Abstract—The problem of testing homogeneity of coefficients in distributed parameter systems (DPS) is stated and motivated by an example from the quality monitoring of the copper hot rolling process. When a system described by partial differential equation(s) (PDE) is observed using sensors or digital cameras and its coefficients are suspected to be inhomogeneous in space, then it is rather difficult to estimate them from noisy observations. Such observations can however be sufficient to test whether these coefficients are constant in space. The test for homogeneity is proposed, which is computationally simple and applicable on-line. The main idea for its construction is based on the expansion of a PDE solution into a series of eigenfunctions. Then, coefficients of this series are estimated from the observations of the process state: directly and indirectly, through estimates of PDE parameters, provided that they are constant. Finally, the squared differences between these two sets of coefficients are used to form the test statistic. The results of verification of the test on simulated and experimental data are also provided.

I. INTRODUCTION

The materials used for production of many kinds of goods are usually required to have a homogeneous internal structure. For example, homogeneity is desirable when producing copper, steel or aluminum alloys. Strict homogeneity requirements are imposed in the copper hot-rolling industry.

Our aim in this paper is to propose a relatively simple test of verifying the homogeneity of a material, based on measurements of its spatial response to a known excitation. In particular, these observations can be provided as images from digital cameras, observing slabs before hot-rolling.

When a material or a process is described by partial differential equations, the notion of its homogeneity is usually interpreted as the requirement that coefficients in these equations are constants, which do not depend on spatial coordinates. Constructing our test for detecting possible inhomogeneities, we understand the homogeneity exactly in this sense.

The idea of constructing the test is based on describing a process in the tested material by a partially differential equation (PDE) with constant coefficients (e.g., by a heat transfer equation) and to estimate them from observations or images. Then, these coefficients are used to estimate eigenvalues of an operator of PDE. On the other hand, direct estimates of the eigenvalues are obtained and compared with those obtained earlier.

In order to reduce the computational burden and to provide the test, which can be applicable on-line, one should avoid iterative procedures such as required by non-linear least-squares methods. For this reason we adopt to the first part of our test the approach recently proposed by the author in [22]. Its essence is in reducing the problem of estimating constant parameters of PDE to the linear least squares fit applied twice.

The reader is referred to [20], [17], [16] for early attempts to identify eigenvalues of operators corresponding to the considered PDE’s, which form foundations to our approach.

It is worth mentioning that an alternative way of verifying the homogeneity of coefficients in PDE’s is to try to estimate them as spatially varying functions and then to check whether their spatial variability is essential. Note, however, that estimation of spatially varying coefficients is much more difficult and data demanding as it follows from the bibliography cited below. In the last twenty years large progress has been made in the theory and algorithms for identifying spatially varying coefficients in PDE’s. We refer the reader to the monograph [1] of Banks and Kunisch and to selected original research papers on estimating spatially varying parameters [2], [3], [11], [12] [13], [14] [8], [6]. One may wish to consult also survey papers on identification of PDE’s [10], [18], [15].

The number of research papers on applications of PDE’s in quality control is much smaller than theoretical works on identification of DPS. Here we mention [4], [7], [5] in which thermal processes are considered, but in a way different from our example in the last section.

The paper is organized as follows. In the next section we start from displaying a general form of the solution for the class of distributed parameter systems described by linear partial differential equations (PDE) of the elliptic type with self-adjoint differential operator and constant coefficients. In Section III the problem of testing the homogeneity of coefficients is stated and in Section IV the proposed test is described. It is based on the algorithm described in [22] to estimate parameters in the above mentioned class of systems. Applying the least square method (LSM) twice we can provide
a physical interpretation of these steps. The first one is used for identifying a finite number of the system eigenvalues (natural frequencies). When eigenvalues are identified then LSM is applied again for estimating parameters, which usually has the interpretation of material constants. Finally, we perform a direct estimation of a system response and the comparison of the above mentioned estimates provides the proposed test. Section V contains extensive discussion on selecting the threshold for the test statistics, which has asymptotically $\chi^2$ distribution. The validity of using this asymptotic distribution is verified by extensive simulation studies, which are reported in Section VI. Then, in Section VII the test is applied to assess the quality of copper before entering a hot-rolling mill. The second application of the test to experimental data is presented in Section VIII, in which the homogeneity of transversally loaded beam was examined. Concluding remarks are collected in Section IX.

II. System’s response when coefficients are constant

Denote by $x = [x^{(1)}, x^{(2)}, \ldots, x^{(d)}]$ the vector of spatial variables. Abusing the notation, we will write $[x, y]$ for a spatial point when $d = 2$. Let $\Omega \subset \mathbb{R}^d$ be an open, connected domain in which our spatially distributed process takes place.

Let us consider spatial differential operator of the following form

$$A_x(a) h(x) = \left( \sum_{i=0}^{r} a_i P_i \right) h(x), \quad x \in \Omega, \quad h \in D(A),$$

(1)

where $a_i, i = 0, 1, \ldots, r$ are constant parameters, which are elements of the vector $a \in \mathbb{R}^{r+1}$. $P_i$’s are partial differential operators w.r.t. components of $x$. Above, $h(x)$ stands for a generic, sufficiently many times differentiable function of spatial variables $x \in \Omega$. Not every such function is admitted to be an element of domain $D(A)$ of operator $A_x(a)$, but only those, which fulfill homogenous boundary conditions imposed on it.

A typical example of (1) is the Laplace operator with the homogenous boundary conditions of the first kind. Then, $P_i = \frac{\partial^2}{\partial x_i^2}$ and $D(A)$ consist of twice differentiable functions, vanishing at the boundary of $\Omega$ (see also equation (28) in Section VII).

Let us consider the class of DPS described by the following PDE with constant coefficients

$$A_x(a) q(x) = u(x), \quad x \in \Omega,$$

(2)

where $q(x)$ is the system state at a spatial point $x \in \Omega \subset \mathbb{R}^d$, while $u(x)$ is a known input signal.

We adopt the following assumptions:

A1) In (2), $\Omega \subset \mathbb{R}^d$ denotes an open, bounded spatial domain with smooth boundary $\Gamma$.

A2) The spatial differential operator $A_x(a)$ is known to within the vector $a \in \mathbb{R}^r$ of unknown, constant parameters. They will be estimated from measurements in order to verify the hypothesis that they are constant indeed.

A3) For every admissible $a \in \mathbb{R}^r$, operator $A_x(a)$ is defined on a dense subset $D(A)$ of $L^2(\Omega)$, which is the Hilbert space of all square integrable functions on $\Omega$.

A4) Homogeneous boundary conditions are taken into account in the definition of $D(A)$.

A5) For every admissible $a \in \mathbb{R}^r$ the operator $A_x(a)$ is selfadjoint with compact resolvent (see e.g. [26] for definitions).

Under these assumptions the following general results for $A_x(a)$ hold (see, e.g., [26] and the bibliography cited there).

P1) There exists a set of eigenvalues $\lambda_k, k = 1, 2, \ldots$ and eigenfunctions $v_k(x), k = 1, 2, \ldots$ such that

$$A_x(a) v_k(x) = \lambda_k v_k(x), \quad k = 1, 2, \ldots$$

(3)

P2) The system $v_k(x), k = 1, 2, \ldots$ is complete orthonormal system in $L^2(\Omega)$ and $\lambda_k \neq 0, k = 1, 2, \ldots$

Possible multiple eigenvalues are included in the sequence $v_k$’s, while in the sequence $\lambda_k, k = 1, 2, \ldots$ a multiple eigenvalue is repeated necessary number of times.

Under the above assumptions a solution of (2) exists and it is unique. Furthermore, the solution of (2), denoted as $q(x; a)$, $x \in \Omega$ to stress the dependence on $a$, is given by

$$q(x; a) = \sum_{k=1}^{\infty} \frac{1}{\lambda_k(a)} \cdot \langle u, v_k \rangle \cdot v_k(x),$$

(4)

where for two functions $g, h$, which are square integrable in $\Omega$, the inner product $(g, h)$ is defined as

$$(g, h) = \int_{\Omega} g(x) h(x) \, dx.$$

In (4) we denoted by $\lambda_k(a), k = 1, 2, \ldots$ the eigenvalues of $A_x(a)$ in order to display their dependence on parameters $a$.

In general both $v_k$’s and $\lambda_k$’s may depend on the vector of parameters $a$. We shall consider an important special case when each $v_k$ is simultaneously the eigenfunction of all operators $P_i, i = 0, 1, 2, \ldots, r$. The following result, although easy to verify, is important for our purposes.

Proposition 1: If for certain constants $\alpha_{ik}$’s eigenfunctions $v_k$’s in (1) are such that

$$P_i v_k(x) = \alpha_{ik} v_k(x), \quad k = 1, 2, \ldots, \quad i = 0, 1, \ldots, r,$$

then $v_k(x), k = 1, 2, \ldots$ do not depend on $a \in \mathbb{R}^r$. Furthermore, $\lambda_k$’s are linear or affine functions of vector $a$, i.e.,

$$\lambda_k(a) = \alpha_k^T \cdot a, \quad k = 1, 2, \ldots,$$

(6)

where

$$\alpha_k \stackrel{def}{=} [\alpha_{0k}, \alpha_{1k}, \ldots, \alpha_{rk}]^T, \quad k = 1, 2, \ldots$$

and they are known, since we assume that partial differential operators $P_i$’s are given.

Note that both $v_k$’s and vectors $\alpha_k$ can be calculated even when $a$ is unknown.
property P2) is also important for our purposes, since by orthonormality of \( v_k \)'s \( (v_k, v_j \geq 0 \) for \( k \neq j \) and 1, otherwise) and their completeness in \( L_2(\Omega) \) we have

\[
\lim_{K \to \infty} \int_{\Omega} \left[ h(x) - \sum_{k=1}^{K} \frac{1}{\lambda_k(a)} \cdot (u, v_k) \cdot v_k(x) \right]^2 \, dx = 0. \quad (7)
\]

For practical reasons it is later assumed that for a certain sufficiently large but finite \( K > 0 \) have

\[
q(x; a) = \sum_{k=1}^{K} \frac{1}{\lambda_k(a)} \cdot (u, v_k) \cdot v_k(x) \quad (8)
\]

### III. Problem Statement

The system state is observed by a sensor(s) with a nonlinear characteristic \( \Phi : R^3 \to R^1 \), i.e., at point \( x \) the output \( w(x; a) \) is given by

\[
w(x; a) = \Phi(q(x; a)) \quad (9)
\]

The observations of \( w \) are corrupted by random noises, which are described below.

Spatial homogeneity of vector \( a \) will be tested using measurements of the system response at isolated spatial points \( x_n \in X, n = 1, 2, \ldots, N \), which are assumed to be of the form

\[
y_n = w(x_n; a) + z_n, \quad n = 1, 2, \ldots, N. \quad (10)
\]

In (10), \( z_n, n = 1, 2, \ldots, N \) are the measurement errors assumed to be uncorrelated random variables with zero mean and variances \( 0 \geq \sigma^2 < \infty \).

We do impose any additional restrictions on the probability distribution of \( z_n \)'s, staying in the framework of the least squares criterion and gaining its computational simplicity. The only restriction is the existence of the first two moments of \( z_n \)'s. The price for admitting so a large class of probability distributions is in the necessity of relying on the asymptotic distribution of the test statistic. In Section V we comment on possible improvements when one can assume that errors have Gaussian distribution.

We state the hypothesis \( H_0 \): parameters \( a_j, j = 0, 1, \ldots, r \) in system (2) are constants, i.e., they are spatially homogenous. Under the above stated assumptions on \( A_x(a) \) and having observations \( (x_n, y_n), n = 1, 2, \ldots, N \), our aim is to construct a test for verifying whether these observations are not in (statistical) contradiction with \( H_0 \).

Let us summarize conclusions from our assumptions concerning the system and its output. From (8) and (9) it follows that, under the hypothesis that the system parameters \( a \in R^{r+1} \) are constant, the output can be expressed as

\[
w(x; a) = \Phi \left( \sum_{k=1}^{K} b_k v_k(x) \right) \quad (11)
\]

where \( b_k = \phi_k(\bar{\alpha}_k^T \cdot a), \quad k = 1, 2, \ldots, K. \quad (12) \)

Above \( \bar{\alpha}_k, k = 1, 2, \ldots, K \) are given, linearly independent \((r + 1)\)-dimensional column vectors with components \( \alpha_{jk}, j = 0, 1, 2, \ldots, r \) (the same as in (5)), while \( T \) denotes transposition. In (11) and (12) functions \( \Phi : R \to R \) and \( \phi_k : R \to R \) are assumed to be known and strictly monotone. Thus, \( \Phi^{-1} \) and \( \phi_k^{-1} \) exist. Now, setting

\[
\phi_k(t) = \frac{(u, v_k)}{t}, \quad t \in R - \{0\}, \quad k = 1, 2, \ldots, K \quad (13)
\]

one can treat the considered class of DPS as a subclass of systems described by (11) and (12).

### IV. Homogeneity Test

To motivate the proposed test, it is expedient to put in parallel the following facts:

**F1)** If parameters \( a \in R^{r+1} \) are constant, then they are related to the output \( w(x; a) \) by (11), (12) and (10).

**F2)** If parameters \( a \in R^{r+1} \) depend on \( x \), then (12) and (13) are no longer valid, but the class of functions of the form

\[
\Phi \left( \sum_{k=1}^{K} \beta_k v_k(x) \right), \quad (14)
\]

with \( K \) large enough, is sufficiently rich in order to approximate a response of system (2) in which inhomogeneous coefficients \( a(x) \) are present. This follows from the invertibility of \( \Phi \) and from (7).

**Homogeneity Test (HT)**

Step 1 Find the estimate \( \hat{\beta} \in R^K \) of \( \beta_k \)'s, which corresponds to possible inhomogeneity of parameters, by the minimization of the function

\[
Q_1(\beta) = \sum_{n=1}^{N} \left( \Phi^{-1}(y_n) - \beta^T v(x_n) \right)^2 \quad (15)
\]

with respect to \( \hat{\beta} \triangleq [\beta_1, \beta_2, \ldots, \beta_K]^T \). Denote by \( \hat{\beta} \triangleq [\hat{\beta}_1, \hat{\beta}_2, \ldots, \hat{\beta}_K]^T \) the minimizer of (15).

Step 2 Find the estimate \( \hat{a} \in R^r \), which minimizes the function

\[
Q_2(a) = \sum_{k=1}^{K} \left( \phi_k^{-1}(\hat{\beta}_k) - \hat{\alpha}_k^T \cdot a \right)^2 \quad (16)
\]

with respect to \( a \).

Step 3 Calculate the second estimate \( \hat{b}_k \) of \( \beta_k \), \( k = 1, 2, \ldots, K \), which corresponds to the homogeneous case,

\[
\hat{b}_k = \phi_k(\hat{\alpha}_k^T \cdot a) = \frac{< u, v_k >}{\lambda_k(a)}, \quad k = 1, 2, \ldots, K. \quad (17)
\]

Step 4 Compare vectors \( \hat{\beta} \) and \( \hat{b} \). A large difference between them is the witness against a homogeneity of coefficients \( a \) (see Remark 1 and Section V for discussion on comparing \( \hat{\beta} \) and \( \hat{b} \)).

**Remark 1:** The above test is based on the comparison of pointwise estimates of \( \hat{\beta} \) and \( \hat{b} \). More detailed information on the differences between them could, in principle, be obtained by comparing the probability distributions of these variables. This is, however, possible if either we have a precise knowledge on the distribution of measurement errors and the dependence of \( w \) on \( a \) can be approximated by a linear one.
or repeated measurements at points \(x_n\)'s are available, which can be used for estimating this distribution.

Remark 2: Instead of estimating \(a\) as in Steps 1 and 2, one can try to use the classical approach. It would lead to a nonlinear version of the least squares method, i.e., to the minimization of the following criterion:

\[
Q(a) = \sum_{n=1}^{N} (y_n - w(x_n; a))^2
\]

with respect to \(a \in \mathbb{R}^p\). In general case, however, a direct, analytical, minimization of \(Q(a)\) is not possible. Furthermore, iterative minimization of \(Q(a)\) may be divergent or convergent to local minima, even if specialized algorithms, such as the Levenberg-Marquardt method, are applied. For this reason, the method recently proposed by the author was used in the above test (see [22] where also some remarks on its asymptotic properties are given).

The few remarks below make the above testing procedure easier to implement.

Remark 3: The implementation of Steps 1 and 2 consists of solving the following two systems of linear equations. The first one corresponds to (15) and reads as follows

\[
V_N V_N^T \hat{\beta} = \sum_{n=1}^{N} \Phi^{-1}(y_n) v(x_n),
\]

where \(V_N \overset{\text{def}}{=} [v(x_1), v(x_2), \ldots, v(x_N)]\) is \(K \times N\) matrix. The uniqueness of the solution of (19) can be guaranteed if and only if matrix \(V_N\) is of the rank \(K (N \geq K)\) is required) and then, the covariance matrix of \(\hat{\beta}\) has the form

\[
\text{cov} (\hat{\beta}) = \sigma^2 (V_N V_N^T)^{-1},
\]

where \(\sigma^2\) is the estimate of the errors variance, which can be obtained in the standard way (see, e.g., [23]) as the sum of squared residuals divided by \((N - K - 1)\). Then, variances of \(\text{var} (\hat{\beta}_k)\)'s are the diagonal elements \(\text{cov} (\hat{\beta})\).

The system of normal equations corresponding to (16) has the form

\[
A^T A \hat{a} = \sum_{k=1}^{K} \phi_k^{-1}(b_k) \hat{a}_k,
\]

where \(A \overset{\text{def}}{=} [\alpha_1, \alpha_2, \ldots, \alpha_K]^{T}\) is \(r \times K\) matrix. Further, we assume \(K \geq r\) and that \(A\) is of the full rank, which implies uniqueness of the solution of (20).

V. THRESHOLD OF THE TEST STATISTICS

The comparison of \(\hat{\beta}\) and \(\hat{b}\) for a relatively small number of observations, is not an easy matter, since both vectors are random and estimated from the same set of data. However, assuming homogeneity of \(a\) and having \(N \to \infty\), one can hope that the estimation accuracy of \(\hat{b}\) is larger than that of \(\hat{\beta}\).

Under this proviso random variables

\[
\xi_k \overset{\text{def}}{=} \frac{b_k - \hat{b}_k}{\sqrt{\text{var} (\hat{\beta}_k)^{1/2}}} \quad k = 1, 2, \ldots, K
\]

(21)
can be treated as being asymptotically normal with the zero mean and the unit variance. Thus, \(\sum_{k=1}^{K} \xi_k^2\) has asymptotically \(\chi^2\) distribution with \(K - (r + 1)\) degrees of freedom, since \(r + 1\) coefficients in \(a\) were estimated from the same data. Now, Step 4 can be quantified by comparison of \(\sum_{k=1}^{K} \xi_k^2\) with the critical value \(C_{\text{crit} (\alpha)}\) calculated as \((1 - \alpha)\) quantile of the \(\chi^2\) distribution with \(K - (r + 1)\) degrees of freedom, for a chosen significance level \(0 < \alpha < 1\) (usually selected as \(0.1, 0.05\) or \(0.01\)). We reduced the number of degrees of freedom by \(r + 1\), since exactly \(r + 1\) parameters is estimated from the same set of observations. Thus, we reject \(H_0\), if criterion sum of squares (CSS)

\[
CSS = \sum_{k=1}^{K} \xi_k^2 > C_{\text{crit} (\alpha)}.
\]

Otherwise, we conclude that observations are not in the contradiction with \(H_0\).

The above considerations are asymptotic and to some extent heuristic in the sense that we suggest utilizing the \(\chi^2\) distribution for \(N\) finite although it is, in general, only the asymptotic distribution of the test statistic. In the rest of the section we describe the way of verifying the validity of such approximation, using simulations, and we specify a class of problems, for which \(\chi^2\) distribution is the exact one, even for a finite \(N\).

If \(N\) is not large enough, then one can establish bounds \(C_{\text{crit} (\alpha)}\) by extensive simulations of the same kind as they are performed for other statistical tests (see, for example, [9], where such bounds are established for simultaneous testing of bivariate independence and normality).

Here, we present a more direct approach for verifying whether \(\chi^2\) distribution can be used when \(N\) is finite. Namely, for a given PDE we select the number of degrees of freedom \(K - (r + 1)\) and establish \(C_{\text{crit} (\alpha)}\) from \(\chi^2\) distribution. Then, we run simulations in the two phases:

A) Assuming that coefficients in our PDE are constant, calculate the fraction of false inhomogeneity detections and verify whether it is less than \(\alpha\), which is the largest level of false detections that we admitted.

B) If in phase A) threshold \(C_{\text{crit} (\alpha)}\) is accepted, then use it for another set of simulation runs – this time conducted for PDE with spatially varying coefficient(s) – and to check whether the fraction of correct inhomogeneity detections is sufficiently high (ideally, close to 1, or the fraction of false positive decisions is close to zero).

Remark 4: If the homogeneity of only one coefficient \(a\) is tested and for constant \(a\) operator \(A_x\) has the form \(A_x(a) = a e^{\lambda_x m}\), \(m \geq 1\), then it is expedient to modify slightly our HT in order to ensure that \(C_{\text{crit} (\alpha)}\) follows the centered \(\chi^2\) distribution for a finite \(N \geq K\). Indeed, in this case

\[
b_k = \frac{u_k v_k}{a \alpha_k} \quad \text{and} \quad \lambda_k (a) = a \alpha_k, \quad k = 1, 2, \ldots, K,
\]

(23)

where \(\alpha_k, k = 1, 2, \ldots\) is a known sequence of numbers, which depends on the form of \(A_x\). We shall give an example of \(A_x\) with \(\alpha_k\)'s of the form (23) in Section VIII. Here we only mention that such \(\alpha_k\)'s typically arise in second and fourth order PDE's, which describe deflections of beams, rods and plates.

Note that \(b_k\)'s in (23) are linear in \(\kappa \overset{\text{def}}{=} 1/a\). Thus, instead of inverting \(\beta_k\)'s in Step 2 of HT, one can estimate \(\kappa\) by the...
minimization of
\[ \hat{Q}_2(\kappa) = \sum_{k=1}^{K} \left( \hat{\beta}_k - \kappa \frac{u_v}{\alpha_k} \right)^2. \]  
(24)

The advantage of this approach is that now the minimizer of \( \hat{Q}_2(\kappa) \), denoted as \( \hat{\kappa} \), is a linear function of \( \hat{\beta}_k \)'s, which are, in turn, linear functions of \( y_n \)'s, provided that the sensor characteristic \( \Phi(t) \) is linear. Now, \( \hat{b}_k \)'s are estimated as
\[ \hat{b}_k = \hat{\kappa} \frac{u_v}{\alpha_k}, \quad k = 1, 2, \ldots, K \]  
(25)
and they are used in Step 4 of HT and in (21), (22) instead of \( \hat{b}_k \)'s.

Now, the linearity of \( \hat{\kappa} \) in \( y_1, y_2, \ldots, N \) implies linearity of \( \hat{b}_k \)'s and CSS is now a quadratic function of the observations. If random errors contained in these observations are Gaussian and uncorrelated, then CSS has \( \chi^2 \) distribution for a finite \( N \). Note however, that even if random errors are not Gaussian, \( \chi^2 \) distribution can be used for \( N \) large enough.

VI. SIMULATION STUDIES

In order to illustrate the above scheme of simulations, the following simple two-point boundary value problem was considered
\[ a_1 \frac{d^2 q(x)}{dx^2} + a_2 q(x) = U \delta(x - x_0), \quad x \in (0, \pi), \]  
(26)
with the boundary conditions \( q(0) = q(\pi) = 0 \) and the Dirac delta \( \delta(x - x_0) \) (pointwise) excitation acting at point \( x_0 \) with intensity \( U \neq 0 \). As is well known, the above equation is used to describe vertical deflections of a string or axial deflections of a rod.

The eigenfunctions and eigenvalues of the corresponding operator \( A_q \) have the form: \( \psi_k(x) = \sqrt{2/\pi} \sin(k x), \lambda_k(a) = a_1 k^2 + a_2, \quad k = 1, 2, \ldots \). In all the simulations reported below, we have selected \( a_1 = 1, a_2 = 0.2, x_0 = 0.23 \pi, U = 50 \).

The observations were simulated according to (10) for \( N = 10 \) equidistantly placed points \( x_n \), while random errors \( \epsilon_n \) were drawn from the Gaussian distribution \( N(0, \sigma) \) or from the contaminated Gaussian distribution, which is the following mixture of normal distributions. Firstly the proportion \( 0 < p < 1 \) of two normal distributions is selected. Then, \( \epsilon_n \) is drawn from \( N(0, \sigma_1) \) distribution with the probability \( 1 - p \) and from \( N(0, \sigma_2) \) distribution with the probability \( p \). We shall refer the contaminated normal distribution as \( N(0, \sigma_1, p, \sigma_2) \). This distribution was chosen in order to verify a robustness of the test against the normality of errors assumption. It is interpreted as the mixture of normal errors \( N(0, \sigma_1) \) and a small proportion \( p \) of gross errors \( N(0, \sigma_2) \). Typically, \( p \) is chosen to be of the order 0.05–0.1 and \( \sigma_2 \) is two–three times greater than \( \sigma_1 \).

The results of phase A) simulations are shown in Table I. Each entry of this table was obtained from \( 10^4 \) simulation runs. For \( \alpha = 0.1 \) the test threshold \( C_{crit}(\alpha) = 2.705 \) was established from the tables of \( \chi^2 \) distribution with 1 degrees of freedom, since \( K = 3 \) eigenfunctions of (26) were taken into account and two parameters in (26) were estimated. The analysis of the first column Table I suggests that for a moderate errors in observations (\( \sigma \leq 1.4 \)) one can safely use the test thresholds based on quantiles of \( \chi^2 \) distribution (each entry of this table is less than \( \alpha = 0.1 \)). Furthermore, the fraction of false inhomogeneity detections is robust against contamination of the observations by 5%–10% of larger errors (see the second and the third columns in Table I), where only slight increase of this fraction can be observed.

In phase B) the following system with inhomogeneous coefficient \( x^{1/4} \) was simulated
\[ \frac{d}{dx} \left( x^{1/2} \frac{d q(x)}{dx} \right) + a_2 q(x) = U \delta(x - x_0), \quad x \in (0, \pi), \]  
(27)
with the same boundary conditions as for (26). Also numerical values of \( a_2, x_0 \) and \( U \) remained the same. The solution of this boundary value problem is plotted in Fig. 1 (solid line) together with the solution of (26) (dashed line). The analytical solution of (27) was obtained using the Mathematica environment but it is too complicated to be reproduced here. As one can observe, the solutions of (26) and (27) are rather close to each other and our aim was to detect the inhomogeneity from simulated noisy observations, which were taken at \( N = 25 \) equidistantly points placed in the interval \( [\pi/5, 4\pi/5] \). We have chosen the equidistant sensors placement as the most common choice, but we refer the reader to the recent monograph [24] for extensive discussions on advantages of more careful sensors’ allocation.

In conducting the test \( K = 5 \) and the test threshold \( C_{crit}(\alpha) = 6.25 \) (\( \alpha = 0.1 \); 3 degrees of freedom) were used. The percentage of correct inhomogeneity detection for various noise dispersions are shown in Table II for normal and contaminated normal distributions. Each numerical value in this table was obtained from \( 10^3 \) simulation runs. Its analysis seems to confirm that the test provides sufficiently high detection power, which is in most cases greater than 90%, with the exception of rather heavy noise and measurements conditions (normal errors with \( \sigma = 1.0 \) contaminated in 5% or 10% by errors with two times larger dispersion) when the percentage of correct decisions drops to about 60% in the two cases displayed in the last row of Table II.

VII. TESTING HOMOGENEITY OF THE HEAT TRANSFER COEFFICIENT IN COPPER BEFORE A HOT-ROLLING MILL

The idea of using the homogeneity test for quality control of a copper slab before entering a hot-rolling mill is rather

\[
\begin{array}{|c|c|c|c|}
\hline
\sigma & N(0, \sigma) & CN(0, \sigma, 0.05, 2 \sigma) & CN(0, \sigma, 0.1, 2 \sigma) \\
\hline
0.4 & 0.000 & 0.000 & 0.001 \\
0.6 & 0.004 & 0.008 & 0.011 \\
0.8 & 0.019 & 0.028 & 0.035 \\
1.0 & 0.047 & 0.055 & 0.061 \\
1.2 & 0.070 & 0.072 & 0.077 \\
1.4 & 0.087 & 0.089 & 0.091 \\
\hline
\end{array}
\]

The fraction of false inhomogeneity detection in the simulation studies described in Section VI.
straightforward. It is based on the proviso that a proper copper slab has the heat transfer coefficient (almost) constant in all the volume of its body. Otherwise, a rolling mill producing, e. g. wires would provide product of low quality.

Thus, HT can be used for verifying homogeneity of the heat transfer coefficient by measuring the temperature distribution on a surface of a copper slab. In other words, inhomogeneity of the surface temperature (or emission energy) is treated as indication that in some parts of a slab the heat transfer coefficient is inhomogeneous, which can lead to malfunctioning of the rolling mill or to a low quality product. At this stage of research we are not able to indicate the reason for a slab inhomogeneity.

To illustrate the above idea, consider a hot copper slab. The heat energy emission from its surface was registered by a camera. In Fig. 2 the registered emission field is displayed for two slabs, where the black color corresponds to the most intensive emission, while domains with the minimal emission are marked as white.

For simplicity, the slab is assumed to be sufficiently thin (in the theoretical calculations zero thickness is used). If the heat transfer coefficient $a_1$, say, and coefficient $a_2$ are constant over the slab, then the temperature $q(x, y)$ of its surface at point $(x, y)$ is described by the following PDE

$$a_1 \frac{\partial^2 q(x, y)}{\partial x^2} + a_2 \frac{\partial^2 q(x, y)}{\partial y^2} + a_0 q(x, y) = -u(x, y), \quad (28)$$

for $(x, y) \in \Omega \triangleq (0, L_1) \times (0, L_2)$,

with the homogeneous boundary conditions:

$$q(x, 0) = q(x, L_2) = 0 \quad \text{for} \quad x \in (0, L_1)$$

and

$$q(0, y) = q(L_1, y) = 0 \quad \text{for} \quad y \in (0, L_2).$$

It is also assumed that $u(x, y) = U = \text{const}$ over the surface of the slab.

According to the Stefan-Boltzmann law, the dependence between the temperature $T$ and the $E$ energy emitted from a surface is $E = 1/(\exp(c/T) - 1)$, where $c > 0$ is a certain constant. Taking into account that we had normalized the observations to $(0, 1)$ range, we mimic the Stefan-Boltzmann law by assuming $\Phi(t) = 1/(\exp(1/t) - 1)$.

It is easy to verify that the eigenfunctions of the operator in the l.h.s. of (28) have the form

$$2 \sin(k \pi x/L_1) \sin(j \pi y/L_2)/\sqrt{L_1 L_2}, \quad (29)$$

TABLE II

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>$N(0, \sigma)$</th>
<th>$CN(0, \sigma, 0.05, 2 \sigma)$</th>
<th>$CN(0, \sigma, 0.1, 2 \sigma)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>100.</td>
<td>100.</td>
<td>100.</td>
</tr>
<tr>
<td>0.4</td>
<td>100.</td>
<td>100.</td>
<td>100.</td>
</tr>
<tr>
<td>0.6</td>
<td>100.</td>
<td>99.8</td>
<td>99.8</td>
</tr>
<tr>
<td>0.8</td>
<td>98.7</td>
<td>93.9</td>
<td>89.3</td>
</tr>
<tr>
<td>1.0</td>
<td>80.9</td>
<td>67.9</td>
<td>59.4</td>
</tr>
</tbody>
</table>

TABLE III

<table>
<thead>
<tr>
<th>$\beta_k$</th>
<th>$b_k$</th>
<th>$\beta_k$</th>
<th>$b_k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.08009</td>
<td>0.08007</td>
<td>2.26708</td>
<td>2.26409</td>
</tr>
<tr>
<td>0.13003</td>
<td>0.13854</td>
<td>0.74150</td>
<td>0.73933</td>
</tr>
<tr>
<td>0.24077</td>
<td>0.24104</td>
<td>0.42699</td>
<td>0.42625</td>
</tr>
<tr>
<td>0.13858</td>
<td>0.13908</td>
<td>0.73486</td>
<td>0.73780</td>
</tr>
<tr>
<td>0.24077</td>
<td>0.24104</td>
<td>0.08009</td>
<td>0.08007</td>
</tr>
</tbody>
</table>
The eigenfunctions and the eigenvalues of the corresponding
where \( q \)
verify whether our test provides the correct answer on the
\( \hat{q} \)
directly. The results of calculations, namely, \( \beta_k \) and \( b_k \), \( k = 1, 2, \ldots \)
and we
in Table III for a
slab named later Slab 02 and in the right panel for Slab 026.
CSS, calculated for \( K = 9 \), equals 1.98 for Slab 02 and
27.14 for Slab 026, respectively. Taking into account that three
parameters were simultaneously estimated, we read \( C_{\text{crit}}(\alpha) \)
from the tables of \( \chi^2 \) distribution with 6 degrees of freedom.
Selecting \( \alpha = 0.05 \), we have \( C_{\text{crit}}(\alpha) = 12.59 \). Thus, \( CSS = 1.98 \)
is apparently below this threshold. We conclude that observations
are not in contradiction with \( H_0 \) and Slab 02 fulfils homogeneity
requirements, although visual inspection shows some irregularities. Note however that we have scaled
the images, what resulted in magnifying irregularities. On the
other hand, \( CSS = 27.14 \) is larger than \( C_{\text{crit}}(\alpha) \) and we conclude that Slab 026 is not homogeneous.

VIII. TESTING HOMOGENEITY OF A BEAM

The aim of the experiment reported in this section was to
verify whether our test provides the correct answer on the
homogeneity in fully controllable conditions, when we know
that a material is indeed homogeneous.

Consider the problem of the static deflection of an elastic
beam of constant cross-section, mass density and constant
Young’s modulus (see, e.g., [25])
\[
q(x) = a \frac{d^4 q(x)}{dx^4} = \frac{d^2 q(x)}{dx^2} \mid_{x=0} = 0, \quad x \in (0, 1), \quad (30)
\]
where \( q(x) \) is the deflection, \( a \) is a constant coefficient, \( U \)
is the transverse pointwise force applied at \( x_0 \) to the beam.
Boundary conditions for the beam pinned at both ends have the
form
\[
q(0) = q(1) = 0, \quad \frac{d^2 q(x)}{dx^2} \mid_{x=1} = 0. \quad (31)
\]
The eigenfunctions and the eigenvalues of the corresponding
operator have the form
\[
u_k(x) = \sqrt{2} \sin(k \pi x), \quad \lambda_k(a) = a \pi^4 k^4, \quad (32)
\]
k = 1, 2, \ldots. The above equations served as the model for the
following experiment. An aluminium beam of 1 meter length,
30 mm width and 3 mm thickness was pinned at both ends
and a load of 1 kg was hanged at the middle of the beam
\( (x_0 = 0.5) \). In order to measure the transverse deflections,
a high resolution digital camera was used. An image from
this camera was digitally processed as follows. Firstly, the
image was converted to a grey scale and then to the black and
white scale. This step provided a 0-1 matrix, where nonzero
elements corresponded to points of the beam. Secondly, the
median from the positions of 1’s (black pixels) in each column
of the matrix was calculated. In this way, we have obtained
a rough curve corresponding to the middle line of the beam.
Finally, the exponentially weighted smoothing (the first order,
low-pass recursive filter) was applied to this curve. For further
calculations we have selected beam deflections sampled from
the smoothed curve at 12 equidistantly placed points. They are
shown as dots in Fig. 3.

Now, the homogeneity test was used in the version modified
according to Remark 4, since we have only one unknown
parameter and \( \lambda_k \)’s have the suitable form. The estimated
parameters of the deflection expansion are shown in Table IV,
where the estimates obtained in Step 1 of HT are displayed
in the second column (the corresponding curve is shown in
Fig. 3 as a solid curve). The third column of this table contains
the estimates as in (25) (see Remark 4). The visual
comparison of the second and third column in Table IV reveals
no essential differences between the estimates. The value of
the test statistics \( CSS = 0.008 \) (for \( K = 2 \)) is much less
than the threshold \( C_{\text{crit}}(\alpha) = 2.7 \) established from the tables of
\( \chi^2 \) distribution with 1 degrees of freedom for \( \alpha = 0.1 \).
We conclude that the observations are not in the contradiction
with the hypothesis that our beam is homogeneous.

IX. CONCLUDING REMARKS

The proposed test of homogeneity of coefficients in PDE’s
is less data demanding than methods of estimating spatially
varying coefficients and for this reason it is worth applying
it before trying to estimate them. The test was found to be
sufficiently sensitive to detect inhomogeneities when applied
to real images of hot copper slabs. It was shown that it does
not reject the hypothesis on homogeneity in the controlled
experiment with bending a beam, which was known to be
homogeneous. The extensive simulation studies reported in
the paper convince us that the test is to some extent robust against
5-10 percent of gross errors in the observations. One can hope
that it will also be useful in other quality monitoring tasks.

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Schemes for Parameter Estimation in Parabolic Equations, Applied
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