Connectivity of Ad Hoc Networks with Link Asymmetries Induced by Shadowing

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***Abstract*—The aim of this letter is to determine the minimum node density to achieve a connected large-scale ad hoc network, where every node has the same transmitting and receiving capabilities. Due to the log-normal shadowing, links are unidirectional in general. Contrary to the prevailing opinion, we argue that such asymmetries result into a "reduced" connectivity graph, which, from the point of view of MAC and routing protocols, is to be considered the true or effective connectivity graph. Accordingly, we derive a new formula for the connection probability between two nodes in order to compute global connectivity. Finally, theoretical findings, borrowed from random graphs theory, are compared to numerical simulation results in synthetic wireless network scenarios.**

***Index Terms*—Ad hoc networks, multi-hop networks, connectivity, log-normal shadowing, random graphs theory.**

I. Introduction

An ad hoc wireless multi-hop network is often represented as a connectivity graph $G(V, E)$, where V is the node set and E is the link set. In such a network, the presence of an edge between two vertices is often modelled as a Bernoulli random variable (r.v.), as a consequence of the random position of nodes in the space. Let $p_\ell(r)$ be the *link probability*, i.e. the probability that a link exists between two nodes at (euclidean) distance r. In the simplest case, it is modelled as a monotone step function of r, i.e. $p_\ell(r) = \mathbf{1}_{r \leq r_0}$, where r_0 is the *transmission range*; the resulting random graph is known as *Geometric Random Graph* (GRG). To account also for the irregularities of the losses along the radio channels, adds some complexity to the characterization of link probability. Here, we consider the case of *log-normal shadowing*, i.e. the path loss is assumed to be a log-normal r.v. [12].

Usually, the connectivity graph is assumed to be undirected, which corresponds to consider links as bi-directional and symmetrical. More generally, they are asymmetrical to the point that, in some cases, as demonstrated by recent measurements on wireless networks [4], a communication is allowed from one node u to another v while being hindered on the converse. This lack of symmetry may adversely affect the performance of MAC and routing protocols, which typically do not make provision for asymmetrical links [11].

In this paper, we assume that the link asymmetries are only due to radio impairments, hence neglecting other sources of dissimilarities, such as uneven power assignment and interference (low traffic regime). We also assume that the link characteristics are stationary, thus ignoring possible time varying behaviors.

Under these assumptions, we are able to determine the minimum node density that can guarantee a large-scale connected network at a given probability level. Since most of available results on the connectivity of GRG refer to undirected graphs [6], [7], [8], [1], we provide some new formulas for the global connection probability of ad hoc networks (directed or not), and we compare said theoretical findings with numerical results obtained in simulated environments.

II. Definitions

Inside the log-normal shadowing model, the path loss (in deciBel) is a normal r.v. $\beta(r)$ whose expectation depends on the length r, according to

$$\beta(r) = \beta_0 + 10\alpha \log(r/d_0) + X, \tag{1}$$

where β_0 and d_0 are constants, α is the attenuation coefficient, $X \sim \mathcal{N}(0, \sigma^2)$ and $\log(\cdot)$ is the base 10 logarithm. Without loss of generality, henceforth $\beta_0 = 0$ dB and $d_0 = 1$ m. Since we assume identical hardwares, equal transmitting powers and homogeneous path loss characteristics, two nodes at distance r are connected if $\beta(r) \leq \beta_{\rm th}$, where $\beta_{\rm th}$ is the *threshold path loss*. Equivalently, we can use the transmission range r_0, defined after (1) as $r_0 \triangleq 10^{\frac{\beta_{\rm th}}{10\alpha}}$, which is the maximum distance between two nodes at which the receiving power is greater than a receiving power threshold, in the absence of log-normal shadowing effect, i.e. for $\sigma^2 = 0$.

It is worth noting that, since all nodes have the same $\beta_{\rm th}$, we implicitly assume there is no power control mechanism deployed in the network. For further details on this mechanism, we refer to [1], [13].

Let $G_u(V, E_u)$, or simply G_u, be an undirected graph and $G_d(V, E_d)$, or simply G_d, a directed graph or *digraph*. They model the connectivity graph of an ad hoc network with symmetrical and asymmetrical links, respectively. In particular, in the former model the log-normal shadowing is supposed to act symmetrically on both directions[1]. A *reduced* graph is thereof introduced, referred to as $G_r(V, E_r)$, or simply G_r: It has an undirected link between nodes $z, v \in V$ iff G_d has arcs in both directions between z and v. G_u is said to be *connected* if, for every node pair, there is at least one sequence of edges (path) connecting them. If this applies to the digraph G_d, by

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[1]An alternative model for G_u is introduced in [2], where G_u is obtained from G_d, and leads to similar results.

considering also arc directions, we say that G_d is *strongly connected*.

Our approach to characterise the connectivity properties of ad hoc networks subject to log-normal shadowing is based on Random Graphs (RGs) theory. By considering the link probabilities p_u, p_d and p_r, for connection graphs G_u, G_d and G_r respectively, the *global connection probabilities* of graphs are introduced: Π_u, respectively Π_r, is the probability that G_u, respectively G_r, is connected; Π_d is the probability that G_d is strongly connected.

Let N be the number of the nodes, and suppose that they are uniformly distributed over a square surface with finite area $L \times L$ so that the node density is $\rho = \frac{N}{L^2}$. In order to avoid border effects, we assume that the network surface is a torus surface, which can be formed by connecting both pairs of opposite edges together. This means that the distance between two points $x = (x_1, x_2)$ and $y = (y_1, y_2)$ is

$$d(x,y) = \{[|x_1 - y_1| \wedge (L - |x_1 - y_1|)]^2 + \\ + [|x_2 - y_2| \wedge (L - |x_2 - y_2|)]^2\}^{\frac{1}{2}}, \qquad (2)$$

where $(a \wedge b) \triangleq \min\{a, b\}$.

III. Connectivity of Networks Subject to Shadowing

In the rest of the paper, connectivity graphs are RGs modeled as GRGs where p_ℓ is obtained by averaging, with respect to r, the probability that two generic nodes at a distance r are connected, namely $p_\ell(r) = Pr\{\beta(r) \leq \beta_{\rm th}\}$. Hence, if r is distributed according to a pdf $f(r)$

$$p_\ell = \int_0^{+\infty} f(r) Pr\{X \leq 10\alpha \log(r_0/r)\} dr = \\ = \frac{1}{2} \int_0^{+\infty} f(r) \,\text{erfc}\left[\frac{10\alpha \log(r/r_0)}{\sigma\sqrt{2}}\right] dr, \qquad (3)$$

where erfc$(\cdot)$ is the complementary error function. As derived in [10], in the toroidal surface defined in Sect. II it is

$$f(r) = \begin{cases} \frac{2\pi r}{L^2} & 0 \leq r \leq L/2, \\ \frac{2\pi r}{L^2} - \frac{8r}{L^2} \arccos\left(\frac{L}{2r}\right) & L/2 \leq r \leq L\sqrt{2}/2, \\ 0 & \text{elsewhere.} \end{cases} \qquad (4)$$

Consider a connectivity graph represented as an undirected RG G_u. In order to find the relationship between node density and global connectivity, we use the property, proved by Erdős and Rényi [5], [3], that, given a constant $c \in \mathbb{R}$, almost every undirected random graph is connected with probability

$$\Pi_u \overset{N\to\infty}{\longrightarrow} e^{-e^{-c}}, \qquad (5)$$

provided the link probability behaves as

$$p_u = \frac{\ln(N) + c + o(1)}{N}. \qquad (6)$$

By eliminating c in (5) and (6) and adopting p_ℓ provided by (3) in place of p_u, we conclude that an undirected GRG is connected with probability Π_u, if node density ρ is such that $\frac{\ln(\rho L^2) - \ln(-\ln \Pi_u)}{\rho L^2} = p_\ell$. Hence, for large values of ρL^2,

$$\Pi_u = e^{-\rho L^2 e^{-\rho L^2 p_\ell}}. \qquad (7)$$

To account for link asymmetries, we adopt a directed GRG G_d as the connectivity graph. By exploiting similar arguments and using Palásti's results [9] (see also [3] for further readings), it is readily seen that, given a constant $\nu \in \mathbb{R}$, a directed RG is strongly connected with probability

$$\Pi_d \overset{N\to\infty}{\longrightarrow} e^{-2e^{-\nu}}, \qquad (8)$$

if the probability that a directed link between two nodes exists is

$$p_d = \frac{\ln(N) + \nu + o(1)}{N}. \qquad (9)$$

Again, by eliminating ν in (8) and (9) and using p_ℓ provided by (3) in place of p_d, we can compute Π_d as a function of ρ as follows

$$\Pi_d = e^{-2\rho L^2 e^{-\rho L^2 p_\ell}}. \qquad (10)$$

Since MAC and routing protocols typically do not make provision for asymmetrical links [11], it may become necessary to model the *real* network connectivity graph as a "reduced" graph G_r, in which an undirected link is present between each node pair if directed links in both directions exist. By assuming that such link probabilities factorise, it is straightforward to realize that

$$p_r = \int_0^{+\infty} f(x) \left(Pr\{X \leq 10\alpha \log(r_0/x)\}\right)^2 dx = \\ = \frac{1}{4} \int_0^{L\sqrt{2}/2} f(x) \,\text{erfc}^2\left(\frac{10\alpha \log(x/r_0)}{\sqrt{2}\sigma}\right) dx. \qquad (11)$$

By similar arguments as above, Π_r can then be expressed in terms of ρ as

$$\Pi_r = e^{-\rho L^2 e^{-\rho L^2 p_r}}. \qquad (12)$$

IV. Numerical results

In order to test the above findings, numerical simulations have been performed on synthetic networks. N isolated nodes were spread uniformly over a torus surface derived from a square with length L on each edge ($L = 400$ m), as described in Sect II. The following system parameters were assumed: $\beta_{\rm th} = 50$ dB, $\alpha = 3$, $\sigma = 3$ and 8. Thereafter a directed or undirected arc – depending on the graph type – was added from one node to another whenever the path loss (randomly generated according to the assumed probabilistic model) turned out lower than $\beta_{\rm th}$. We then checked whether each resulting graph G_u, G_d and G_r was connected or not. Finally, the global connection probabilities were evaluated by means of Monte Carlo simulations (1000 trials).

In Fig. 1 the simulation results are compared to Π_u, Π_d and Π_r, computed from (7), (10) and (12), having obtained p_ℓ in (3) and p_r in (11) by numerical integration procedures. The agreement between numerical and theoretical results is quite satisfactory and improves as σ increases: This is in keeping with the general acknowledgment that the higher σ the better

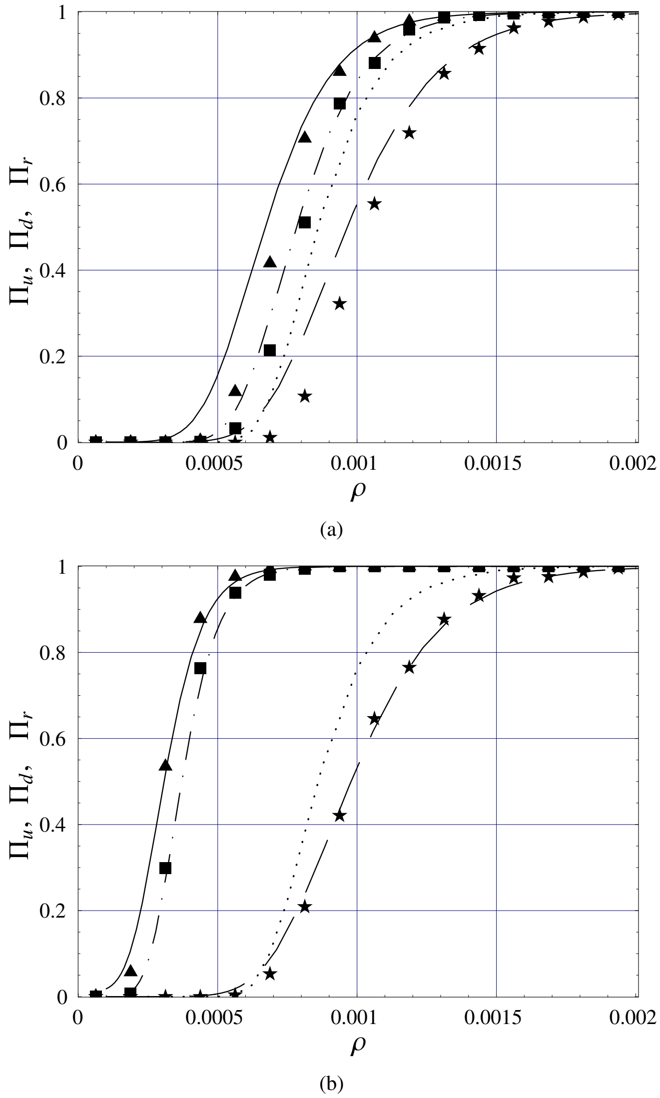

Fig. 1. Global connection probabilities Π_u, Π_d and Π_r vs node density ρ of ad hoc networks in different log-normal shadowing environments. Both theoretical Π_u (straight line), Π_d (dash-dotted) and Π_r (dashed) and numerical results (triangles, boxes and stars, respectively) are reported, together with global connection probability in absence of the shadowing effect (dotted line). The parameter values adopted are: $L = 400$ m, $\beta_{\rm th} = 50$ dB, $\alpha = 3$, *(a)* $\sigma = 3$ dB and *(b)* $\sigma = 8$ dB.

the ad hoc networks are modelled as the classical RGs (e.g. [1]). It is worth noting that this is the most frequent case in typical wireless communication scenarios [12], where $\sigma/\alpha > 1$.

From a Layer-2 and Layer-3 protocols perspective, G_r is to be considered the "true" connectivity graph for ad hoc networks. Hence, Fig. 1 shows that, to achieve a given a global connection probability level, a minimum node density ρ, also referred to as the *critical node density*, is required which is significantly larger than that predicted by adopting G_u or G_d connectivity graph models. Thus, we suggest that the critical node density shall be conservatively determined according to (12).

Furthermore, if σ increases, Π_u and Π_d also increase markedly, whereas Π_r is relatively unaffected. In other words, we can not expect much advantage from shadowing effects toward connectivity of real wireless multi-hop networks.

V. Concluding Remarks

Motivated by recent measurements on real testbeds, we addressed the impact of unidirectional links on the connectivity properties of an ad hoc network and introduced some new formulas to compute global connection probability. From the point of view of MAC and routing protocols, we argued that *i)* the connectivity properties of networks should be predicted by considering link direction and, consequently, network design tools should be revised accordingly and *ii)* the log-normal shadowing model for path loss (in particular its standard deviation) does not improve such connectivity properties. This study can be applied also to wireless sensor networks, where the approximation of zero interference holds due to the low duty cycle of the sensors.

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