Structures and chromaticity of some extremal 3-colourable graphs

F.M. Dong, K.M. Koh *

Department of Mathematics, National University of Singapore, Lower Kent Ridge Road, Singapore 119260, Singapore

Received 30 January 1998; revised 4 November 1998; accepted 16 November 1998

Abstract

Given a graph $G$ and a positive integer $r$, let $s_r(G) = P(G, r)/r!$. Thus $\chi(G) = r$ and $s_r(G) = 1$ iff $G$ is uniquely $r$-colourable. It is known that if $G$ is uniquely 3-colourable, then $e(G) \geq 2v(G) - 3$. In this paper, we show that if $G$ is a 3-colourable connected graph with $e(G) = 2v(G) - k$ where $k \geq 4$, then $s_3(G) \geq 2^{k-3}$; and if, further, $G$ is 2-connected and $s_3(G) = 2^{k-3}$, then $t(G) \leq v(G) - k$ where $t(G)$ denotes the number of $C_3$'s in $G$. We proceed to determine the structures of all 3-colourable 2-connected graphs $G$ with $e(G) = 2v(G) - k$, $s_3(G) = 2^{k-3}$ and $t(G) = v(G) - k$. By applying this structural result, we finally study the chromaticity of such graphs and produce new chromatically equivalent classes. © 1999 Elsevier Science B.V. All rights reserved.

Keywords: Graph; 3-colourable; Chromatic polynomial

1. Introduction

Given a (simple) graph $G$ with vertex set $V(G)$ and edge set $E(G)$, we denote by $v(G)$, $e(G)$, $t(G)$, $\chi(G)$ and $P(G, \lambda)$, respectively, its order, size, number of triangles, chromatic number and chromatic polynomial. Let $\mathcal{N}$ be the set of natural numbers. For $k \in \mathcal{N}$, define $s_k(G) = P(G, k)/k!$. Thus, $s_k(G) = 0$ iff $k < \chi(G)$; and for $k = \chi(G)$, $s_k(G)$ is the number of $k$-colourings of $G$ which produce different colour classes. In particular, $s_k(G) = 1$ where $k = \chi(G)$ iff $G$ is uniquely $k$-colourable.

Two graphs $G$ and $H$ are said to be $\chi$-equivalent, written $G \sim H$, if $P(G, \lambda) = P(H, \lambda)$. Obviously, the relation ‘$\sim$’ is an equivalence relation on the class of graphs. Any equivalence class under ‘$\sim$’ is called a $\chi$-equivalence class. We denote by $[G]$ the equivalence class determined by $G$. A family $\mathcal{F}$ of graphs is said to be $\chi$-closed if $[G] \subseteq \mathcal{F}$ for any $G \in \mathcal{F}$; i.e., $\mathcal{F}$ is a union of some $\chi$-equivalence classes. The

* Corresponding author.
E-mail address: matkohkm@nus.edu.sg (K.M. Koh)
determination or classification of \(\chi\)-equivalence classes of a family of graphs is referred to as the chromaticity of the family of graphs (see, for instance [4,5]).

We shall focus in this paper on 3-colourable graphs. In [1,2], we studied the structures and chromaticity of certain extremal uniquely 3-colourable graphs. In this paper, we shall investigate the structures and chromaticity of some extremal non-uniquely 3-colourable graphs. If \(G\) is uniquely 3-colourable, then as shown in Lemma 3.1 below, \(e(G) \geq 2v(G) - 3\). Now, let \(G\) be a 3-colourable connected graph with \(e(G) = 2v(G) - k\) where \(k \geq 4\). Then \(G\) is non-uniquely 3-colourable, i.e., \(s_3(G) > 1\). Is there any sharp lower bound for \(s_3(G)\)? The first objective of this paper is to establish the following:

(i) \(s_3(G) \geq 2^{k-3}\);
(ii) if \(s_3(G) = 2^{k-3}\), then \(t(G) \leq v(G) - k + 1\);
(iii) if \(G\) is 2-connected and \(s_3(G) = 2^{k-3}\), then \(t(G) = v(G) - k\).

Let \(\mathcal{G}_k\) be the family of 3-colourable 2-connected graphs \(G\) such that \(e(G) = 2v(G) - k\), \(s_3(G) = 2^{k-3}\) and \(t(G) = v(G) - k\), where \(k \geq 4\). Our second objective is to investigate the structures of graphs in \(\mathcal{G}_k\). Finally, these findings are applied to study the chromaticity of certain non-uniquely 3-colourable graphs. We show that the family \(\mathcal{G}_k\) is \(\chi\)-closed and discover also a subfamily of \(\mathcal{G}_k\) which forms itself a \(\chi\)-equivalent class, for each \(k \geq 4\).

2. 2-trees

In this section, we shall focus on a special class of graphs, called 2-trees, which will play an important role in the determination of the structures of 3-colourable graphs in the remaining sections.

For a graph \(G\) and a vertex \(x\) in \(G\), let \(N(x)\) denote the set of vertices adjacent to \(x\), \(d(x) = |N(x)|\) and \(G - x\) the subgraph of \(G\) obtained from \(G\) by deleting \(x\). A vertex \(x\) is called a simplicial vertex of \(G\) if either \(d(x) = 0\) or \(N(x)\) is a clique of \(G\). For integer \(q \geq 1\), the \(q\)-tree of the least order is \(K_q\). A graph \(G\) of order \(n\) \((n > q)\) is a \(q\)-tree if \(G\) contains a simplicial vertex \(x\) such that \(d(x) = q\) and \(G - x\) is a \(q\)-tree.

We shall now provide some useful properties of 2-trees. Let \(G\) be a 2-tree. It is clear that \(v(G) \geq 2\) by definition. If \(v(G) = 2\), then \(G \cong K_2\); and if \(v(G) = 3\), then \(G \cong K_3\). In general, we have the following result, which can be proved by induction.

**Lemma 2.1.** If \(G\) is a 2-tree with \(v(G) \geq 3\), then

(i) \(G\) is 2-connected;
(ii) any edge of \(G\) is contained in some triangle;
(iii) \(G\) is uniquely 3-colourable; and
(iv) \(e(G) = 2v(G) - 3\) and \(t(G) = v(G) - 2\).

For \(e \in E(G)\), we shall denote by \(G - e\) the subgraph of \(G\) obtained from \(G\) by deleting \(e\).
Lemma 2.2. Let $G$ be a 2-tree with $v(G) \geq 3$ and $uv$ be an edge in $G$.

(i) If $uv$ is contained in exactly one triangle of $G$, then $G - uv$ is connected with two blocks which are 2-trees as shown in Fig. 1(a).

(ii) If $uv$ is contained in exactly two triangles $xuv$ and $yuv$, then $G - uv$ is a connected graph having a structure as shown in Fig. 1(b), where all $G_i$’s are 2-trees, and $ux, uy, vx, vy$ are edges of $G$.

Proof. We shall prove the lemma by induction on $v(G)$. The result is trivial if $v(G) \leq 4$. Now assume that $v(G) \geq 5$. By definition, $G$ contains a simplicial vertex $w$ with $d(w) = 2$ such that $G - w$ is a 2-tree. By the induction hypothesis, the results hold for $G - w$. We split our discussion into some cases.

If $w = u$ or $w = v$, then $uv$ is contained in exactly one triangle of $G$, and it is easy to see that $G - uv$ has exactly two blocks, respectively isomorphic to $K_2$ and $G - w$.

If $N(w) = \{u, v\}$, then $uv$ is contained in at least two triangles of $G$. Only when $uv$ is contained in exactly one triangle of $G - w$, $uv$ is contained in exactly two triangles of $G$. But in this case, $G - w - uv$ has exactly two blocks which are 2-trees, and hence $G - uv$ is a graph having a structure as shown in Fig. 1.

Otherwise, $w$ is a simplicial vertex of degree 2 in $G - uv$. As the results hold for $G - w$, it is obvious that the results hold also for $G$. $\square$

3. Some extremal 3-colourable graphs

We shall begin with the following observation which gives a sharp lower bound for $e(G)$, where $G$ is a uniquely $r$-colourable graph. For $A \subseteq V(G)$, we shall denote by $[A]$ the subgraph of $G$ induced by $A$, and write $e[A]$ for $e([A])$.

Lemma 3.1. Let $G$ be a uniquely $r$-colourable graph, where $r \geq 2$. Then

$$e(G) \geq (r - 1)v(G) - \left(\begin{array}{l}r \\ 2 \end{array}\right).$$
Proof. Let $A_1, A_2, \ldots, A_r$ be the colour classes of a $r$-colouring of $G$. Since $G$ is uniquely $r$-colourable, $[A_i \cup A_j]$ is connected for every pair $i$ and $j$ with $1 \leq i < j \leq r$, by Theorem 12.16 in [3]. Thus $e[A_i \cup A_j] \geq |A_i| + |A_j| - 1$ for $1 \leq i < j \leq r$, and so

$$e(G) = \sum_{1 \leq i < j \leq r} e[A_i \cup A_j] \geq \sum_{1 \leq i < j \leq r} (|A_i| + |A_j| - 1) = (r - 1) \sum_{1 \leq i \leq r} |A_i| - \binom{r}{2} = (r - 1)v(G) - \binom{r}{2}.$$ 

By Lemma 3.1, if $G$ is uniquely 3-colourable, then $e(G) \geq 2v(G) - 3$. Dong and Koh [2] showed that if $G$ is a uniquely 3-colourable graph with $e(G) = 2v(G) - 3$, then $t(G) \leq v(G) - 2$, and also characterized such graphs $G$ when $v(G) - 3 \leq t(G) \leq v(G) - 2$ as stated below.

**Theorem 3.1.** Let $G$ be a uniquely 3-colourable graph with $e(G) = 2v(G) - 3$. Then $t(G) \leq v(G) - 2$. Further,

(i) if $t(G) = v(G) - 2$, then $G$ is a 2-tree;
(ii) if $t(G) = v(G) - 3$, then $G$ is a graph having a structure as shown in Fig. 2, where all $G_i$’s are 2-trees.

It follows from Lemma 3.1 that if $e(G) < 2v(G) - 3$, then $G$ cannot be uniquely 3-colourable, i.e., $s_3(G) > 1$. For a 3-colourable connected graph $G$ with $e(G) = 2v(G) - k$, where $k \geq 3$, we shall give below a sharp lower bound for $s_3(G)$ in terms of $k$. 

![Fig. 2.](image)
Theorem 3.2. Let $G$ be a 3-colourable connected graph with $e(G) = 2v(G) - k$, where $k \geq 3$. Then

(i) $s_3(G) \geq 2^{k-3};$
(ii) when $s_3(G) = 2^{k-3}$, $t(G) \leq v(G) - k + 1.$

To prove Theorem 3.2, we shall make use of the following result, known as the Fundamental Reduction Theorem [6] for chromatic polynomials (see also Theorem 1 in [4]).

Lemma 3.2. Let $G$ be a graph and $e = uw \in E(G)$. Then

$$P(G, \lambda) = P(G - e, \lambda) - P(G \cdot e, \lambda),$$

where $G \cdot e$ is the graph obtained from $G$ by contracting $u$ and $v$ and removing all but one of the multiple edges if they arise.

For $u, v \in V(G)$, if $uv \in E(G)$, let $G + uv = G$; otherwise, let $G + uv$ denote the graph obtained from $G$ by adding a new edge $uv$.

Proof of Theorem 3.2. We shall prove the results by induction on $v(G)$ and $k$. It is obvious that $s_3(G) \geq 2^{k-3}$ when $v(G) \leq 4$ or $k = 3$. It is also clear that (ii) holds when $v(G) \leq 4$. If $k = 3$ and $s_3(G) = 1$, we have $t(G) \leq v(G) - 2$ by Theorem 3.1. Now assume that $k \geq 4$ and $v(G) \geq 5$. Suppose that the results hold for any 3-colourable connected graph $H$ with either $e(H) = 2v(H) - l$ where $l < k$, or with $v(H) < v(G)$.

Let $A_1, A_2, A_3$ be the colour classes of $G$. Since $e(G) < 2v(G) - 3$, as shown in the proof of Lemma 3.1, one of $[A_1 \cup A_2]$’s (say $[A_1 \cup A_2]$) is disconnected. As $G$ is connected, there are vertices $u, v$ and $w$ in $G$ such that $u, v \in N(w)$ and that $u$ and $v$ are in different components of $[A_1 \cup A_2]$. Of course, $uw \notin E(G)$. Thus both $G + uv$ and $G \cdot uw$ are connected and 3-colourable. By Lemma 3.2, we have

$$P(G, \lambda) = P(G + uv, \lambda) + P(G \cdot uw, \lambda)$$

and, in particular,

$$s_3(G) = s_3(G + uv) + s_3(G \cdot uw).$$

Consider the graph $G + uv$. Since $e(G + uv) = e(G) + 1 = 2v(G) - k + 1 = 2v(G + uv) - (k - 1),$ we have $s_3(G + uv) \geq 2^{k-4}$ by the induction hypothesis. For the graph $G \cdot uw$, we have

$$e(G \cdot uw) \leq e(G) - 1 = 2v(G) - k - 1 = 2v(G + uv) - (k - 1).$$

Let $k' = 2v(G + uv) - e(G \cdot uw)$. Then $k' \geq k - 1$. As $v(G \cdot uw) < v(G)$, by the induction hypothesis, we have $s_3(G \cdot uw) \geq 2^{k-4} \geq 2^{k-4}$. Hence

$$s_3(G) = s_3(G + uv) + s_3(G \cdot uw) \geq 2^{k-4} + 2^{k-4} = 2^{k-3}.$$

Suppose now that $s_3(G) = 2^{k-3}$. Then $s_3(G + uv) = s_3(G \cdot uw) = 2^{k-4}$. Since $e(G + uw) = 2v(G) - k + 1$, we may apply induction on $G + uv$ and obtain

$$t(G + uv) \leq v(G + uv) - (k - 1) + 1 \leq v(G) - k + 2.$$
As $uvw$ is a triangle in $G + uv$ but not in $G$, $t(G) \leq t(G + uv) - 1 \leq v(G) - k + 1$. The proof is thus complete. □

Assume further that $G$ is 2-connected. We shall now establish a tighter upper bound for $t(G)$ and determine the structure of $G$ when this bound for $t(G)$ is attained.

**Theorem 3.3.** Let $G$ be a 3-colourable 2-connected graph with $e(G) = 2v(G) - k$ and $s_3(G) = 2^{k-3}$, where $k \geq 4$. Then

(i) $t(G) \leq v(G) - k$ and

(ii) when $t(G) = v(G) - k$, $G$ has a structure as shown in Fig. 3, where $G_i$ is a 2-tree for $i = 1, 2, \ldots, k$.

**Proof.** We shall prove the theorem by induction on $k$. Assume that $k = 4$. Then $s_3(G) = 2$. Let $A_1, A_2$ and $A_3$ be the three colour classes of a 3-colouring of $G$. As $e(G) < v(G) - 3$, some $[A_1 \cup A_2]$ (say $[A_1 \cup A_2]$) is disconnected. Since $G$ is connected, there exist a vertex $w \in A_3$ and two vertices $u, v$ in $N(w)$ which are in different components of $[A_1 \cup A_2]$. As $s_3(G) = s_3(G + uv) + s_3(G \cdot uv)$ and both $G + uv$ and $G \cdot uv$ are 3-colourable, we have $s_3(G + uv) = s_3(G \cdot uv) = 1$. Since $e(G + uv) = 2v(G + uv) - 3$, by Theorem 3.1, $t(G + uv) \leq v(G + uv) - 2$.

**Case 1:** $t(G + uv) = v(G + uv) - 2$.

By Theorem 3.1, $G + uv$ is a 2-tree. Since $G$ is 2-connected, $uv$ is contained in at least 2 triangles of $G + uv$ by Lemma 2.2. This implies that $t(G) \leq v(G) - 4$. If $t(G) = v(G) - 4$, then $uv$ is contained in exactly two triangles of $G + uv$, and by Lemma 2.2 again, $G$ has a structure as shown in Fig. 3 with $k = 4$.

**Case 2:** $t(G + uv) \leq v(G + uv) - 3$.

As $uvw$ is a triangle in $G + uv$, $t(G) \leq v(G) - 4$. Suppose that $t(G) = v(G) - 4$. Then $t(G + uv) = v(G + uv) - 3$ and $uv$ is contained in exactly one triangle of $G$. By
Theorem 3.1, \( G + uv \) has its structure shown in Fig. 2 with three 2-trees \( G_1, G_2 \) and \( G_3 \). Assume \( uv \in E(G_1) \). By Lemma 2.2, \( G_1 - uv \) is a connected graph with two blocks of 2-trees. Hence \( G \) has a structure as shown in Fig. 3 with \( k = 4 \).

We have thus proved the theorem for \( k = 4 \). Now assume that the theorem holds for all \( k < k' \), where \( k' \geq 5 \). Let \( A_1, A_2, A_3 \) be the three colour classes of a 3-colouring of \( G \). One of \([A_i \cup A_j]\)'s (say \([A_1 \cup A_2]\)) is disconnected.

As \( G \) is connected, there exist a vertex \( w \in A_3 \) and \( u, v \in N(w) \) such that \( u \) and \( v \) are in different components of \([A_1 \cup A_2]\). Then both \( G + uv \) and \( G \cdot uv \) are 3-colourable, \( G + uv \) is 2-connected and \( G \cdot uv \) is connected. Observe that

\[
e(G + uv) = e(G) + 1 = 2v(G) - k + 1 = 2v(G + uv) - (k - 1)
\]
and

\[
e(G \cdot uv) \leq e(G) - 1 = 2v(G) - k - 1 = 2v(G \cdot uv) - (k - 1).
\]

By Theorem 3.2, we have \( s_3(G + uv) \geq 2^{k-4} \) and \( s_3(G \cdot uv) \geq 2^{k-4} \). Since \( uv \notin E(G) \), \( s_3(G) = s_3(G + uv) + s_3(G \cdot uv) \). As \( s_3(G) = 2^{k-3} \), we have \( s_3(G + uv) = 2^{k-4} \). By the induction hypothesis, \( t(G + uv) \leq v(G + uv) - (k - 1) \). Thus \( t(G) \leq t(G + uv) - 1 \leq v(G + uv) - (k - 1) - 1 = v(G) - k \). This proves (i).

Now assume that \( s_3(G) = 2^{k-3} \) and \( t(G) = v(G) - k \). Then \( s_3(G + uv) = 2^{k-4} \). Since \( t(G) \leq t(G + uv) - 1 \) and \( t(G + uv) \leq v(G + uv) - (k - 1) \), we have \( t(G + uv) = v(G + uv) - (k - 1) \). By the induction hypothesis, \( G + uv \) has a structure as shown in Fig. 4, where all \( G_i' \)'s are 2-trees.

Without loss of generality, assume that \( uv \in E(G'_{k-1}) \). Since \( t(G) = t(G + uv) - 1 \), \( uv \) is contained in exactly one triangle of \( G \). Of course, this triangle is in \( G'_{k-1} \). By Lemma 2.2, \( G'_{k-1} - uv \) is a connected graph with 2 blocks of 2-trees. Let \( x_k \) be the cut-vertex of \( G'_{k-1} - uv \). Since \( G \) is 2-connected, \( x_k \) is not a cut-vertex of \( G \). Hence \( G \) has its structure shown in Fig. 3. This proves (ii).
4. Chromaticity of non-uniquely 3-colourable graphs

In this section, we shall apply the structural results obtained in Section 3 to study the chromaticity of certain non-uniquely 3-colourable graphs.

Before we proceed, let us consider the following question related to Theorem 3.3(ii). Suppose that $G$ is a graph having its structure shown in Fig. 3, where $G_i$ is a 2-tree for each $i = 1, 2, \ldots, k$. Must $s_3(G)$ be equal to $2^{k-3}$? We shall answer this question and shall find the answer useful to serve our purpose.

A subset $A$ of $V(G)$ is called a cut-set of $G$ if the subgraph of $G$ obtained by removing all vertices in $A$ from $G$ contains more components than $G$.

**Lemma 4.1.** Let $H$ be a 3-colourable graph having a structure as shown in Fig. 5(a), where $\{x, y\}$ is a cut-set of $H$ and $E(H_1) \cap E(H_2) = \emptyset$. If $xy \in E(H_1)$, then

$$s_3(H) = s_3(H_1)s_3(H_2 + xy)$$

and if $xy \notin E(H)$, then

$$s_3(H) = s_3(H_1 + xy)s_3(H_2 + xy) + 2s_3(H_1 \cdot xy)s_3(H_2 \cdot xy).$$

**Proof.** (i) Assume that $xy \in E(H_1)$. Then by Zykov’s theorem ([7], see also Theorem 3 in [4]), we have

$$P(H, \lambda) = \frac{P(H_1, \lambda)P(H_2 + xy, \lambda)}{\lambda(\lambda - 1)}$$

and, in particular,

$$s_3(H) = \frac{P(H_1, 3)P(H_2 + xy, 3)}{3 \times 2 \times 3!} = s_3(H_1)s_3(H_2 + xy).$$

(ii) Assume that $xy \notin E(H)$. We have, by Lemma 3.2,

$$P(H, \lambda) = P(H + xy, \lambda) + P(H \cdot xy, \lambda)$$

$$= \frac{P(H_1 + xy, \lambda)P(H_2 + xy, \lambda)}{\lambda(\lambda - 1)} + \frac{P(H_1 \cdot xy, \lambda)P(H_2 \cdot xy, \lambda)}{\lambda}$$

and, in particular,

$$s_3(H) = s_3(H_1 + xy)s_3(H_2 + xy) + 2s_3(H_1 \cdot xy)s_3(H_2 \cdot xy).$$

**Corollary.** Let $H$ be a 3-colourable graph having its structure shown in Fig. 5(a), where $\{x, y\}$ is a cut-set of $H$ and $H_1$ is uniquely 3-colourable.

(i) If $x, y$ are in different colour classes of any 3-colouring of $H_1$, then $s_3(H) = s_3(H_2 + xy)$.

(ii) If $x, y$ are in the same colour class of any 3-colouring of $H_1$, then $s_3(H) = 2s_3(H_2 \cdot xy)$. \qed
Lemma 4.2. Let $G$ be a graph having a structure as shown in Fig. 3, where $G_i$ is a 2-tree for all $i = 1, 2, \ldots, k$ and $k \geq 3$. Let $r$ be the number of $G_i$'s such that both $x_i$ and $x_{i+1}$ are in the same colour class of any 3-colouring of $G_i$. Then

(i) $r \neq k - 1$ and
(ii) $s_3(G) = (2^{k-1} + (-1)^{k-r}2^r)/3$.

Proof. (i) Suppose that $x_i$ and $x_{i+1}$ are in the same colouring class for any $i = 1, 2, \ldots, k - 1$. Then for any 3-colouring of $G$, $x_k$ and $x_1$ are in the same colour class too. This shows that if $r \geq k - 1$, then $r = k$. Hence (i) holds.

(ii) If $k = 3$, then by the corollary to Lemma 4.1,

\[
s_3(G) = \begin{cases} 
1 & \text{if } r = 0, \\
2 & \text{if } r = 1, \\
4 & \text{if } r = 3. 
\end{cases}
\]

Thus (ii) holds when $k = 3$.

Now suppose that $k \geq 4$. If $r = 0$, then by the corollary to Lemma 4.1,

\[
s_3(G) = s_3(C_k).
\]

As $P(C_k, \lambda) = (\lambda - 1)((\lambda - 1)^{k-1} + (-1)^k)$, we have $s_3(C_k) = (2^{k-1} + (-1)^k)/3$. Assume that $r \geq 1$ and that $x_1$ and $x_2$ are in the same colour class of a 3-colouring of $G_1$. Then by the corollary to Lemma 4.1, $s_3(G) = 2s_3(G')$, where $G'$ is the graph obtained from $G$ by deleting all vertices in $G_1 \setminus \{x_1, x_2\}$ and identifying $x_1$ and $x_2$. Thus, by the induction hypothesis, we have

\[
s_3(G) = 2s_3(G') = 2 \cdot \frac{2^{k-2} + (-1)^{k-r}2^r-1}{3} = \frac{2^{k-1} + (-1)^{k-r}2^r-1}{3}.
\]

Corollary. Let $G$ be a graph having its structure shown in Fig. 3, where $G_i$ is a 2-tree for all $i = 1, 2, \ldots, k$ and $k \geq 3$. Then $s_3(G) \geq 2^{k-3}$, and $s_3(G) = 2^{k-3}$ if there are exactly 3 $G_i$'s such that $x_i$ and $x_{i+1}$ are in different colour classes of the 3-colouring of $G_i$. 

Fig. 5.
Now for $k \geq 4$, let $\mathcal{X}(k)$ be the family of graphs $G$ having a structures as shown in Fig. 3, where all $G_i$'s are 2-trees and there are exactly 3 $G_i$'s such that $x_i$ and $x_{i+1}$ are in different colour classes of any 3-colouring of $G_i$. We shall show that the family $\mathcal{X}(k)$ is $\chi$-closed.

**Theorem 4.1.** Let $G$ be a graph and $k \geq 4$. Then $G \in \mathcal{X}(k)$ iff $G$ is 2-connected, 3-colourable, $e(G) = 2v(G) - k$, $s_3(G) = 2^{k-1}$ and $t(G) = v(G) - k$.

**Proof.** The sufficiency follows directly by Theorem 3.3 and the corollary to Lemma 4.2.

We shall now prove the necessity. Let $G \in \mathcal{X}(k)$. Obviously, $G$ is 2-connected. By the definition of $\mathcal{X}(k)$, $G$ is 3-colourable and $s_3(G) = 2^{k-1}$. As $e(G) = 2v(G) - 3$ and $t(G) = v(G) - 2$, we have

$$e(G) = \sum_{1 \leq i \leq k} e(G_i) = \sum_{1 \leq i \leq k} (2v(G_i) - 3)$$

$$= 2 \sum_{1 \leq i \leq k} v(G_i) - 3k = 2(v(G) + k) - 3k = 2v(G) - k$$

and

$$t(G) = \sum_{1 \leq i \leq k} t(G_i) = \sum_{1 \leq i \leq k} (v(G_i) - 2) = v(G) - k.$$

This completes the proof. □

**Corollary.** For every $k \geq 4$, the family $\mathcal{X}(k)$ is $\chi$-closed.

Finally, we consider a subfamily of $\mathcal{X}(k)$. For $k \geq 4$, let $\mathcal{X}^+(k)$ be the family of graphs $G \in \mathcal{X}(k)$ having the structures as shown in Fig. 3, where three of the $G_i$'s are isomorphic to $K_2$ and the remaining $G_i$'s are isomorphic to the 2-tree of order 4, and $x_i$ and $x_{i+1}$ are simplicial vertices of $G_i$. For instance, the graph of Fig. 6 is a member of $\mathcal{X}^+(k)$. Our final aim is to show that the family $\mathcal{X}^+(k)$ forms itself a $\chi$-equivalent class.

**Lemma 4.3.** Let $G \in \mathcal{X}(k)$, where $k \geq 4$. Then $v(G) \geq 3k - 6$, and the equality holds iff $G \in \mathcal{X}^+(k)$.

**Proof.** The graph $G$ has a structure as shown in Fig. 3, where all $G_i$'s are 2-trees. By the definition of $\mathcal{X}(k)$, there are exactly $(k - 3)$ $G_i$'s such that both $x_i$ and $x_{i+1}$ are in the same colour class of any 3-colouring of $G_i$. For $1 \leq i \leq k$, if $x_i$ and $x_{i+1}$ are in the same colour class of any 3-colouring of $G_i$, then $v(G_i) \geq 4$, where equality holds iff $G_i$ is a 2-tree of order 4; otherwise, $v(G_i) \geq 2$, where equality holds iff $G_i \cong K_2$. The result thus follows. □

By Theorem 4.1, we have the following.
Lemma 4.4. Let $G$ be a 3-colourable graph and $k \geq 4$. Then $G \in \mathcal{X}^*(k)$ iff $G$ is 2-connected, $\nu(G) = 3k - 6$, $e(G) = 5k - 12$, $t(G) = 2k - 6$ and $s_3(G) = 2^{k-3}$.

Given $k \geq 4$, it is obvious that all graphs in $\mathcal{X}^*(k)$ have the same chromatic polynomial. In fact, for any $G \in \mathcal{X}^*(k)$,

$$P(G, \lambda) = (\lambda - 1)^k(\lambda - 2)^{2(k-3)} - 2^{k-3}(\lambda - 2)^{k-3}(\lambda - 1).$$

Now, by Lemma 4.4, we eventually arrive at the following.

Theorem 4.2. For any $k \geq 4$, the family $\mathcal{X}^*(k)$ is a $\chi$-equivalent class.

Acknowledgements

The authors would like to express their sincere thanks to the referees for their helpful comments.

References