infty} \frac{\sin \alpha}{\alpha} d\alpha = \pi/2
\] (15)
as tabulated, for example, by Gradshteyn and Rydzig [1980], by performing the derivative along \( z_0 \), we obtain the final result:
\[
\Delta g_r(x_0) = \frac{-Gmz_0}{|x_0|^3},
\] (16)
which is the correct expression for the vertical gravity anomaly generated by a point-like mass. The field is well-behaved even if the separate Fourier transform of the density distribution is an impulsive, discontinuous function, given by the Dirac delta and the Green’s function \( 4\pi r^2 \) is similarly divergent at the origin. The physical integrals defining the field are instead convergent.

4. Magnetic Fourier Equations

4.1. Introduction

[20] The expression of the total field magnetic anomaly of a magnetized source is given by Blakely [1995]:
\[
\Delta T(x_0) = -\frac{\mu_0}{4\pi} \hat{F} \cdot \nabla \left[ \int_{R'} M(x) \cdot \nabla \left( \frac{1}{|x-x_0|} \right) dx \right].
\] (17)
where (1) \( \mu_0 = 4\pi \times 10^{-7} \) Henry/m is the vacuum magnetic permeability; (2) \( \nabla \) and \( \nabla_0 \) are the gradients with respect to \( x \) and \( x_0 \), respectively, and for the last integral the relation \( \nabla = -\nabla_0 \) holds; (3) \( \hat{F} \) is the unitary vector representing the main direction of the ambient external geomagnetic field; and (4) \( M(x) \) is the magnetization vector distribution in Am.

[21] The integral of equation (17) is only an approximation, which is valid if the magnetic anomaly caused by the perturbing source is small relative to the ambient magnetic field. This hypothesis may fall very near strongly magnetic sources, but in the following we will assume its general validity or we will use the magnetic field at points some distance away from magnetic sources, such that the approximations of equation (19) are valid. If the magnetization direction is uniform throughout the source volume, although its intensity may vary, we can write
\[
M(x) = M(x)\hat{M},
\] (18)
where \( \hat{M} \) is the unitary vector representing the direction of the magnetization distribution with modulus \( M(x) \). In light of these new definitions, the last equation takes on the following form:
\[
\Delta T(x_0) = -\frac{\mu_0}{4\pi} \hat{F} \cdot \nabla \left[ \int_{R'} M(x)\hat{M} \cdot \nabla \left( \frac{1}{|x-x_0|} \right) dx \right].
\] (19)

[22] It is important to note here that the last expression is correct only outside of the source magnetized volume; otherwise it should be multiplied for the proper relative permeability \( \mu_r \) of the source. This is the main difference with respect to the similar gravity expression of equation (5). In any case, a similar expression is generally valid in \( \mathbb{R}^3 \) if we deal with the magnetic induction vector \( \mathbf{H} \):
\[
\mathbf{H}(x_0) = \frac{1}{4\pi} \nabla \left[ \int_{R'} M(x)\hat{M} \cdot \nabla \left( \frac{1}{|x-x_0|} \right) dx \right].
\] (20)
rather than the magnetic field \( \mathbf{B} \). The induction field is characterized by
\[
\mathbf{B} = \mu_0\mu_r\mathbf{H},
\] (21)
where \( \mu_r = 1 \) in nonmagnetic materials. Again, the Fourier transform of equation (20) is a convolution product, and, for the same properties of the derivatives and from the result of equation (4), we reach the following compact expression:
\[
\mathcal{F}_{3-D}[\mathbf{H}] = -\frac{\hat{M} \cdot \mathbf{k}}{|k|^2} \mathcal{F}_{3-D}[M],
\] (22)
with steps similar to those shown in equation (7). The correct spatial expression of the magnetic induction field \( \mathbf{H} \) is obtained by an inverse transformation of equation (22), and is valid all over the space. Outside of magnetized material, the magnetic field is rapidly obtained as \( \mathbf{B} = \mu_0\mathbf{H} \), and thus the magnetic anomaly is obtained by projection on the ambient geomagnetic field. With a slight improper notation, but in compact form, we can write
\[
\mathcal{F}_{3-D}[\Delta T] = -\mu_0 \frac{\hat{F} \cdot \mathbf{k}}{|k|^2} \mathcal{F}_{3-D}[M].
\] (23)
implicitly assuming that the inverse Fourier transform of this expression gives the correct magnetic anomaly only outside of magnetized material. If we are interested in the field inside magnetic sources, we must perform an inverse transformation of equation (22) and then multiply it by \( \mu_0\mu_r \) for inside magnetized material and by \( \mu_0 \) otherwise.

[23] We have again found a very compact algebraic expression in the Fourier domain, where a magnetic anomaly can be seen as a filtered version of the corresponding magnetization distribution. The high-frequency part of the spectrum, in this case, is less suppressed than the corresponding gravity field. In the Fourier domain, the magnetic anomaly is obtained by a simple algebraic multiplication of the magnetization distribution for the proper filter. This makes it easy and fast to calculate the magnetic anomaly given knowledge of the magnetization distribution. Again, the Fourier derivative rule gives similar compact expressions like equation (8) for the magnetic gradients.

4.2. Horizontal Slab Magnetic Anomaly

[24] To demonstrate the correctness of equation (23), we will derive the particular well-known case of the 2-D magnetic anomaly at a given altitude \( z_0 \) for a horizontal
It can be easily shown that the term proportional to $\delta(k_z)$ vanishes, so we are left with

$$F_{2-D}[\Delta T] = -\mu_0 F_{2-D}[M_0]$$

$$\times \int_{-\infty}^{\infty} \frac{dk_z}{2\pi} \left( \vec{F} \cdot \vec{k} \right) \left( \vec{M} \cdot \vec{k} \right) e^{ik_z z_0}$$

$$\times \left( e^{-ik_z z_1} - e^{-ik_z z_2} \right) \left[ \pi \delta(k_z) + iP \left( \frac{1}{k_z} \right) \right].$$

(27)

We shall thus evaluate

$$I = \int_{-\infty}^{\infty} \frac{dk_z}{2\pi} \left( \vec{F} \cdot \vec{k} \right) \left( \vec{M} \cdot \vec{k} \right)$$

$$\times e^{ik_z z_0} \left( e^{-ik_z z_1} - e^{-ik_z z_2} \right).$$

(29)

[25] This integral of the Jordan kind can be solved in the complex plane using the Cauchy theorem on the closed half-circular path shown in Figure 2 [Churchill et al., 1974]. If we continue the integrand function in the complex domain C, the Cauchy theorem states that the integral along a closed path in C is proportional to the sum of the residues at each pole inside the closed path.

[26] The closed path in Figure 2 is given by the linear path on $\text{Re}(k_z)$ from $-R$ to $R$, together with the corresponding half-circular path of radius $R$. Letting $R \to \infty$, the integral I on the real axis approaches the integral of equation (29). The exponential term ensures that the integral contribution $I_1$ given by the half-circular path vanishes when $R \to +\infty$, leaving only the integral $I$. This is due to the following inequality:

$$|I| \leq \lambda \left| e^{-i(z_1 - z_0)R} - e^{-i(z_2 - z_0)R} \right| \to 0,$$

(30)

when $R \to +\infty$, where $\lambda$ is a generic positive and sufficiently large number. This can be easily demonstrated by substituting $k_z = R e^{i\theta}$ and letting $R \to \infty$. The integral is thus given by the residues of the integrand function on its internal poles. Within the chosen path, we have only a pole of first order when $k_z = -i\sqrt{k_x^2 + k_y^2}$, while the apparent pole $k_z = 0$ actually disappears because it is canceled by the factor $(e^{-ik_z z_1} - e^{-ik_z z_2})$, which vanishes as $k_z \to 0$. The pole $k_z = -i\sqrt{k_x^2 + k_y^2} = -ik_0$ has the relevant property of deriving from the harmonic behavior of the magnetic field. So, according to the Cauchy theorem, we have

$$I + I_1 = I = 2\pi i \text{Res}(w_0),$$

(31)

where the expression of the residue $\text{Res}(w_0)$ at the pole $w_0$ of a generic complex function $f(w)$ is given by

$$\text{Res}(w_0) = \frac{1}{(n-1)!} \left\{ \frac{\partial^{n-1} (w - w_0)^n f(w)}{\partial w^{n-1}} \right\}_{w = w_0},$$

(32)

where $n$ is the order of the pole and the 0th order derivative of a function corresponds to the function itself. In our case, it practically corresponds to multiplying the integrand.
function in (29) by \((2\pi i)(k_z + ik_h)\), and then, by substituting \(k_z = -ik_h\), the following expression is obtained:

\[
I = \frac{\tilde{F}_x k_x + \tilde{F}_y k_y - i\tilde{F}_z k_h}{2k_h^2} \left( \tilde{M}_x k_x + \tilde{M}_y k_y - i\tilde{M}_z k_h \right) e^{ik_h z_x} (e^{ik_h z_1} - e^{-ik_h z_2}),
\] (33)

and thus, by substituting the terms

\[
\Phi_F = \frac{\tilde{F}_z + i\tilde{F}_x k_x + \tilde{F}_y k_y}{k_h},
\] (34)

and

\[
\Phi_M = \frac{\tilde{M}_z + i\tilde{M}_x k_x + \tilde{M}_y k_y}{k_h},
\] (35)

with the same convention as Blakely [1995], we obtain the final result:

\[
\mathcal{F}_{2-D} [\Delta T] = \frac{\mu_0}{2} \Phi_F \Phi_M \mathcal{F}_{2-D} [M_h] e^{ik_h z_x} (e^{ik_h z_1} - e^{-ik_h z_2}),
\] (36)

which is consistent with the expression by Schouten and McCamy [1972], in the form of equation 11.35 of Blakely [1995]. A similar derivation can be obtained for the gravity case, and, moreover, other particular cases such as the dipole or the magnetized line may be recovered by introducing the proper Fourier transforms. We will demonstrate in Appendix B that the Cauchy theorem allows us to recover a more general class of models by equations (7) and (23).

5. Practical Implementations in the Real Case: Possible Errors

5.1. Introduction

[27] In the real case, the potential field anomalies are calculated over a discrete data set. This means that the continuous expressions of equations (7) and (23) should be replaced by the corresponding discrete approximations, and the continuous Fourier transform is replaced by its discrete FFT counterpart [Cooley and Tukey, 1965]. The real 3-D infinite spatial domain is thus described by a prismatic finite-size 3-D grid of discrete, regularly spaced points. The magnetization or density distributions are thus defined within this grid by assigning a given value to each point of the grid. Typically, the upper layers of the grid, where \(z < 0\), define the harmonic domain outside the subsurface, where the density or magnetization distributions vanish and the
calculated field is properly defined. The wave number domain is thus automatically defined by the number and spatial separations of the points of the spatial grid, up to the corresponding Nyquist frequencies.

The replacement of the Fourier operator by its discrete FFT approximation is characterized by errors due to aliasing, imposed periodicity, and edge effects. The aliasing effect can be reduced by using a large number of cells, thus increasing the corresponding Nyquist frequencies. The computations by FFT are very fast, and thus using a large number of cells does not dramatically increase the execution times, as happens in the discrete linear approach.

Replacing the continuous theoretical Fourier operator with the discrete approximation implies that the source distribution is periodic in all directions. This imposed periodicity means that density or magnetization distributions are repeated above, below, and off the sides of the original 3-D grid (Figure 3). These “ghost” sources may interfere with the real source, especially for calculating anomalies that are not in the immediate vicinity of the real source. Another source of error may be generated by edge discontinuities when we impose periodicity with sharp edges between each period. While the latter problem may be reduced by a tapering Hamming window, which makes the source vanish at the edges of the original 3-D grid, the former problem requires an expansion of the 3-D grid used for the computation, so that the ghost sources are moved far from the real source, reducing their interferences (Figure 3).

Practically, we expand the 3-D grid used to define the density (magnetization) and anomaly fields in each direction, by adding at each side (lateral and vertical) the same number of cells, so that each linear dimension is doubled. The densities are padded with zeros outside of the original region. The tapering window reduces the edge effects. This expanded grid is the real 3-D grid used for the FFT computations, and the potential field anomalies are calculated fields thus are not exactly defined, introducing some errors. Figure 7 shows the corresponding magnetic anomalies that are not in the immediate vicinity of the real source, reducing their interferences (Figure 3).

These corrections are essential, since preliminary synthetic tests (Figure 4) have shown that the errors due to the imposed periodicity may be as large as 20% in the immediate vicinity of the source, becoming worse with distance. In the subplots of Figure 4, we show the errors coming from our Fourier method in the case of a simple prismatic source, whose field will be exactly mapped in the following section. These subplots are differentiated as a function of distance from the source and percent grid expansion, which ranges from zero, when the grid is not expanded, to 1, when the original grid dimensions are effectively doubled. By introducing these corrections, the mean relative errors in the gravity case are reduced from 30% to 3–5% near the source, reaching 10% at greater distances, where the effects due to the ghost sources are probably more intense. We note that the RMS errors for gravity anomalies seem larger than those for magnetic anomalies. This is probably due to the differences in the decay with distance between gravity and magnetic fields, so that “ghost” source effects are more suppressed in the case of magnetics rather than gravity.

The synthetic tests in the following subsection provide a spatial distribution of these errors. We think, however, that these corrections make our method usable at large distances from the source, at least until the anomaly field is not drastically suppressed. Otherwise, more “traditional” upward continuation filters may be used.

We also note that the method provides potential fields defined on a regular 3-D grid. If we are interested in obtaining the fields on a scattered set of data points, for example on a topographic surface, we should interpolate these data from the 3-D grid to the uneven data set. The calculated fields thus are not exactly defined, introducing another error of interpolation. In any case, we think that this is not a very limiting factor, given the accuracy of 3-D interpolating algorithms.

5.2. Synthetic Examples

Figure 5 shows a synthetic model adopted for testing both gravity and magnetic calculations. The values of the density and magnetization distributions are in $g/cm^3$ and $A/m$, respectively. The model has been randomly generated according to a scaling Gaussian noise [Turcotte, 1997], which allows us to represent complex geologic models of density or magnetization [Maus and Dimri, 1994]. This method consists of generating a regular 3-D grid $W_{ijk}$ of values according to a Gaussian probability distribution function. Then, a 3-D discrete Fourier transform $\tilde{W}_{ijk}$ of complex coefficients for this grid is calculated, and the corresponding scaling distribution in the Fourier domain is obtained as $\tilde{W}_{ijk}/|k|^{5/2}$, where $*$ represents the complex conjugate, $k$ is the wave vector, and $\beta$ is the scaling exponent. This is a rapid method for generating complex 3-D distributions of physical crustal parameters at minimum amount of user-dependent information. It is thus useful for testing and displaying calculations of potential field anomalies by our method.

In Figure 5 we have generated a random grid made of $51 \times 51 \times 51$ equally spaced points with a scaling exponent of $\beta = 5$. The resulting model has been confined to the bottom half-space to simulate a real subsurface model, while the top half-space represents the observation domain. Figure 6 shows the resulting gravity anomaly, whose values are defined in the whole grid, included the source volume. The field decay with depth is clear, and the anomaly does not show rings or edge effects that suggest possible FFT errors. Figure 7 instead shows the corresponding magnetic anomaly, under an inducing field characterized by an inclination of $90^\circ$ and a declination of $0^\circ$. The field decay with depth correctly increases in this case, and again no rings or edge effects are visible. We note here that calculations of these potential field anomalies on a 3-D grid of about $10^6$ data, given the density/magnetization distribution, took about 3 sec, which is a very short time for this kind of calculation.

The results of another synthetic test are shown in Figures 8 and 9, where we can see the gravity and magnetic anomalies, respectively at three observation levels. Figures 8a and 9a, 8b and 9b, and 8c and 9c are obtained by calculating the fields by our FFT method, while Figures 8d and 9d, 8e and 9e, and 8f and 9f represent the error maps coming from the comparisons with the exact fields given by the equations of Bhattacharyya [1964] for the magnetic case and Nagy [1966] for the gravity case. The generating source is a buried prism with its top 300 m deep and its bottom
Figure 5. Three-dimensional synthetic density/magnetization distribution generated to evaluate an example of gravity/magnetic anomaly by Fast Fourier Transform (FFT) calculation.

Figure 6. Three-dimensional gravity anomaly generated by the 3-D density distribution of Figure 5.
600 m deep. It is horizontally centered within the grids of Figures 8 and 9, extending from −400 to 400 m northward and −200 to 200 m eastward. This is the same prism as that used for the error plots in Figure 4. This simple source permits a quantitative analysis of the spatial distribution of errors. Moreover, it is a severe test for the FFT calculation, since it is characterized by sharp behavior of the density/magnetization, which discontinuously vanish outside the prism volume. A comparison of these anomalies suggests good agreement between both of the calculations. 

For the gravity case, the relative error of the amplitude estimations is about 4–5% near the source, even using a relatively small number of $20 \times 20 \times 20$ points. At greater distances, the errors of the estimated amplitudes do not exceed 10–15%. For the magnetic case, the situation is even better, with errors of the amplitudes of 1–2% near the

Figure 8. (a, b, c) Comparison between gravity anomalies at different observation levels of a right rectangular prism and (d, e, f) their associated error maps.
source, up to 6–8% at the greatest heights. We note that the lateral zones, where anomalies tend to vanish, are more affected than the local maxima or minima. This is probably due to the effects of horizontal side ``ghosts'', which can generate small offsets. In any case, the worse errors in the gravity data are about 30% at greater heights, and 10% near the sources. The average errors are those shown in Figure 4 at 100% of grid expansion. On the other hand, this small round-off error is trivial in the real case, if compared with the typical accuracy of real magnetic or gravity data.

Moreover, the execution times appear dramatically suppressed. Figure 10 shows a comparison of execution times for calculating the potential field anomalies of randomly generated synthetic sources, as a function of the number of data. The FFT algorithm requires about 3 s of execution times for a grid of $10^3$ data. A similar discretization cannot be achieved with a simple PC using the
prismatic approximation. We note that the execution time for a grid made of $10^3$ cells was 0.032 s for the FFT calculation and 9 s for the prismatic approximation, but with $20^3$ cells it became 0.14 s for FFT calculation and 473 sec for the prismatic approximation. The maximum number of points that we were able to compare with our PC was 50.$^3$. In this case, it took about 1.29 s for the FFT calculation and about $10^7$ s for the prismatic approximation.

Our method thus seems particularly useful for interactive forward modeling of potential field data at various altitudes, since it allows for proper real-time calculation and mapping of the field generated by a complex source, which can be approximated by a large number of discrete points.

6. Palinuro Seamount Magnetic Model

The Palinuro Seamount is a volcanic complex located in the southern Tyrrenian Sea in Italy. This region is characterized by several Pleistocene volcanic centers [Beccaluva et al., 1985], including the neighboring Aeolian Archipelago and the Marsili Seamount (Figure 11). The southern Tyrrenian Sea is interpreted as a back-arc basin, caused by the eastward retreat of the Ionian slab [Malinverno and Ryan, 1986].

While the genesis of the Aeolian Archipelago is clearly interpreted as a volcanic arc due to the interaction between the European Plate and the subducting Ionian slab [Beccaluva et al., 1985], the Marsili Seamount is superimposed on a spreading-like system triggered by crustal depletion that occurred in a roll-back regime due to Ionian subduction [Marani and Trua, 2002].

In this context, the origin of the Palinuro volcanic complex is not yet clear, especially when compared with the neighboring volcanic centers of Marsili and the Aeolian complex. The Palinuro complex is developed along an E–W trend, probably located on a strike-slip fault with the same orientation [Colantoni et al., 1981; Del Ben et al., 2008]. This tectonic structure and its connected magmatism represent the northernmost evidence of the southern Tyrrenian Sea volcanism, potentially related to the subduction system.

The Palinuro volcanic edifice rises from 3000 to 80 m b.s.l. (Figure 12). It consists of some separate volcanic bodies that are basally connected to form a continuous volcanic ridge [Marani and Gamberi, 2004]. The E–W orientation of these small summit cones may represent evidence of the supposed strike-slip fault which may have triggered the genesis of the Palinuro complex. It appears that the topography of the seamount shows a N–S asymmetric shape, with the crest separating a southern portion, characterized by steep scarps reaching a depth of about 3000 m, from the northern portion that decreases to about 1800 m with a lower topographic gradient. This may suggest that the Palinuro complex is located at the border of the Marsili Basin (Figure 11), where there is a transition from oceanic basin to continental shore.

Detailed morphological analysis based on swath bathymetry shows the existence of important structures related to instability of the edifice, such as a caldera rim.
located in the western part of the volcanic complex and a set of flanking faults [Marani et al., 2004]. Basalts and basaltic andesites dredged from the top of the complex at about 80 m b.s.l. were dated at about 0.3 Ma [Colantoni et al., 1981], which suggests that the magnetization distribution of the volcanic complex should be parallel to the field of the actual geocentric dipole. The magnetic method can thus be a useful way to obtain

Figure 12. Bathymetric map of the Palinuro Seamount. The dashed line is a morphological trend following some cuspidal points in the bathymetry contours, while the triangled line shows the caldera rim. The main volcanic bodies are labeled from A to E.

Figure 13. Magnetic map (total intensity) of the Palinuro Seamount.
Figure 14. Residual error maps at varying magnetizations (a–f). Figure 14c, obtained using a bulk magnetization of 8 A/m, appears to have the minimum RMS error.
additional information about the characteristics of this complex, since volcanic rocks below the Curie isotherm are typically characterized by high magnetization values.

[46] We recently (May 2008) investigated details of the Palinuro Seamount magnetic anomaly during a cruise in cooperation with the Hydrographic Institute of Italian Navy, with the low-magnetic fiberglass R/V Arethusa. The planned survey consisted of a set of parallel S–N lines, with an average spacing of 1 km, and some orthogonal control tie lines. The total-intensity anomaly map, processed for diurnal variations by coherent magnetograms and after IGRF2008 subtraction, is shown in Figure 13.

[47] In the magnetic anomaly map, a similar E–W alignment is also visible in the very sharp E–W gradient at a latitude 39°29’, with longitudes ranging from 14°45’ to 14°50’. This feature appears aligned with the crest of the seamount, but it does not seem to be a topographic anomaly. The linearity of this anomaly looks fault-like, but there is no clear shallow evidence of any E–W displacement. This kind of linearity is probably due to the superposition of the magnetic anomalies coming from the set of small E–W oriented summit cones. This magnetic lineament divides the Palinuro volcano within a southern, high-magnetic N–S elongated portion, from a northern low-magnetic area. This magnetic characterization reflects the morphological evidence of an asymmetric N–S development, previously explained from the morphological data as due to the transition from oceanic basin to continental shore.

[48] The data have been further processed to extract information regarding the average magnetization of the Palinuro Seamount. To obtain a quantitative result, the terrain corrections to the reduced-to-the-pole magnetic anomaly were computed by the Fourier forward model described in the previous sections, for different values of the magnetization, which is assumed to be uniform for the entire seamount. The base of the model was assumed to be flat and horizontal at a depth of 3500 m. At that depth, we found that the choice of bottom depth actually does not influence the results as long as the magnetization is uniform, since the shallow portion of the outcropping volcano represents the main magnetic component.

[49] The RMS error (i.e., the average difference between the observed reduced-to-the-pole magnetic anomaly and the calculated terrain magnetic anomaly) was studied to extract information about the bulk magnetization. We assume that the average magnetization can be obtained as the optimal value that minimizes the RMS error. Figure 14 shows the residual magnetic anomaly maps computed for various values of terrain magnetization. The optimal value that produces a minimum RMS error is at about 8 A/m, as can be seen in the minimization plot of Figure 15. In particular, we should say that the RMS error is quite large, suggesting that the uniform magnetization is probably a raw approximation. This is also demonstrated by the persistence of the negative residual central anomaly in all the subplots of Figure 14. This suggests that there may be a relatively low magnetization in the western part of Palinuro volcano, as can be seen in Figure 16. However, we can interpret these results in terms of relative variations.

[50] Figure 16 shows the residual magnetic anomaly, reduced to the pole and corrected for the average terrain magnetization of 8 A/m, spread over the bathymetric surface. This map allows a joint interpretation of the morphological and magnetic characteristics of the volcanic complex. It is important to highlight that low magnetic values should be referred to the offset average magnetization of 8 A/m. A forward model of the residual anomaly showed that a relatively negative magnetization, smaller than 8 A/m, can fit the −490 nT minimum of Figure 13. The global magnetization of the seamount thus remains positive, since it is superimposed on the average magnetization of 8 A/m.

[51] The eastern portion of Palinuro volcano, marked by the letter E, is the morphological evidence of an outcropping crater [Marani and Gamberi, 2004], and is characterized by a negative residual anomaly. We interpret this characteristic feature as due to a partial filling of the crater by nonmagnetic sediments. The neighboring summit ridge (80 m b.s.l.), marked by the letter D, is crossed by the linear N–S trending morphological structure, identified by a dashed line, which has clear magnetic evidence. A narrow stripe of positive residual magnetic anomaly follows the orientation of this morphological lineament, which separates two intense positive anomalies on the southern flank of the volcano. Joint analysis of morphological and magnetic data suggests that this lineament can be interpreted as a dike-like structure that dissects the former volcanic center.

[52] The western sector of the Palinuro Seamount is morphologically characterized by a caldera rim. The residual magnetic anomaly is characterized by a negative circular pattern that follows the rim structure (letter B). As shown by
the rock samples of Colantoni et al. [1981], this region is strongly altered by hydrothermal effects, which may have lowered the magnetization values. The negative magnetic pattern extends up to the westernmost cone A.

The entire Palinuro structure can be thus grossly subdivided into a western subregion, characterized by a negative residual anomaly, where hydrothermal processes may have played a relevant role in terms of rock differentiation, and an eastern subregion, characterized by shallow, fresh lavas, with an intense positive residual magnetic anomaly.

7. Conclusions

We have developed a set of forward model equations in the Fourier domain for rapid computation of 3-D potential field anomalies generated by complex 3-D sources. The fields and sources are defined within a discrete 3-D grid that approximates the real continuous space by a set of regularly spaced points. Potential field anomalies can thus be calculated by our model even within a set of scattered points by interpolation. The method is very fast, allowing for proper modeling of potential field anomalies in real-time. Gradients or other field components can be calculated using the same approach by properly changing the wave vector factor.

However, as our method is based on Fourier calculation, it can be complicated by edge effects or aliasing problems. We have shown how these problems can be managed by increasing the Nyquist frequencies and enlarging the volume adopted for the calculations, with a balance between round-off errors and efficiency/speed of the method.

Application of this method to the magnetic anomaly of the Palinuro Seamount in southern Italy, indicated an average magnetization of 8 \( \mu \)A/m for this volcanic complex, with a subsequent joint interpretation of the residual magnetic anomaly and the bathymetry, in terms of local magnetization variations.

Appendix A: Fourier Transform of \( 1/|x| \)

The Fourier transform of \( 1/|x| \) exists in the space of the distributions, because this is a function that decreases more rapidly than a positive-degree polynomial function. The integral

\[
\int_{\mathbb{R}^3} \frac{1}{|x|} e^{-ik \cdot x} dx,
\]

(A1)

which defines the Fourier transform, is finite for any value of \( k \neq 0 \) because of the oscillatory behavior of the exponential function, which gives a finite result. This is similar to integrating the function \( \sin(kx)/x \) from \( -\infty \) to \( +\infty \), which is finite even if \( 1/x \) is not integrable in the same domain because it is logarithmically divergent. When \( k = 0 \), the integral in equation (A1) diverges. This is properly expressed in the form of the Fourier transform \( 4\pi/|k|^2 \), which diverges as \( k \to 0 \).

The result of equation (4) can be demonstrated in polar coordinates \((r, \theta, \phi)\) as follows:

\[
\mathcal{F}_{3-D} \left[ \frac{1}{|x|} \right] = \frac{4\pi}{|k|^2} \int_{0}^{2\pi} \int_{0}^{\pi} \frac{1}{r} \sin \theta e^{-ik \cos \theta} \sin \theta d\theta d\phi,
\]

(A2)

because the integral is rotation-invariant, and thus can be computed for the particular case where \( k \) is aligned with the \( z \) axis. We can integrate over \( \theta \) and \( \phi \); we obtain

\[
\mathcal{F}_{3-D} \left[ \frac{1}{|x|} \right] = \frac{2\pi}{-ik} \int_{0}^{+\infty} e^{-(\mu + i\gamma)r} - e^{-(\mu - i\gamma)r} dr,
\]

(A3)
where we introduced the factor $\mu$ to make the integral convergent. The exact result for the integral of equation (4) is obtained when $\mu \to 0$, as follows:

$$
\mathcal{F}_{3-D}\left[ \frac{1}{|k|} \right] = \frac{2\pi}{-ik} \lim_{\mu \to 0} \left[ \frac{1}{\mu + ik} - \frac{1}{\mu - ik} \right] = \frac{4\pi}{k^2}. \quad (A4)
$$

### Appendix B: Derivation of Parker’s Gravity Equations

[59] To demonstrate the generality and validity of our equations, we will show how we can recover the particular well-known case of Parker [1972]. We have a density layer contained within two undulating surfaces $z_1(x,y)$ and $z_2(x,y)$, and thus the density is given by

$$
\rho(x) = \rho_0(x,y), \quad (B1)
$$

where $z_1(x,y) < z < z_2(x,y)$, and vanishes outside. Let us start from the gravity equation (7):

$$
\mathcal{F}_{3-D}[\Delta g_z] = \frac{i4\pi G \rho_0}{k_x^2 + k_y^2 + k_z^2} \times \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \rho_0(x,y)e^{-i(k_x x + k_y y)}dzdy \int_{z_1(x,y)}^{z_2(x,y)} e^{-ik_z z} dz. \quad (B2)
$$

[60] The integral along $z$ is easily calculated:

$$
\mathcal{F}_{3-D}[\Delta g_z] = \frac{i4\pi G \rho_0}{k_x^2 + k_y^2 + k_z^2} \times \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dx dy \rho_0(x,y)e^{-i(k_x x + k_y y)} \times \frac{1}{k_z} \left[ e^{-ik_z z_2(x,y)} - e^{-ik_z z_1(x,y)} \right]. \quad (B3)
$$

and thus we obtain the final expression:

$$
\mathcal{F}_{3-D}[\Delta g_z] = \frac{-4\pi G}{k_x^2 + k_y^2 + k_z^2} \times \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dx dy \left[ e^{-ik_z z_2(x,y)} - e^{-ik_z z_1(x,y)} \right] \rho_0(x,y)e^{-i(k_x x + k_y y)}. \quad (B4)
$$

[61] If we are interested in the 2-D Fourier transform of the magnetic anomaly at given altitude $z_0$, we must perform an inverse Fourier transform along the $k_z$ direction:

$$
\mathcal{F}_{2-D}[\Delta g_z] = -\frac{4\pi G}{k_x^2 + k_y^2 + k_z^2} \times \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dx dy \rho_0(x,y)e^{-i(k_x x + k_y y)} \times \frac{dk_z}{2\pi} e^{ik_z z_0} \left[ e^{-ik_z z_2(x,y)} - e^{-ik_z z_1(x,y)} \right]. \quad (B5)
$$

[62] Adopting the similar integration path of Figure 2 in the complex plane, we find again the internal pole when $k_z = -ik_h = i\sqrt{k_x^2 + k_y^2}$. No other poles are present, and thus the Cauchy theorem, with steps similar to those of section 4.2, states that

$$
\mathcal{F}_{2-D}[\Delta g_z] = -\frac{2\pi G}{k_h} e^{ik_h z_0} \times \mathcal{F}_{2-D}\left[ \rho_0(x,y) \left( e^{-ik_z z_2(x,y)} - e^{-ik_z z_1(x,y)} \right) \right]. \quad (B6)
$$

and by substituting the exponential term with its infinite Taylor series, we obtain

$$
\mathcal{F}_{2-D}[\Delta g_z] = 2\pi Ge^{ik_h z_0} \times \mathcal{F}_{2-D}\left[ \rho_0(x,y) \left( e^{-ik_z z_2(x,y)} - e^{-ik_z z_1(x,y)} \right) \right]. \quad (B7)
$$

which is consistent with the expression of Parker [1972], in the form of equation 11.41 of Blakely [1995]. With similar steps it is possible to obtain the corresponding equation for the magnetic anomaly.

### Acknowledgments

We thank AE Stefan Maus and the reviewers Richard J. Blakely and Jeffrey D. Phillips for their comments, which greatly improved the quality of this paper. The data acquisition was possible because of CONAGEM, an agreement between INGV and the Hydrographic Institute of Italian Navy. We also thank Captains M. Demarte and G. Lanuto, Officer C. Marchi, and the crew of the R/V Aretusa of the Hydrographic Institute of the Italian Navy for their logistical support during the cruise. We thank Mr. Marani as well, who kindly provided us with the bathymetric grid of the Tyrrhenian Sea.

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