Effect of noise correlation and input entanglement on the capacity of the quantum bosonic Gaussian channel

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We present an algorithm for the calculation of the classical capacity of a quantum bosonic memory channel with additive Gaussian noise in the full input energy domain. The method is applicable to all channels with noise correlations obeying certain conditions. As a generic example, we consider a memory channel with Gauss-Markov noise [Phys. Rev. A 80, 062313 (2009)]. We determine the optimal input states and resulting capacity, and address the limiting case of maximal noise correlations. We evaluate the enhancement of the transmission rate when using these optimal entangled input states by comparison with a product coherent-state encoding, and find out that this simple coherent-state encoding achieves not less than 90% of the capacity.

I. INTRODUCTION

A central problem of information theory is the capacity of communication channels, which is the supremum on the rate of information that can be transmitted per channel use, given the energy spent on information transmission. For quantum channels which may transmit quantum or classical information one can define the quantum or classical capacity. We focus here on the latter.

The additivity of quantum channels which plays a key role in this problem means that the capacity of a channel system composed of several channels is equivalent to the sum of the capacity of its individual channels. In general quantum channels are not additive [1], however for several memoryless and forgetful channels this property was proven to hold [2–5].

For some channels with memory, in particular, channels with noise correlations between the uses of the channel, it has been shown that the input states that achieve the classical capacity are entangled [6, 7]. For discrete channels it has been shown that entangled states improve the transmission rate for a memory above a certain threshold, whereas below it product states are optimal [7–9]. Bounds on the classical and quantum capacities of a qubit channel with finite memory were derived [9–12]. Classical and quantum capacities of memory channels were discussed in a general framework [5].

For quantum memory channels with continuous alphabet, the first studies considered two uses of a Gaussian channel with correlated additive noise [13] and a lossy Gaussian channel with correlated noise [14]. The classical capacity was achieved for input states with some degree of entanglement. Later, lower and upper bounds on the classical capacity were derived for a lossy optical memory channel [15].

It was shown that for a certain class of Gaussian channels with correlated noise the capacity is equivalent to the capacity of a product channel [16,17], obtained by orthogonal and symplectic diagonalization of the noise covariance matrix of the channel. All this results rely on the generally accepted conjecture that Gaussian states achieve the capacity of Gaussian channels. For a particular case, i.e. the lossy Gaussian channel with vacuum noise this conjecture was proven to hold [6]. We will take this conjecture for granted in this paper.

In a recent paper [16] we have evaluated the classical capacity of a Gaussian channel with additive Markov correlated noise. Our study was focused on an input energy domain above a certain threshold, where we found a quantum water-filling solution, similar to classical water-filling [20]. In the quantum case we have taken into account that a part of the input energy has to be spent to the creation of quantum carries of information. For the first time the notion of quantum water-filling appeared in the discussion of the capacity of a memoryless, phase-dependent Gaussian channel [21], where the quantum carriers of information were considered as a part of the channel and only the classical modulation was restricted by an energy constraint. The quantum water-filling solution for a lossy Gaussian channel with a non-Markovian noise model was obtained independently in [17].

Diagonalization of the covariance matrix of the noise leads to a tensor product channel with a phase dependent noise spectrum [16,18]. The optimal input state in the new representation is a product of single mode pure states. Above the input energy threshold where the quantum water-filling solution holds these single mode states are found to be squeezed states, with the squeezing matching the anisotropy of the noise. The energy spent to optimal modulation equalizes the overall output energy of all modes. When the optimal input state is rotated back to the original basis where the noise is correlated, it is entangled. This is one more example of a channel for which optimal input states may be entangled.

In the present paper we extend our results to the full input energy domain and study the properties of the capacity of the Gauss-Markov channel. The solution of the optimization problem via the method of Lagrange multipliers is now extended to the range of the input energy values where the extremum lays outside of the physically acceptable region. Then the maximum is attained at the boundary of the region. We present an algorithm for a numerical solution of the arising system of equations including the limit of infinite number of modes. Although the algorithm was derived independently, the validity of this method is based on the arguments which are essen-
tially equivalent to ones previously presented in [17] for an algorithm that solves a similar system of equations for a lossy channel.

The paper is organized as follows. First, we introduce the notion of the classical capacity and of Gaussian channels in section II. Then in section III we present the general solution to the capacity for the one mode case, in section IV we present the solution for an arbitrary number of modes. Finally, in section V we analyze the capacity of the Gaussian Markov channel as well as the optimal states and input energy distribution in the full range of the correlation strength. We evaluate the gain from using the optimal input states which are entangled and therefore may be complicated to produce, with respect to easily generated coherent product states.

II. CLASSICAL CAPACITY OF QUANTUM GAUSSIAN CHANNELS

In order to transmit classical information via a quantum channel one defines an alphabet with letters associated to quantum states $\rho_i^m$. The quantum channel $T$ is a completely positive, trace-preserving linear map acting on the input “letter” states:

$$\rho_i^{\text{out}} = T(\rho_i^m),$$  

resulting in output states $\rho_i^{\text{out}}$. On average the “letters” $\rho_i^m$ appear in the transmitted messages with a priori probabilities $p_i$ so that the overall modulated input state is $\rho^m = \sum_i p_i \rho_i^m$. By linearity, the action of $T$ on the overall modulated input reads $T(\rho^m) = \sum_i p_i \rho_i^{\text{out}} \equiv \overline{\rho}$, where $\overline{\rho}$ will be referred to as the overall modulated output. The state $\rho^m$ is physical only if it has finite energy. Therefore it has to obey the energy constraint

$$\sum_i p_i \text{Tr}(\rho_i^m \hat{a}^\dagger \hat{a}) \leq \overline{n},$$  

where $\overline{n}$ is the maximum mean photon number per use of the channel, and $\hat{a}, \hat{a}^\dagger$ are the annihilation and creation operators.

The classical capacity $C(T)$ of the channel $T$ is the supremum on the amount of classical bits which can be transmitted per invocation of the channel via quantum states in the limit of an infinite number of channel uses. This quantity can be calculated with the help of the so-called `one-shot’ capacity given by the Holevo bound [24]

$$C_1(T) = \sup_{\rho_p^m, \rho_p^m} \left\{ S \left( \sum_i p_i T[\rho_i^m] \right) - \sum_i p_i S(T[\rho_i^m]) \right\},$$  

with the von Neumann entropy $S(\rho) = -\text{Tr}(\rho \log \rho)$ where log denotes the logarithm to base 2. The supremum in (3) is taken over all ensembles of $\{p_i, \rho_i^m\}$ of probability distributions $p_i$ and pure input states $\rho_i^m$, because it was proven in [23] that the optimal input states for noisy quantum channels are pure.

The term “one-shot” means that only one invocation of $T$ is needed to calculate Eq. (3). Furthermore, a number of $n$ consecutive uses of the channel $T$ can be equivalently considered as one parallel $n$-mode channel $T^{(n)}$, which is used only once. Then an upper bound to the capacity $C(T)$ is given by the limit:

$$C(T) = \lim_{n \to \infty} \frac{1}{n} C_1(T^{(n)}).$$  

It has been shown that the latter is the capacity for particular memoryless [22, 23] and forgetful channels [5], but generally is only an upper bound on the capacity. In the following we find the optimum among the Gaussian states which realizes the limit (4). Therefore, we find the lower bound to the capacity and as we rely on the conjecture that Gaussian states are optimal we refer to it as the capacity.

Let us now consider an $n$-mode optical additive channel $T^{(n)}$. In the following, the number of modes of this channel corresponds to the number of mono-modal channel uses. Each mode $j$ is associated with the annihilation and creation operators $\hat{a}_j, \hat{a}_j^\dagger$, respectively, or equivalently to the quadrature operators $q_j = (\hat{a}_j + \hat{a}_j^\dagger)/\sqrt{2}$, $p_j = i(\hat{a}_j - \hat{a}_j^\dagger)/\sqrt{2}$ which obey the canonical commutation relation $[q_j, p_j] = i\delta_{ij}$, where $\delta_{ij}$ denotes the Kronecker delta. By defining the vector $\hat{R} = (\hat{q}_1, ..., \hat{q}_n; \hat{p}_1, ..., \hat{p}_n)^T$, we can express the displacement vector $m$ and covariance matrix $\gamma$ that uniquely determine an $n$-mode Gaussian state $\rho$ as

$$m = \text{Tr}[\rho \hat{R}],$$  

$$\gamma = \text{Tr}[(\hat{R} - m) \rho (\hat{R} - m)^T] = \frac{1}{2} J, \quad J = i \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$  

where $J$ is the symplectic or commutation matrix with the $n \times n$ identity matrix $I$.

We consider a continuous encoding alphabet, where instead of a discrete index we use a complex number. A message of length $n$ is encoded in a 2$\pi n$ real vector

$$\alpha = (R\{\alpha_1\}, R\{\alpha_2\}, ..., R\{\alpha_n\}, \Im\{\alpha_1\}, ..., \Im\{\alpha_n\})^T.$$

Physically, this corresponds to a displacement of the $n$-partite Gaussian input state defined by the covariance matrix $\gamma_m$ (and zero mean) in the phase space by $\alpha$ and is denoted by $\rho^m_\alpha$. The Wigner function of $\rho^m_\alpha$ reads

$$W^m_\alpha(\hat{R}) = \exp\left[-\frac{1}{2} (\hat{R} - \sqrt{2}\alpha)^\dagger \gamma_m^{-1} (\hat{R} - \sqrt{2}\alpha)\right] \frac{1}{(2\pi)^n \sqrt{\det(\gamma_m)}}.$$

In the following we will refer to this pure state as quantum input. Relying on the “Gaussian conjecture” we only consider Gaussian distributions so that the overall modulated input state is a Gaussian mixture $\rho^m = \int d^2\alpha f(\alpha) \rho^m_\alpha$, where $d^2\alpha = d\Re\{\alpha_1\} d\Im\{\alpha_1\} ... d\Re\{\alpha_n\} d\Im\{\alpha_n\}$ with (classical) Gaussian distribution $f(\alpha)$ with zero mean and covariance matrix $\gamma_m$. In the following we will refer to this covariance
matrix as classical input or classical modulation. We can set the displacement of the non modulated input state and the mean of the classical modulation without loss of generality to zero, because displacements do not change the entropy $S(\rho)$.

The action of the channel $T^{(n)}$ on an $n$-mode input state carrying message $\alpha$ reads as in [13]

$$T^{(n)}[\rho^{in}_\alpha] = \rho^{out}_\alpha = \int d^{2n}\beta \ f_{env}(\beta) \ \times \ \hat{D}(\beta_1) \otimes \ldots \otimes \hat{D}(\beta_n),$$

(7)

with $\beta = (\mathcal{R}\{\beta_1\}, \ldots, \mathcal{R}\{\beta_n\}, \mathcal{S}\{\beta_1\}, \ldots, \mathcal{S}\{\beta_n\})^T$ and the displacement operator $\hat{D}(\beta_j) = e^{\beta_j \hat{a}_j^\dagger - \beta_j^* \hat{a}_j}$. The displacement is applied according to the (classical) Gaussian distribution of the noise $f_{env}(\beta)$ with covariance matrix $\gamma_{env}$ (which also will be referred to as “environment”). If this matrix is not diagonal, then the environment introduces correlations between the successive uses of the channel. These correlations model the memory of the channel.

The covariance matrices of the non-modulated output state $\rho^{out}_\alpha$ and the overall modulated output state $\bar{\rho}$, read, respectively,

$$\gamma_{out} = \gamma_{in} + \gamma_{env},$$

$$\gamma = \gamma_{out} + \gamma_{mod}.\tag{8}$$

Note that the covariance matrix does not depend on the displacement as seen in definition [6]. Therefore, the covariance matrix of $\rho^{out}_\alpha$ is the same for all $\alpha$ and the output entropy of all displaced states is identical. Thus, the one-shot capacity [3] of this additive channel reduces to

$$C_1(T^{(n)}) = \sup_{\gamma_{in},\gamma_{mod}} \{S(\bar{\rho}) - S(\rho^{out}_\alpha)\}.\tag{9}$$

We will use in the following the reduced Holevo $\chi$-quantity that reads

$$\chi = S(\bar{\rho}) - S(\rho^{out}_\alpha).\tag{10}$$

The von Neumann entropy of a Gaussian state $\rho$ is expressed in terms of the symplectic eigenvalues $\nu_j$ of its covariance matrix:

$$S(\rho) = \sum_{j=1}^n g\left(\nu_j - \frac{1}{2}\right)\tag{11}$$

$$g(x) = \begin{cases} (x+1) \log(x+1) - x \log x, & x > 0 \\ 0, & x = 0. \end{cases}$$

We note that for Gaussian states the symplectic eigenvalues are always greater or equal than $1/2$.

### III. OPTIMIZATION PROBLEM

#### A. One mode channel

In this subsection we consider the case of a single use of the channel $T^{(1)}$ in a similar fashion as in [16]. We start our discussion with the results obtained in [16]. However, we present the solution for the whole input energy domain. We consider the noise covariance matrix to have different variances in the quadratures, where we choose without loss of generality $\gamma_{env}^q > \gamma_{env}^p$ and vanishing off diagonal terms $\gamma_{env}^{q,p} = 0$. Any $2 \times 2$ covariance matrix with non identical eigenvalues can be reduced to this form by a symplectic and orthogonal transformation, which changes neither the entropy of the state nor the energy constraint. Therefore, by our choice we do not lose generality. As already discussed before we restrict ourselves to the optimization over Gaussian states.

In the following we determine the one-shot capacity under the following constraints. The first is the condition that $\rho^{in}$ is a pure state which together with the definition [5] and the commutation relation imply

$$\det \gamma_{in} = \frac{1}{4}.\tag{12}$$

The second is the input energy constraint [2], that reads

$$\gamma_{in}^q + \gamma_{in}^p + \gamma_{mod}^q + \gamma_{mod}^p = 2\pi + 1 = \lambda,\tag{13}$$

where $\gamma_{in}^q, \gamma_{mod}^q$ are the diagonal elements of the matrices $\gamma_{in}$ and $\gamma_{mod}$ and $\lambda$ will be referred to as “input energy” in the following. Furthermore, in order for $\gamma_{in}^q, \gamma_{mod}^q$ to be physical they have to be positive. The optimization problem is solved by using the Lagrange multipliers method, with the total Lagrangian being

$$L = g\left(\nu - \frac{1}{2}\right) - g\left(\nu_{out} - \frac{1}{2}\right) - \tau \left(\gamma_{in}^q \gamma_{in}^p - (\gamma_{in}^q)^2 - \frac{1}{4}\right) - \mu (\gamma_{in}^q + \gamma_{in}^p + \gamma_{mod}^q + \gamma_{mod}^p - \lambda),\tag{14}$$

where

$$\nu = \sqrt{\gamma_{in}^q \gamma_{in}^p - (\gamma_{in}^q)^2},$$

$$\nu_{out} = \sqrt{\gamma_{out}^q \gamma_{out}^p - (\gamma_{out}^q)^2},\tag{15}$$

and $\tau, \mu$ are Lagrange multipliers. In the following we summarize the solution of the system of equations which correspond to the stationary point of $L$. The details are presented in appendix [A]. We prove also in appendix [A] that $L$ is concave at this stationary point which implies that the found solution is indeed the maximum of $L$.

#### 1. Quantum-waterfilling solution

First, we find that the solution implies that the input and modulation covariance matrix cross terms vanish, that is $\gamma_{in}^{q,p} = \gamma_{mod}^{q,p} = 0$. Therefore, the diagonal elements $\gamma_{in}^q, \gamma_{mod}^q$ are the eigenvalues of $\gamma_{in}, \gamma_{mod}$. These equations result in the quantum water-filling condition

$$\gamma_{in}^q + \gamma_{mod}^q = \gamma_{mod}^p + \gamma_{env}^p.\tag{16}$$
which means that the total output energy
\[ \lambda \equiv \lambda + \gamma^q_{\text{env}} + \gamma^p_{\text{env}} \] (17)
is uniformly distributed between the quadratures, that is
\[ \pi^{(q)} = \pi^{(p)} = \frac{\lambda}{2} = \pi_{\text{wf}} \] (18)

We see that this value is also equal to the overall output symplectic eigenvalue which will be referred to as water-filling level. The optimal quantum input is found to be determined by the noise ratio, i.e.
\[ \frac{\gamma^q_{\text{in}}}{\gamma^p_{\text{in}}} = \frac{\gamma^q_{\text{env}}}{\gamma^p_{\text{env}}} \] (19)

One of the equations links the Lagrange multiplier \( \mu \)
\[ g' (\pi_{\text{wf}} - \frac{1}{2}) = 2 \mu \Rightarrow \pi_{\text{wf}} = \frac{1}{2} \coth (\mu \ln 2) \] (20)
to the water-filling level \( \pi_{\text{wf}} \), where \( g'(x) \) denotes the derivative of \( g(x) \) with respect to \( x \) and \( \ln (x) \) is the natural logarithm. In order for the solution to be physical, all eigenvalues in (16) have to be positive. This requires an input energy \( \lambda \) being above the threshold \( \lambda_{\text{thr}} \), i.e.
\[ \lambda \geq \lambda_{\text{thr}} = \sqrt{\frac{\gamma^q_{\text{env}}}{\gamma^p_{\text{env}}} + \gamma^q_{\text{env}} - \gamma^p_{\text{env}}} \] (21)

The optimal eigenvalues which are the solutions of (16), (19) are schematically depicted in Fig. 1 (a) for a particular noise covariance matrix. The one-shot capacity above the threshold reads
\[ C_1 = g \left( \pi + \frac{\gamma^q_{\text{env}} + \gamma^p_{\text{env}}}{2} \right) - g \left( \sqrt{\frac{\gamma^q_{\text{env}}}{\gamma^p_{\text{env}}} \gamma^p_{\text{env}}} \right) \] (22)

It was shown in [25] that the one-shot capacity is additive for such a tensor product channel and therefore \( C_1 \) is the capacity \( C \) above threshold, provided that Gaussian inputs are optimal.

### 2. Solution below the threshold

If \( \lambda \) is below the threshold (21), then the solution of the equations (16), (19) and the purity constraint (12) might result in negative modulation eigenvalues, which would be unphysical. In this case the solution of the optimization problem lays on the border of the valid physical region. From the solution given by Eqs. (16), (19) and depicted in Fig. 1 (a) one can see that when \( \lambda \) decreases the water-filling level \( \pi_{\text{wf}} \) also decreases and at \( \lambda = \lambda_{\text{thr}} \) it crosses the level \( \gamma^q_{\text{in}} + \gamma^p_{\text{env}} \) so that for lower \( \lambda \) the modulation eigenvalue \( \gamma^q_{\text{mod}} \) becomes negative. Therefore, one has to set \( \gamma^q_{\text{mod}} = 0 \) and to solve the new optimization problems with a modified \( \mathcal{L} \) taking into account this new condition. Here we summarize the results which are presented in details in appendix A 2. We find that the off-diagonal terms again vanish, i.e. \( \gamma^p_{\text{in}} = \gamma^q_{\text{mod}} = 0 \).

Now, one obtains a transcendental equation that solves the problem below the threshold, i.e.
\[ g' (\pi - \frac{1}{2}) (\gamma^q - \gamma^p) = \frac{g' (\gamma_{\text{out}} - \frac{1}{2}) (\gamma^p - \gamma^q_{\text{mod}})}{2 \nu_{\text{out}}} \] (23)

We find that from this equation it follows that
\[ \frac{1}{2} \leq \gamma^q_{\text{in}} < \frac{1}{2} \sqrt{\frac{\gamma^q_{\text{env}}}{\gamma^p_{\text{env}}} \gamma^p_{\text{env}}} \] (24)

which means that the more noisier quadrature is always anti-squeezed, and therefore the less noisy quadrature is squeezed. We remark that in the limit \( \lambda \to 1 \) and thus corresponds to vanishing modulation eigenvalues (absence of information transmission). We note that equation (24) which determined the water-filling level reads now
\[ g' (\pi - \frac{1}{2}) \gamma^q = \mu. \] (25)

Although we clearly have no longer a water-filling solution, we can calculate the quantity \( \pi_{\text{wf}} \) following (20) with the same \( \mu \), but refer to it as a "virtual" water-filling level. This quantity will have an important meaning when evaluating the solution for the multimode channel. The reason is that equations (20) and (25) in the multimode channel problem will govern the distribution of input energy between the channels, because \( \mu \) is a monotonically decreasing function of \( \lambda \). For \( \lambda \geq \lambda_{\text{thr}} \) this can be seen from the fact that \( \mu \) is a monotonically decreasing function of \( \pi_{\text{wf}} \) (see Eq. 20) and \( \pi_{\text{wf}} \) is monotonously increasing function of \( \lambda \). For \( \lambda < \lambda_{\text{thr}} \) this is proven in
This property allows us to relate $\lambda_{\text{thr}}$ via Eq. \((20)\) to
\[
\mu_{\text{thr}} = \frac{1}{2} g' \left( \frac{1}{2} (\lambda_{\text{thr}} + \gamma_{\text{env}}^q + \gamma_{\text{env}}^p) - \frac{1}{2} \right)
\]  
\(\text{(24)}\)
such that if $\lambda \geq \lambda_{\text{thr}}$, then the corresponding $\mu \leq \mu_{\text{thr}}$. Moreover, for the lowest input energy $\lambda = 1$ we can define using Eq. \((23)\) an upper bound $\mu_0$ for all possible values of $\mu$ that correspond to $\lambda > 1$, that reads
\[
\mu_0 = \frac{1}{2} g' \left( \sqrt{\left( \gamma_{\text{env}}^q + \frac{1}{2} \right) \left( \gamma_{\text{env}}^p + \frac{1}{2} \right) - \frac{1}{2}} \right) \sqrt{\frac{\gamma_{\text{env}}^q}{\gamma_{\text{env}}^p}}
\]  
\(\text{(25)}\)

In appendix A 2 and A 3 we draw additional conclusions from Eqs. \((22)\) and \((23)\). We show that $\gamma^p < \gamma^q$ and $\gamma^p \geq 1/2$. Furthermore, we find that $d\gamma_{\text{env}}^q/d\lambda > 0$ which means that the anti-squeezing is increasing with the input energy. Moreover, we find that $d\gamma_{\text{env}}^p/d\lambda > 0$ which implies together with $d\gamma_{\text{env}}^q/d\lambda > 0$ that the modulation in the less noisy quadrature is increasing with $\lambda$.

The optimal eigenvalues follow from the solution of \((22)\) and are schematically depicted in Fig. 1 (b) for a particular chosen noise. The one-shot capacity below the threshold is then calculated by inserting the optimal eigenvalues into Eq. \((10)\).

**B. Multimode channel**

We consider in this paper channels with correlation of the noise only between the uses of the channel without the threshold is then calculated by inserting the optimal input and modulation covariance matrices.

The energy (or mean photon number) constraint \((2)\), \((13)\) can now be written as
\[
\lambda \equiv \sum_{i=1}^{n} \lambda_i.
\]  
\(\text{(29)}\)

The total output energy $\lambda$ of the $n$-mode channel is the sum of the input energy $\lambda$ and the total energy of the noise $\gamma_{\text{env}}$ (environment)
\[
\lambda = \lambda + \gamma_{\text{env}}, \quad \gamma_{\text{env}} = \sum_{i=1}^{n} \left( \gamma_{\text{env},i}^q + \gamma_{\text{env},i}^p \right).
\]  
\(\text{(30)}\)

In [16] the equipartition of the total output energy between the modes was obtained as the solution for such a model which we called *global water-filling* similar to the classical *water-filling* solution of $n$ parallel classical Gaussian channels [20]. As in the one mode case this solution only holds above a certain input energy threshold. Now we extend the discussion to energies below this threshold.

The maximum of the Holevo quantity of $n$ parallel channels is again determined using the Lagrange multipliers method (as in the one mode case), where we have now $n$ purity constraints
\[
\gamma_{\text{env},i}^q \gamma_{\text{env},i}^p = 1/4,
\]  
\(\text{(31)}\)

and the overall input energy constraint \((29)\). The Lagrangian is then constructed by a sum of $n$ Holevo $\chi$-quantities \((10)\) for corresponding modes, $n$ Lagrange multipliers $\tau_i$ for the purity constraints and only one common multiplier $\mu$ for the input energy constraint. The fact that the solution of the system of equations which results from the Lagrange multipliers method maximizes the Holevo quantity under the given constraints follows from the results on convex separable minimization subject to bounded variables found in [20]. This was first pointed out for a lossy channel in [17]. In connection to their problem it follows that the maximum is attained by our solution in the multi-mode case provided that for the one-mode case $\chi$ is a concave function of $\lambda$ on the solution. The proof of the concavity of $\chi$ in $\lambda$ on the one-mode solution is presented in appendix B.

In general, the maximum of the Lagrangian corresponds to a partition of $n$ modes into three different sets corresponding to one of three types of input energy distributions within each mode: the case of a quantum water-filling solution with 4 positive eigenvalues (see appendix III A 1), the case of one vanishing modulation eigenvalue with 3 positive eigenvalues (see appendix III A 2) or the case when both modulation eigenvalues vanish and the mode does not participate in information transmission (i.e. unmodulated vacuum is sent). We denote the corresponding sets by: $N_1$, $N_2$ and $N_3$. We denote the
number of modes in the sets as \( n_1, n_2 \) and \( n_3 \), with \( n = n_1 + n_2 + n_3 \). Furthermore, we denote the input energies per set by \( \lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)} \), where

\[
\lambda^{(1)} = \sum_{i \in N_1} \lambda_i, \quad \lambda^{(2)} = \sum_{i \in N_2} \lambda_i, \quad \lambda^{(3)} = \sum_{i \in N_3}, \quad (32)
\]

which sum up to the total input energy \( \lambda \) \(^{(29)}\).

1. **Set \( N_3 \): modes with water-filling solution**

For all modes that belong to \( N_3 \), the water-filling solution described in section 2 holds. This means that the input energy allocated to each mode cannot be lower than its corresponding energy threshold, i.e.

\[
\lambda_i \geq \lambda_{i,\text{thr}}, \quad i \in N_3. \quad (33)
\]

where \( \lambda_{i,\text{thr}} \) reads as in 21 (for all \( i \)). Then for all \( i \in N_3 \), the energy equipartition 16 holds. Moreover, Eq. 20 guarantees that \( \varpi_{\text{wf}} \) is the common water-filling level for all modes due to the common Lagrange multiplier \( \mu \) that is a monotonously decreasing function of \( \varpi_{\text{wf}} \), which now reads

\[
\varpi_{\text{wf}} \equiv \varpi_{q,p}^{i} = \frac{\lambda_3}{2n_3}, \quad i \in N_3, \quad (34)
\]

where \( \lambda_3 \) is the total energy at the output of the modes belonging to set \( N_3 \).

As the partition of the input energy between the modes is \textit{a priori} not known the distribution of the modes between the sets is also not defined. However, we can determine whether a particular mode belongs to set \( N_3 \) using the Lagrange multiplier \( \mu \). This is possible because as mentioned before, \( \mu \) is a monotonously decreasing function of the input energy \( \lambda \) and for \( \lambda_i \geq \lambda_{i,\text{thr}} \) we have \( \mu \leq \mu_{i,\text{thr}} \) defined in 21 and depends only on the noise eigenvalues of mode \( i \). Then we can formalize the definition of \( N_3 \) using \( \mu_{i,\text{thr}} \) as

\[
N_3 = \{ i \mid \mu_{i,\text{thr}} \geq \mu \}. \quad (35)
\]

If \( \mu_{i,\text{thr}} \geq \mu \) for all \( i \) then the set \( N_3 \) contains all modes and we have

\[
\lambda^{(3)} = \lambda. \quad (36)
\]

In this case, Eqs. 16, 19 and 34 determine the global water-filling solution with

\[
\varpi_{\text{wf}} = \frac{\lambda}{2n}. \quad (37)
\]

If the condition 33 is not satisfied for at least one quadrature of any mode then this global water-filling solution has no physical meaning because it will lead to negative values of some modulation eigenvalues.

2. **Set \( N_1 \): modes excluded from information transmission**

Modes for which both modulation eigenvalues are zero do not contribute to the Holevo quantity and consequently to the information transmission. The vanishing modulation eigenvalues \( \gamma_{\text{mod},i} = 0, i \in N_1 \) imply

\[
\frac{\varpi_{i}^{p}}{\varpi_{i}^{q}} = \frac{\gamma_{\text{out},i}}{\gamma_{\text{in},i}}, \quad i \in N_1. \quad (38)
\]

Obviously if the mode is not modulated there is no reason to spend input energy on the squeezing of this mode, which results in the vacuum state being the optimal input state

\[
\gamma_{\text{in},i} = \frac{1}{2}, \quad i \in N_1. \quad (39)
\]

This is consistent with 22 from which we obtain Eq. 39 for \( \lambda_i = 1 \).

In order to deduce the set of modes that are excluded from information transmission we can use the threshold value \( \mu_{0,i} \) defined according to Eq. 25 which corresponds to \( \lambda_i = 1 \) (vacuum energy)

\[
N_1 = \{ i \mid \mu \geq \mu_{0,i} \}. \quad (40)
\]

3. **Set \( N_2 \): single quadrature-modulated modes**

For the modes for which \( 1 < \lambda < \lambda_{\text{thr},i} \) the water-filling solution no longer holds and we have to set the modulation eigenvalue of the noisier quadrature to zero in order to find the optimal input state, i.e.

\[
\gamma_{\text{mod},i}^{q} = 0, \quad i \in N_2. \quad (40)
\]

This implies

\[
\varpi_{i}^{p} = \frac{\gamma_{\text{out},i}}{\gamma_{\text{in},i}}, \quad i \in N_2. \quad (41)
\]

With the functions \( \mu_{\text{thr},i} \) and \( \mu_{0,i} \) defined in 24, 25 we can simply define this set, i.e.

\[
N_2 = \{ i \mid \mu_{\text{thr},i} < \mu < \mu_{0,i} \}. \quad (42)
\]

The eigenvalues that solve the optimization problem for the modes of this set are found using Eq. 22, 23.

We note that both, \( \mu_{0,i} \) and \( \mu_{\text{thr},i} \) depend only on the noise eigenvalues. Therefore, the partition into the three sets is completely determined by only one parameter \( \mu \). Furthermore, we recall that \( \mu \) is the common parameter which enters the equations for sets \( N_2, N_3 \).

IV. SOLUTION FOR ARBITRARY NUMBER

OF MODES

A. Finite number of modes

Recall that the solution of the problem for \( n \) modes is given by the optimal distribution of the input energy between the modes. The optimal energy distribution within
one mode depends on its corresponding set which is given by the noise spectrum and the global parameter $\mu$. Now we express the fact that for the modes belonging to the second set $\gamma_{mod,i} = 0$, we rewrite
\[
\gamma_i^q - \gamma_i^p = \lambda_i - 2\gamma_{in,i}^q - \gamma_{env,i}^q + \gamma_{env,i}^p, \quad i \in N_2.
\] (44)

Furthermore, from (23) we express
\[
g' \left( \frac{\nu_{out,i} - \frac{1}{2}}{2\nu_{out,i}} \right) + \frac{\nu_{out,i}^2}{2\nu_{out,i}} = \frac{\mu}{\gamma_i^q}.
\] (45)

Then we insert Eqs. (44) and (46) together into the left hand side of Eq. (22) and obtain with $\gamma_i^q = \gamma_{out,i}^q$ and
\[
\lambda_i(\gamma_{in,i}^q, \mu) = \frac{\gamma_{out,i}^q}{\mu} f(q_{in,i}) + 2\gamma_{in,i}^q + \gamma_{env,i}^q - \gamma_{env,i}^p. \quad (46)
\]

This means that we have established a relation between the optimal input eigenvalues $\gamma_{in,i}^q$, $i \in N_2$, the global parameter $\mu$ and the optimal input energy distribution $\lambda_i$ between the modes in $N_2$. Using Eq. (46) and definition (28) we can now eliminate variable $\lambda_i$ from Eq. (23). Thus, we obtain a transcendental equation that determines the optimal input eigenvalues $\gamma_{in,i}^q$ as an implicit function of $\mu$:
\[
g' \left( \frac{\nu_{out,i}}{2\nu_{out,i}} \sqrt{1 + \frac{f(q_{in,i})}{\mu}} - \frac{1}{2} \right) = 2\mu \sqrt{1 + \frac{f(q_{in,i})}{\mu}}.
\] (47)

Now we are ready to calculate the input energies of all three sets for a given $\mu$. First, we evaluate the total input energy of “water-filling” modes, i.e. modes in $N_3$. Using Eqs. (25), (32) and (34) we deduce the total input energy used for the modes in $N_3$ as a function of $\mu$
\[
\lambda^{(3)}(\mu) = \sum_{i \in N_3} \left( 2\nu_{in,i}(\mu) - \gamma_{env,i}^q - \gamma_{env,i}^p \right). \quad (48)
\]

Second, the total (vacuum) energy of modes belonging to $N_1$, using (25), reads
\[
\lambda^{(1)}(\mu) = \sum_{i \in N_1} 1. \quad (49)
\]

Functions $\lambda^{(1)}(\mu)$ and $\lambda^{(3)}(\mu)$ depend on $\mu$ through $N_1,N_3$ and $\nu_{in,i}$ which are only functions of $\mu$ and the noise eigenvalues. The total input energy for modes in $N_2$ is the sum
\[
\lambda^{(2)}(\mu) = \sum_{i \in N_2} \lambda_i(\gamma_{in,i}^q, \mu). \quad (50)
\]

Now we apply the overall input energy constraint
\[
\lambda^{(1)}(\mu) + \lambda^{(2)}(\mu) + \lambda^{(3)}(\mu) = \lambda. \quad (51)
\]

Thus, we obtain a closed equation on $\mu$ which we can solve by iterations. The solution of this system of equations provides us with $n_2$ optimal eigenvalues $\gamma_{in,i}^q$ and $\mu$ which determine all other eigenvalues. Once the optimal spectra are obtained one can calculate the one-shot capacity of $n$ modes ($n$ successive uses) of the channel using equations (9) and (11)
\[
C_1(T^{(n)}) = \sum_{i=1}^{n} \left( g \left( \nu_i - \frac{1}{2} \right) - g \left( \nu_{out,i} - \frac{1}{2} \right) \right), \quad (52)
\]

where $\nu_i, \nu_{out,i}$ contain the obtained optimal input and modulation spectra.

B. Infinite number of modes

In order to make the transition to an infinite number of channel uses we have to consider a parallel channel with an infinite number of one mode channels, $n \to \infty$. In this limit all functions previously labeled by $i$ depend now on a continuous parameter $x$ defined on a proper domain which depends on the particular model. All sums that run from $i = 1, \ldots, n$ now become integrals over the whole domain of $x$. The three sets become now sets of continuous variables and cover the whole domain of $x$, they read
\[
N_1 = \{ x | \mu_0(x) < \mu \},
\]
\[
N_2 = \{ x | \mu_{thr}(x) \leq \mu \leq \mu_0(x) \},
\]
\[
N_3 = \{ x | \mu_0(x) \geq \mu \},
\]

where $\mu_{thr}(x), \mu_0(x)$ are defined as in (24), (28) where index $i$ is replaced by $x$. Equations (47)-(50) remain the same, except for the replacements $\gamma_{in,i}$ by $\gamma_{in}(x)$ and the sums over $i$ by integrals over $x$.

Once the $\mu$ is found which is the solution of (51) we can determine the optimal spectra $\gamma_{in}^q(x)$ and $\gamma_{in}^p(x)$. The found optimal spectra are used to evaluate the capacity
\[
C = \lim_{n \to \infty} \frac{1}{n} C_1(T^{(n)}) = \frac{1}{|A|} \int_A dx \left( g \left( \nu(x) - \frac{1}{2} \right) - g \left( \nu_{out}(x) - \frac{1}{2} \right) \right), \quad (54)
\]

where $A$ is the spectral domain and $|A|$ is its size.
Let us note that for the whole class of noises, with correlations given by stationary (shift invariant) Gauss processes, the covariance matrix that describes such noise is a symmetric Toeplitz matrix. Therefore our conclusions derived in what follows are valid for all such channels.

The matrix (55) can be diagonalized in the limit of $n \to \infty$ using a passive symplectic transformation, which allows us to study the channel system in the diagonal, non-correlated basis, since entropies remain unchanged by such transformations.

The spectra of the noise quadratures in the limit of infinite uses of (55) read

$$\gamma_{\text{env}}(x) = N \frac{1 - \phi^2}{1 + \phi^2 + 2\phi \cos(x)}, \quad x \in [0, \pi]$$

and is mirror symmetric with respect to $x = \pi/2$, i.e. $\gamma_{\text{env}}(x) = \gamma_{\text{env}}(\pi - x)$.

In order to find the capacity of the channel for given noise parameters $N, \phi$ we first check whether the threshold condition (53) (here with continuous parameter $x$ replacing index $r$) is fulfilled for all $x$. If this is the case, then the solution is a global water-filling, depicted in Fig. 2(a), where optimal eigenvalue spectra are obtained by (19), (16) (for all $x$), and the capacity can be simplified to (16)

$$C = g(\pi - N) - \frac{1}{\pi} \int_0^\pi dx g \left( \sqrt{\gamma_{\text{env}}(x) \gamma_{\text{env}}(x)} \right),$$

and

$$\pi \geq \frac{2}{1 - \phi} \left( N + \frac{1}{2} \right).$$

If the threshold condition is violated then we can apply the algorithm which is presented in [V].

We note that for different noise parameters the threshold functions $\mu_0(x), \mu_{\text{thr}}(x)$ may have a complicated profile as depicted in Fig. [V] for different noise parameter values. In Fig. [V] we illustrate for a particular choice of the noise parameters $(N, \phi)$ different partitions of the spectral domain between the sets for different input energies $\lambda$ corresponding to different $\mu$.

Our result confirms that the modes belonging to $\mathcal{N}_2$ are squeezed in the less noisy quadrature which is the one that is modulated, as depicted in Fig. [V]. An example plot of optimal input and modulation spectra for $\lambda < \lambda_{\text{thr}}$ is shown in Fig. 2(b). We see the naturally expected behavior of the capacity in Fig. [V]. It decreases with increasing noise variance $N$ and increases with increasing noise correlations $\phi$. We note that the capacity increases with $\phi$ up to the noiseless capacity at “full correlations” ($\phi \to 1$). This limit will be discussed in section [VIII].

A. Optimal quantum input state

An important question is what is the covariance matrix of the optimal input state in the original, “correlated” basis. We know that in the basis where the noise,
modulation and input matrices are diagonal, the optimal input spectrum defines a product of one mode squeezed states. By using general properties of Toeplitz matrices we conclude in appendix 3 that in the infinite limit the optimal input covariance matrix in the original basis is also Toeplitz and we found that its $k$th diagonal reads

$$
\gamma_{\text{in},k}^{q,p} = \frac{1}{2\pi} \int_0^{2\pi} dx e^{ikx} \gamma_{\text{in}}^{q,p}(x), \quad k = 0, 1, 2, \ldots, \infty. \quad (59)
$$

We can express $\gamma_{\text{in}}^{q,p}(x)$ exactly in the case of global water-filling ($\lambda \geq \lambda_{\text{thr}}$) which we consider for the rest of this subsection, that is

$$
\gamma_{\text{in}}^{q,p}(x) = \frac{1}{2} \frac{\gamma_{\text{env}}^{q,p}(x)}{\gamma_{\text{env}}^{q}(x)}. \quad \text{(60)}
$$

Inserting the definition of the noise spectrum of the Gaussian Markov channel (57) we deduce the spectra for the $q$ and $p$-quadrature of the optimal input matrix, i.e.

$$
\gamma_{\text{in},k}^{q,p} = \frac{1}{4\pi} \int_0^{2\pi} dx e^{ikx} \frac{1 + \phi^2 \pm 2\phi \cos(x)}{\sqrt{1 + \phi^2 + 2\phi \cos(x)}}, \quad (61)
$$

where the upper sign is for $q$ and the lower for $p$. We plot 60 in Fig. 7 for the $p$-quadrature for different $\phi$ together with the noise block of the $p$-quadrature. In order to verify whether the overall state is entangled we can check whether the reduced single mode states are mixed, i.e. whether for the reduced covariance matrix we have

$$
\det \gamma_{\text{in}} = \gamma_{\text{in},0}^{q}\gamma_{\text{in},0}^{p} > \frac{1}{4}, \quad \phi > 0. \quad (61)
$$

Integration over the whole domain 0 to $2\pi$ leads to $\gamma_{\text{in}}^{q} = \gamma_{\text{in},0}^{q}$. Then we find for $\gamma_{\text{in},0}^{q}(\phi = 0) = 1/2$, that means that in the absence of correlations the optimal input state is a set of coherent states and not entangled. The limit of $\phi \rightarrow 1$ is unphysical because in this limit each single mode state becomes a thermal state with its temperature tending to infinity. This corresponds to an overall maximally entangled state. It is easy to show that 61 is monotonously increasing from $\phi = 0$ to $\phi = 1$ and

FIG. 4: Functions $\mu_{\text{thr}}(x)$ (lower dotted curve), $\mu_{0}(x)$ (upper dotted curve) and values for $\mu$ (solid bars) for different input energies $\lambda$ and noise parameters $\phi = 0.85$, $N = 1$. From top to bottom the values are $\mu = 1.45(\lambda = 1.006), 1.34(\lambda = 1.04), 0.42(\lambda = 3), 0.04(\lambda = 35)$. The numbers indicate the intervals on the $x$-axis that belong to sets $N_1, N_2$ or $N_3$.  

FIG. 5: Optimal input $\gamma_{\text{in}}^{q}(x)$ (solid curve) and modulation $\gamma_{\text{mod}}(x)$ (dashed curve) eigenvalue spectra of the $q$-quadrature ($p$-quadrature spectra are the same but mirrored with respect to a vertical line at $\pi/2$) vs. spectral parameter $x$, for $\phi = 0.85$, $N = 1$ and $\lambda < \lambda_{\text{thr}}$. The partitioning in sets is taken from Fig. 4. In (a): $\lambda = 3$ which corresponds to $\mu = 0.42$, in (b): $\lambda = 1.04$ which corresponds to $\mu = 1.34$.  

FIG. 6: (a) Capacity $C$ vs. correlation $\phi$, where from top to bottom $N = 1, 2, 3$. (b) Capacity $C$ vs. noise variance $N$, where from top to bottom $\phi = 0.9, 0.7, 0.5$. The input energy is $\lambda = 3$ for both plots. The dashed part of the curves corresponds to the global water-filling solution with $\lambda > \lambda_{\text{thr}}$. One observes that the capacity for full correlations $\phi \rightarrow 1$ tends to the capacity of the ideal noiseless channel $N = 0$.  

Therefore we conclude that for all $\phi > 0$ the optimal input state is entangled.

In order to express the covariance matrix of the overall modulated output, let us recall that in the global water-filling case the overall modulated output eigenvalues are identical ($\gamma_{in,k}^p$ is constant in $x$) and we can express them using Eq. (37) as

$$\gamma_{in,k}^p(x) = \pi + N + \frac{1}{2}.$$  

Therefore, from Eq. (59) we easily see that only the main diagonal ($k = 0$) has non vanishing values and these values are identical. Then the covariance matrix $\gamma$ is proportional to the identity matrix $I$ and therefore it is diagonal in the initial as well as in the rotated basis:

$$\gamma^p = I(\pi + N + \frac{1}{2}).$$

This means that the sum of the optimal input and modulation covariance matrix has to cancel the correlations of the noise.

### B. Full correlations

We observe in Fig. 6 that for fixed $N$ and $\lambda$ the higher the correlations are the higher is the capacity. Furthermore, for $\phi \to 1$ the capacity tends to the capacity of the noiseless channel

$$\lim_{\phi \to 1} C = C_0 = g(\pi),$$  

where the noiseless capacity $C_0$ was calculated in [6] where we set here the channel transmittance to one. Eq. (62) can be deduced by the following reasoning. For any $0 < \phi < 1$ the capacity $C$ is upper bounded by $C_0$. In addition, $C$ is lower bounded by the optimal transmission rate when using coherent states, which is in the following denoted by $R$. Thus, we need to show that for $\phi \to 1$, both bounds fall together.

$R$ is easily calculated, as the restriction to coherent input states basically leads to the discussion of a classical multimode Gaussian channel with a new noise $\gamma_{env}(x) = \gamma_{env}(x) + 1/2$. Clearly, the solution of the optimization problem completely reduces now to the classical water-filling [20] which determines the optimal modulation spectrum. For the noise spectrum of the Gauss-Markov channel [57], we find the global water-filling solution

$$R = g(\pi + N) - \frac{1}{\pi} \int_0^\pi dx \, g^\frac{1}{2} \left( \sqrt{\gamma_{env}(x) \gamma_{env}(x)} + \frac{1}{2} \right),$$  

$$\pi \geq N \frac{2\phi}{1 - \phi}.$$  

(63)

Since for global water-filling the overall modulated output state is identical in $R$ and $C$ [58] the difference to the capacity comes from the difference in the non-modulated output only and is remarkably little. Indeed if we look at the output eigenvalue spectrum for $R$ in [63] which reads

$$\nu_{out}(x) = \sqrt[4]{1 + \gamma_{env}(x) \gamma_{env}(x) + \gamma_{env}(x) \gamma_{env}(x)}$$

instead of

$$\nu_{out}(x) = \sqrt[4]{1 + \gamma_{env}(x) \gamma_{env}(x) + \gamma_{env}(x) \gamma_{env}(x)}$$

for $C$ [58] we see that the two formulas simply differ by the terms which are the arithmetic mean of the noise eigenvalues $\gamma_{env}(x)$, $\gamma_{env}(x)$ in the first one and the geometric mean in the second one. As the geometric mean is always less or equal than the arithmetic mean one confirms that $C \geq R$.

Below the energy threshold one has to solve

$$\frac{1}{\pi} \int_\alpha dx \gamma_{env}(x) = \frac{1}{\pi} \int_\alpha dx \gamma_{env}(x) + \pi$$  

for $\alpha$ (depicted in Fig. 2 (b) but with $\gamma_{in}^p(x) = 1/2$) which is the $x$ value that defines the sets $N_2$ and $N_1$ when restricted to coherent states. For the found value of $\alpha$ we determine the optimal modulation eigenvalues

$$\gamma_{mod}(x) = \theta(x - \alpha)(\gamma_{env}(\alpha) - \gamma_{env}(x))$$

$$\gamma_{mod}(x) = \theta(\pi - x - \alpha)(\gamma_{env}(\alpha) - \gamma_{env}(x)),$$  

(65)

where $\theta(x)$ is the Heaviside step function. By inserting $\gamma_{mod}^p(x) = 1/2$, $\gamma_{mod}^q(x)$ and $\gamma_{mod}^q(x)$ in [54] one obtains $R$ for $\pi < 2N\phi/(1 - \phi)$.

In the limit of full correlations $\phi \to 1$ the noise spectra $\gamma_{env}(x)$ tend to zero for $0 < x < \pi$ and to infinity for $x = 0$ (for the $q$-spectrum) and $x = \pi$ (for the $p$-spectrum). Due to the finite energy of the noise $\frac{1}{\pi} \int_0^\pi dx \gamma_{env}(x) = N$ these functions become delta-like distributions. In this
limit $\alpha \rightarrow 0$ and the solution to $R$ is given by a classical water-filling over a vacuum noise spectrum with infinite edges which though can be shown to give an infinitesimally small contribution and therefore can be neglected. Thus (62) is proven. The same result was obtained in [27] for a channel with additive Markov noise, that becomes a collection of thermal channels when the noise is diagonalized.

C. How useful are the optimal input states?

In this subsection we evaluate the gain $G$ from the use of the optimal input states with respect to coherent product states for the Gauss-Markov channel for two modes and an infinite number of modes. Our motivation here is that the optimal input states are entangled and therefore may be not easy to generate. On the contrary, coherent states are easily accessible in the lab by standard tools of quantum optics. The gain $G$ is given by the ratio of the capacity $C$ to the optimal transmission rate using coherent states $R$ (discussed in section V B)

$$G \equiv \frac{C}{R}. \quad (66)$$

The gain was already discussed in [13] for the case of a two mode additive channel which is identical to our Gauss-Markov channel with noise covariance matrix (55) taking $n = 2$. We remark here, that the capacity of two modes with correlations $\phi$ (in $q$ and $p$, but no $q - p$ correlations) is identical to the capacity of the mono-modal phase dependent channel discussed in section III A, because in the two mode case the diagonalized noise spectrum leads to two phase-dependent mono-modal channels with inverse variance in $q$ and $p$. As it was shown [13] the gain for such a channel exhibits a maximum with respect to the input energy constraint $\bar{\pi}$ for fixed Signal-to-Noise Ratio

$$SNR = \frac{\pi}{N}$$

and correlation $\phi$. Furthermore, it was deduced that the gain increases with increasing correlations $\phi$ between the two modes.

In the case of an infinite number of modes, we know already that in the absence of correlations the optimal input states are coherent states, and therefore there is no gain ($G = 1$). For full correlations, the behavior is essentially different from the two mode case: since the channel becomes effectively noiseless, coherent input states are optimal in this limit as well, whereas for two modes the highest available squeezing is best. Therefore, an interesting question is where we find the maximum gain with respect to the noise correlations in the limit of infinite uses of the channel. In Fig. 8 we plotted the gain $G$ vs. $\pi$ for fixed $SNR$ and different $\phi$ for an infinite number of modes and for two modes. We see that unlike in the

![FIG. 8: Gain $G$ vs. $\pi$ for an infinite number of modes and for two modes (inset). For both plots we took $SNR = 3$ and $\phi = 0.7$ (solid), $\phi = 0.9$ (dashed), $\phi = 0.99$ (dotted).](image1)

![FIG. 9: (Color online) Contour plot of the maximal gain $\max_{\pi} G$ vs. $\phi$, $SNR$ for two modes. In the area above the dotted line the quantum water-filling solution holds.](image2)

![FIG. 10: (Color online) Contour plot of the maximal gain $\max_{\pi} G$ vs. $\phi$, $SNR$ for an infinite number of modes. In the area left to the dotted line the global quantum water-filling solution holds.](image3)
case of two modes where the gain with higher correlations is always higher, in the infinite case the maximum of the gain is found for some intermediate correlations. However, in this plot one does not see the dependency on $\text{SNR}$. So the question that follows is: what is the dependence of the maximal gain (with respect to $\text{SNR}$) on $\phi$ and $\text{SNR}$. In order to answer this question we make a contour plot of $\max_{\text{SNR}} G$ vs. $\phi$ and $\text{SNR}$. In the case of two modes we see in Fig. [9] that the optimal gain is obtained at $\phi = 1$ for a certain $\text{SNR}$. In addition, in this case, the increase in gain at high correlations is very strong compared to lower correlations. Furthermore, a low $\text{SNR}$ seems to benefit from entanglement more than higher $\text{SNR}$.

For an infinite number of modes the situation is different as we can see in Fig. [10]. Instead of a sharp edge towards high correlations we see an almost flat area of maximal gain in the region of high correlations and low $\text{SNR}$. This holds on one hand for high correlation and low $\text{SNR}$, but on the other hand also for less correlated noise and higher $\text{SNR}$. Furthermore, the enhancement is rather robust and does not drop as sharply with decreasing correlations as in the two mode case. However, as the region of high gain has input energies below the global water-filling threshold, where the optimal input squeezing becomes quite complex (as depicted for example in Fig. [5]) a modulation of coherent states might be practically more favorable, because it is already quite efficient and the gain due to entanglement does not exceed 10%.

VI. CONCLUSIONS

We presented an algorithm for the calculation of the classical capacity of the Gaussian channel with additive correlated noise, under the conjecture that Gaussian states are optimal. This method is applicable to the case when there are no $q - p$ correlations in the noise covariance matrix and moreover the $q$ and $p$ blocks commute at least asymptotically in the limit of an infinite number of uses. This applies in particular to the whole class of channels in which noise correlations are given by a stationary Gauss process.

We applied this method to channel with a Gaussian-Markov noise which has asymptotically commuting block matrices. We found that in the limit of full correlations the capacity tends to the noiseless capacity. We calculated the covariance matrix of the optimal input state not only in the eigenbasis of the noise covariance matrix but also in the original, correlated basis. In the correlated basis the optimal input state is entangled and we found that the degree of entanglement scales with the correlation parameter of the noise from no entanglement (i.e. a set of coherent states) to a maximally entangled state.

Furthermore, we discussed the gain from using optimal entangled input states with respect to coherent product states in the case of two modes and an infinite number of modes. We found that in contrary to two modes, where the gain always strongly increases with correlations for any Signal-to-Noise Ratio, for an infinite number of modes a high gain is achieved in a region of high correlations and low $\text{SNR}$ and of lower correlations and higher $\text{SNR}$. In addition, the gain in the limit of infinite modes does not drop as sharply with decreasing correlations as in the two mode case. We also observed that a Gaussian modulated coherent state encoding already achieves not less than 90% of the classical capacity.

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Appendix A: Equations of the Lagrange multipliers method for one-mode

In the following we present the solution via the Lagrange multipliers method for the optimization problem for the one-mode channel introduced in section [III].

1. Search for the extremum

The extremum of the Lagrangian $\mathcal{L}$ defined in (14) must satisfy

$$\nabla \mathcal{L} = 0,$$

where

$$\nabla = \left( \frac{\partial}{\partial \gamma_{\text{in}}^p}, \frac{\partial}{\partial \gamma_{\text{mod}}^p}, \frac{\partial}{\partial \gamma_{\text{mod}}^q}, \frac{\partial}{\partial \gamma_{\text{in}}^q}, \frac{\partial}{\partial \mu}, \frac{\partial}{\partial \tau} \right) ^T.$$

This corresponds to six equations:

\begin{align}
\kappa(\mathcal{P}) \gamma_{\text{out}}^p - \kappa(\mu_{\text{out}}) \gamma_{\text{out}}^q \gamma_{\text{in}}^p - \mu - \tau \gamma_{\text{in}}^p &= 0, \\
\kappa(\mathcal{P}) \gamma_{\text{mod}}^p - \kappa(\mu_{\text{out}}) \gamma_{\text{mod}}^q - \mu - \tau \gamma_{\text{mod}}^q &= 0, \\
\kappa(\mathcal{P}) \gamma_{\text{in}}^p - \mu &= 0, \\
\kappa(\mathcal{P}) \gamma_{\text{mod}}^p - \mu &= 0, \\
-\kappa(\mathcal{P}) (\gamma_{\text{in}}^q + \gamma_{\text{mod}}^q) + \kappa(\mu_{\text{out}}) \gamma_{\text{in}}^q + \tau \gamma_{\text{mod}}^q &= 0, \\
-\kappa(\mathcal{P}) (\gamma_{\text{in}}^p + \gamma_{\text{mod}}^p) &= 0,
\end{align}

where $\mu, \tau$ are Lagrange multipliers and

$$\kappa(x) = \frac{g(x - \frac{1}{2})}{2x},$$

$$\kappa_{\text{mod}}(x) = \frac{g(x - \frac{1}{2})}{2x},$$

$$\kappa_{\text{in}}(x) = \frac{g(x - \frac{1}{2})}{2x}.$$
From Eqs. (A3) and (A4) we derive Eqs. (16) and (20). Since \( \kappa(x) > 0 \) for all \( x \geq 1/2 \) we find with Eq. (A6) that \( \gamma_{in}^{q} = -\gamma_{mod}^{q} \). Therefore, Eq. (A5) simplifies to

\[
\gamma_{in}^{q}(\kappa(\nu_{out}) - \tau) = 0. \tag{A7}
\]

If one assumes that \( \gamma_{in}^{q} \neq 0 \) then the resulting multiplier \( \tau \) leads to a contradiction when inserted in Eqs. (A1) and (A2). Thus, we conclude that \( \gamma_{in}^{q} = \gamma_{mod}^{q} = 0 \). Finally, by combining equations (A1)-(A3) one deduces Eq. (19).

In order for the solutions to be physical the modulation eigenvalues \( \gamma_{in}^{q} \) of \( \gamma_{mod}^{q} \) have to be positive, which is the case for an input energy above the energy threshold \( (21) \). For such \( \gamma \) we will show now that \( L \) is concave on the solution which proves that we found indeed the maximum. We can prove concavity by checking whether the Hessian is negative definite. First, we see immediately from Eqs. (A5) and (A6) that all cross derivatives with respect to \( \gamma_{in}^{q} \) and \( \gamma_{mod}^{q} \) and the four variances vanish on the solution \( \gamma_{in}^{q} = \gamma_{mod}^{q} = 0 \). Therefore, we can consider separately the \( 2 \times 2 \) block of the Hessian that contains second derivatives of \( L \) with respect to \( \gamma_{in}^{q} \). This can be simplified by using \( \gamma_{in}^{q} = \gamma_{mod}^{q} = 0 \) and Eqs. (A1), (A2) to

\[
H_{qp} = -\begin{pmatrix} A + B & A \\ A & A \end{pmatrix}, \tag{A8}
\]

where \( A = \kappa(\nu) \), \( B = \kappa(\nu_{out})(c - 1) \), with

\[
c = 1 + 2\sqrt{\gamma_{env}^{q} + \gamma_{env}^{p} - \gamma_{env}^{q} - \gamma_{env}^{p}}. \tag{A9}
\]

The eigenvalues of (A8) read

\[
h_{qp}^{1,2} = -A \pm \frac{B}{2} \pm \sqrt{A^2 + \frac{B^2}{4}}, \tag{A10}
\]

which are both negative, since \( A, B > 0 \).

Now we show that the eigenvalues of the Hessian of the remaining four variables are also negative. However, instead of considering the Hessian of \( L \) we consider equivalently the Hessian of \( \chi \) with embedded constraints \( (12), (13) \) which becomes then a function of only two variables \( \gamma_{in}^{q}, \gamma_{mod}^{q} \). Then we find a Hessian of the same shape as (A8), where now

\[
A = \frac{g(\nu_{out} - \frac{1}{2})}{\nu_{out}^4}, \quad B = \frac{g(\nu_{out} - \frac{1}{2})}{\nu_{out}^4} \gamma_{in}^{q}, \tag{A11}
\]

Since \( A, B > 0 \) we conclude again that both eigenvalues (that read like (A10)) are negative. Therefore, the total Hessian is negative definite, which proves the concavity of \( L \) for an input energy above the threshold \( (21) \).

2. Below the threshold

If the input energy is below the threshold \( (21) \) then the extremum of the Lagrangian lays outside of the physical region. In this case we set the modulation eigenvalue of the noisier quadrature (we take without loss of generality \( \gamma_{env}^{q} > \gamma_{env}^{p} \)) to zero, that is \( \gamma_{mod}^{q} = 0 \) which replaces Eq. (A3). Therefore, this degree of freedom does no longer exist and there is no \( q - p \) correlation in the modulation: \( \gamma_{mod}^{q} = 0 \). Now, we have to find the extremum of the Lagrangian again, where the new gradient reads

\[
\nabla = \begin{pmatrix} \partial \gamma_{in}^{q} \\ \partial \gamma_{mod}^{q} \end{pmatrix} = \begin{pmatrix} \partial \gamma_{in}^{q} \\ \partial \gamma_{mod}^{q} \end{pmatrix} \tag{A5}
\]

The previous equation (A5) is now simplified to

\[
-\gamma_{in}^{q}(\kappa(\nu) - \kappa(\nu_{out}) - \tau) = 0. \tag{A5}
\]

Again, if one takes \( \gamma_{in}^{q} \neq 0 \) then the solution for \( \tau \) when inserted into Eqs. (A1) and (A2) leads to a contradiction, namely \( \gamma_{mod}^{q} < 0 \), which is unphysical. Therefore, we conclude that \( \gamma_{in}^{q} = 0 \).

The optimal eigenvalues are then found by solving the transcendental equation \( (22) \) and the Lagrange multiplier \( \mu \) is given by \( (A1) \) leading to Eq. (23).

In the following we will verify that \( L \) is also concave for an input energy below the threshold \( (21) \). First, we see again that all cross derivatives with respect to \( \gamma_{in}^{q} \) and the three variances vanish on the solution. Therefore, we consider in the Hessian of \( L \) the second derivative of \( L \) with respect to \( \gamma_{in}^{q} \) separately. We find by using Eqs. (A2), (A3)

\[
\frac{\partial^2 L}{\partial(\gamma_{in}^{q})^2} = -\kappa(\nu) - \kappa(\nu_{out}) \left( \frac{\gamma_{out}^{q} - \gamma_{in}^{q}}{\gamma_{in}^{q}} - 1 \right) < 0, \tag{A12}
\]

since \( \gamma_{out}^{q} > \gamma_{in}^{q} \).

Again, for the remaining three variables instead of considering the Hessian of \( L \) we consider equivalently the Hessian of \( \chi \) with embedded constraints \( (12), (13) \). Now, \( \chi \) becomes a function of only one variable \( \gamma_{in}^{q} \). We proof in appendix A3 that

\[
\frac{\partial^2 \chi}{\partial(\gamma_{in}^{q})^2} = \frac{\partial F}{\partial \gamma_{in}^{q}} < 0, \tag{A13}
\]

where \( F \) is given by (A25). Therefore, the full Hessian of \( L \) is negative definite for input energies below the threshold as well. Thus, we have shown that \( L \) is concave on the solution, which proves that we indeed found the maximum of \( L \).

a. Bounds below the threshold

Here we present bounds on the optimal input squeezing and optimal overall modulated output variances that can be deduced from (22). First, from the fact that

\[
g(\nu_{out} - \frac{1}{2}) \leq \frac{g(\nu_{out} - \frac{1}{2})}{\nu_{out}^4}, \tag{A13}
\]
as $1/x g'(x - 1/2)$ is a monotonously decreasing function and $\nu \leq \nu_{\text{out}}$, it follows that
\[ |\nu^p - \nu^q| > |\nu_{\text{env}}^p - \frac{\gamma_{\text{env}}^q}{4(\gamma_{\text{env}}^q)}|, \]  
where additionally the expressions on the left hand side and right hand side inside the absolute values have the same sign. Then one can directly deduce that
\[ \frac{1}{2} \leq \gamma_{\text{in}}^q < \frac{1}{2} \frac{\sqrt{\gamma_{\text{env}}^q}}{\gamma_{\text{env}}^q}, \]  
(A15)
because other values of $\gamma_{\text{in}}^q$ lead contradictions in (A14). From Eqs. (A14), (A15) it follows directly that
\[ \nu^p < \nu^q. \]  
(A16)

Furthermore, we can find a lower bound on $\nu^p$. Suppose $\nu^p < 1/2$, then we have
\[ \frac{1}{p} = \frac{1}{\sqrt{\gamma^p}} > \frac{1}{\sqrt{\frac{1}{2} \gamma^q}} \]  
\[ \Rightarrow g'\left(\sqrt{\frac{1}{2} \gamma^p} - \frac{1}{2}\right) > g'\left(\sqrt{\frac{1}{2} \gamma^q} - \frac{1}{2}\right). \]  
(A17)
since $g'(x - 1/2)$ is a monotonically decreasing function of $x$. Thus, we can rewrite Eq. (22)
\[ g'\left(\sqrt{\frac{1}{2} \gamma^q} - \frac{1}{2}\right) > \left(\sqrt{\frac{1}{2} \gamma^p} - \frac{1}{2}\right) \]  
and by using (22) and the fact that below the threshold $\nu^q = \gamma_{\text{out}}^q$ we find
\[ g'(\nu_{\text{out}} - \frac{1}{2}) \Sigma > g'\left(\sqrt{\frac{1}{2} \gamma_{\text{out}}^q} - \frac{1}{2}\right) \left(\gamma_{\text{out}}^q - \frac{1}{2}\right) \]  
(A18)
where
\[ \Sigma = \frac{\gamma_{\text{env}}^q}{4(\gamma_{\text{in}}^q)^2} - \frac{\gamma_{\text{env}}^p}{\gamma_{\text{env}}^q}. \]  
(A20)
Thus, our assumption $\nu^q < 1/2$ leads to an inequality which depends solely on $\gamma_{\text{in}}^q, \gamma_{\text{env}}^q, \gamma_{\text{env}}^p$ with the constraints on $\gamma_{\text{env}}^q$ given by (A15) and for the noise variances that $\gamma_{\text{env}}^q > \gamma_{\text{env}}^p, 0 \leq \gamma_{\text{env}}^p \leq 1/2 - 1/(4\gamma_{\text{env}}^q)$. Since, inequality (A19) is always violated for the given constraints, we come to a contradiction which proves that $\nu^p \geq 1/2$.

3. Monotonicity of $\mu$

In the following we prove that the Lagrange multiplier $\mu$ is monotonically decreasing in the input energy $\lambda$, when $\lambda$ is below the threshold $\lambda_{\text{thr}}$ i.e.
\[ \frac{d\mu}{d\lambda} < 0. \]  
(A21)
Using Eq. (23) we can write
\[ \frac{d\mu}{d\lambda} = g''\left(\frac{\nu - 1}{2}\right) \frac{d\nu}{d\lambda} + g'\left(\nu - \frac{1}{2}\right) \left(\frac{d\nu}{d\lambda} \nu - \nu^q \frac{d\nu}{d\lambda}\right), \]  
(A22)
We can upper bound this quantity if we replace $g''(\nu - 1/2)$ by $-1/\nu g'(\nu - 1/2)$ which leads to
\[ \frac{d\mu}{d\lambda} > \frac{d\nu^q}{d\lambda} \]  
(A23)
Since all terms in (A23) except for $d\nu^q/d\lambda$ are clearly positive it remains to show that
\[ \frac{d\nu^q}{d\lambda} = 1 - \frac{d\gamma_{\text{in}}^q}{d\lambda} > 0 \Rightarrow \frac{d\gamma_{\text{in}}^q}{d\lambda} < 1. \]  
(A24)
This derivative can be expressed in terms of Eq. (22)
\[ F \equiv g'(\mu - \frac{1}{2}) (\nu^p - \nu^q) - \frac{g'(\nu_{\text{out}} - \frac{1}{2})}{2\nu_{\text{out}}} \left(\frac{\nu^p}{\gamma_{\text{env}}^q} - \frac{\gamma_{\text{env}}^q}{\gamma_{\text{env}}^q} \nu_{\text{out}}\right), \]  
(A25)
that is
\[ \frac{d\nu^q}{d\lambda} = -\frac{\partial F}{\partial \nu^q}. \]  
(A26)
Therefore, the monotonicity of $\mu$ in $\lambda$ can be proven by showing
\[ |\frac{\partial F}{\partial \lambda}| > |\frac{\partial F}{\partial \nu^q}| < 0. \]  
(A27)
We observe that
\[ \frac{\partial F}{\partial \lambda} = \frac{g''(\mu - \frac{1}{2})}{4\nu_{\text{out}}^2} \nu^q (\nu^p - \nu^q) \]  
\[ + \frac{g'(\nu_{\text{out}} - \frac{1}{2})}{4\nu_{\text{out}}} \left(1 + (\nu^q)^2 \frac{\nu_{\text{out}}}{\nu_{\text{in}}^2}\right) > 0 \]  
(A28)
then it suffices to prove that
\[ \frac{\partial F}{\partial \lambda} + \frac{\partial F}{\partial \nu_{\text{in}}} < 0. \]  
(A29)
By carrying out the partial derivatives in (A29) we arrive at a sum of six terms which can be simplified to
\[ -\frac{\eta}{4\nu_{\text{out}}^3} \nu^q (\nu^p - \nu^q) T_1 - \frac{1}{4\nu_{\text{out}}^3} T_2, \]  
(A30)
where
\[ T_1 = g''\left(-\frac{1}{2}\right) \frac{\nu}{\eta} + g'\left(-\frac{1}{2}\right), \]  
(A31)
\[ T_2 = g''\left(-\frac{1}{2}\right) \frac{\nu_{\text{out}}}{\nu_{\text{in}}} + g'\left(-\frac{1}{2}\right) \left(\frac{\gamma_{\text{env}}^q}{\gamma_{\text{in}}^q} \nu_{\text{out}}^2 - \zeta\right), \]  
(A32)
and
\[ \eta = \frac{\gamma^p - \gamma^q}{\gamma^q - \gamma^p}, \quad \zeta = \left( \frac{\gamma^p_{\text{env}} - 4 \gamma^p_{\text{env}}}{4(\gamma^p_{\text{env}})} \right)^2. \tag{A32} \]

The factors \( \eta \) and \( \zeta \) are positive, since \( \gamma^q, \gamma^p, \nu_{\text{out}} > 0 \). In the following we treat \( T_1 \) and \( T_2 \) separately, showing that each of them is positive which proves (A21). The positivity of \( T_1 \) can be verified via its partial derivatives with respect to \( \gamma^q, \gamma^p \) which lead to two polynomials:
\[ \frac{\partial T_1}{\partial \gamma^q} = p_1(\gamma^q - \gamma^p (3 - 4 \gamma^p (\gamma^p + 3 \gamma^q))) \tag{A33} \]
\[ \frac{\partial T_1}{\partial \gamma^p} = p_2(\gamma^q - \gamma^p), \tag{A34} \]

where \( p_1, p_2 \) are positive polynomials of \( \gamma^q, \gamma^p \). Clearly (A34) is always positive since \( \gamma^q - \gamma^p > 0 \). Then we see that (A33) is negative if \( \gamma^p \geq 1/2, \gamma^q \geq 1/2 \). The first condition was shown in (A2) and the latter always holds by definition of \( \gamma^q \) in \( \mathcal{N}_2 \) and (A15). Therefore, we have to check whether
\[ T_1(\gamma^q) = \frac{1}{2} \sqrt{\frac{\gamma^q_{\text{env}}}{\gamma^p_{\text{env}}} + \gamma^q_{\text{env}}, \gamma^q = \frac{1}{2}} > 0 \tag{A35} \]
for all values of \( \gamma^q_{\text{env}}, \gamma^p_{\text{env}} \). We derive \( T_1(\gamma^q, \gamma^p) \) with respect to \( \gamma^q_{\text{env}}, \gamma^p_{\text{env}} \) and we find that
\[ \partial T_1(\gamma^q, \gamma^p)/\partial \gamma^q_{\text{env}} < 0 \] and \( \partial T_1(\gamma^q, \gamma^p)/\partial \gamma^p_{\text{env}} > 0 \). Therefore, \( T_1 \) can be bounded from below by its value at the point where \( \gamma^q_{\text{env}} \) is maximal and \( \gamma^p_{\text{env}} \) is minimal from all possible values. This leads to
\[ T_1(\gamma^q, \gamma^p)\big|_{\gamma^q_{\text{env}} \to \infty, \gamma^p_{\text{env}} \to 0} \to 0, \]
where the limit is reached from above. This proves that \( T_1 > 0 \).

Now we show that \( T_2 > 0 \) as well. First, we simplify \( T_2 \) by taking out positive pre factors and dropping positive terms to
\[ \beta(\nu_{\text{out}})\left(\frac{\gamma^q_{\text{inh}}}{16}\right)^3 \left( \frac{(\gamma^p_{\text{env}})^2}{\gamma^q_{\text{env}}} - \frac{(\gamma^q_{\text{env}})^2}{16(\gamma^p_{\text{env}})^4} \right) + \frac{4 + \beta(\nu_{\text{out}})}{16} + \gamma^p_{\text{env}} \gamma^q_{\text{env}} \left( \frac{\gamma^p_{\text{env}}}{4} + \frac{2\beta(\nu_{\text{out}})}{4} \frac{\gamma^q_{\text{env}}}{4(\gamma^p_{\text{env}})^2} \right), \tag{A36} \]

where
\[ \beta(\nu_{\text{out}}) = \nu_{\text{out}}^2 - \frac{3}{2} < 0. \tag{A37} \]

From (A36) we easily conclude that \( T_2 \) is certainly positive if
\[ \beta(\nu_{\text{out}}) \geq -2, \Rightarrow \nu_{\text{out}} \geq \nu_{\text{crit}} = 0.63. \tag{A38} \]

Below this value we will use a different approach. We can rewrite \( T_2 \) as
\[ T_2 = \xi \left( x^2 - 1 \right) \left( \frac{g''(\nu_{\text{out}} - \frac{1}{2})}{g'(\nu_{\text{out}} - \frac{1}{2})} \nu_{\text{out}} - 1 \right) + \frac{8\nu_{\text{out}}^{2}x}{\nu_{\text{env}}} \tag{A39} \]

where
\[ x = \frac{2\gamma^q}{s}, \quad s = \sqrt{\frac{\gamma^q_{\text{env}}}{\gamma^p_{\text{env}}}}, \tag{A40} \]
\[ \xi = \frac{\nu_{\text{out}}\nu_{\text{env}}}{x^2 g'(\nu_{\text{out}} - \frac{1}{2})} > 0. \]

Since \( \nu_{\text{out}} \) depends only on \( x \) and \( \nu_{\text{env}} \) and since it is bounded from above by \( \nu_{\text{max}} \) we can find the values of \( x \) for which this maximum is attained. This leads to the boundaries
\[ x_{\text{min}} < x \leq 1, \]
\[ x_{\text{min}} = \frac{\nu^2_{\text{max}} - \nu^2_{\text{env}}}{\nu_{\text{env}}} - \frac{1}{4} - \frac{\nu^2_{\text{max}} - \nu^2_{\text{env}} - \frac{1}{4}}{\nu_{\text{env}}} - 1. \tag{A41} \]

where \( x = 1 \) corresponds to the value of \( \gamma^q_{\text{in}} \) at quantum water-filling. It is easy to check that \( x_{\text{min}} \) is monotonously increasing in \( \nu_{\text{env}} \) and taking account that \( \nu_{\text{env}} > 0 \) we find the interval \( 0 < \nu_{\text{env}} < \nu_{\text{max}} - 1/2 \) where at \( \nu_{\text{env}} = \nu_{\text{max}} - 1/2 \) the bound \( x = 1 \) is reached. Thus, we bounded the valid domain for \( x \) and \( \nu_{\text{env}} \). We find that in this domain the maximum of \( T_2 \) lies in the irregular limit when both \( x \) and \( \nu_{\text{env}} \) go to zero. In the following we treat this limit along the lines \( x = k\nu_{\text{env}} \) for positive \( k \). We note that the derivative of \( x_{\text{min}} \) with respect to \( \nu_{\text{env}} \) at \( \nu_{\text{env}} = 0 \) under the condition \( \nu_{\text{max}} > 1/2 \) provides the lower bound on \( k \), i.e.
\[ \lim_{\nu_{\text{env}} \to 0} \frac{d}{d\nu_{\text{env}}} x_{\text{min}} = \frac{2}{4\nu^2_{\text{max}} - 1}, \quad \nu_{\text{max}} > \frac{1}{2} \tag{A42} \]

The latter is a decreasing function of \( \nu_{\text{max}} \in [0, \nu_{\text{crit}}] \) and is bounded by 3.4. Thus, this interval of \( \nu_{\text{max}} \) corresponds to \( k > 3.4 \). Finally, we evaluate the limit
\[ \lim_{\nu_{\text{env}} \to 0, x \to 3.4\nu_{\text{env}}} T_2 = 0.27. \tag{A43} \]

Thus, \( T_2 \) is greater then zero which proves the desired inequality (A29) which means that (A24) holds and thus, the desired inequality (A21) is proved. Additionally by combining Eqs. (A26), (A28) and (A29) we conclude that
\[ \frac{d\gamma^q}{dx} > 0. \tag{A44} \]

This means that the anti-squeezing in the more noisy quadrature is always increasing until the squeezing value at \( \lambda = \lambda_{\text{thr}} \) is reached.
Appendix B: Concavity of the Holevo $\chi$-quantity in $\lambda$

In the following we will verify that the Holevo $\chi$-quantity \cite{10} is a concave function of $\lambda$.

For $\lambda \geq \lambda_{thr}$ we find that $\gamma_{in}^q$ is given by \cite{19} and independent of $\lambda$ and therefore, from Eq. (18) that on the solution $\chi$ is a function of only one variable $\lambda$. Then, at the extremum of $L$ which reads has to take into account this dependence. Now this reads

$$\frac{d^2\chi}{d\lambda^2} = \frac{\partial^2\chi}{\partial^2\lambda} = \frac{\gamma''(\tau_{\text{sat}} - \frac{1}{2})}{4} < 0.$$ \hspace{1cm} (B1)

Thus, we have shown that above the threshold $\chi$ is a concave function of $\lambda$.

For an input energy $\lambda$ below the threshold, $\gamma_{in}^q$ depends on $\lambda$ via the implicit function given by Eq. (22). Therefore, the total second derivative of $\chi$ with respect to $\lambda$ has to take into account this dependence. Now this reads

$$\frac{d^2\chi}{d\lambda^2} = \frac{\partial^2\chi}{\partial^2\lambda} + \frac{\partial^2\chi}{\partial\gamma_{in}^q} \frac{d^2\gamma_{in}^q}{d\lambda^2} + \partial\chi \frac{d^2\gamma_{in}^q}{d\lambda^2} + \left( \frac{\partial^2\chi}{\partial\lambda^2} + \frac{\partial^2\chi}{\partial\gamma_{in}^q} \frac{d\gamma_{in}^q}{d\lambda} \right) \frac{d\gamma_{in}^q}{d\lambda}.$$ \hspace{1cm} (B2)

One can easily show using Eq. (A4) that $\partial\chi/\partial\lambda = \mu$ and by Eq. (A25) it follows that $\partial\chi/\partial\gamma_{in}^q = F$. Thus, Eq. (B2) simplifies on the solution to

$$\frac{d^2\chi}{d\lambda^2} = \frac{d\mu}{d\lambda} < 0,$$ \hspace{1cm} (B3)

as proven in appendix A.3. Thus, we proved that $\chi$ is a concave function of $\lambda$ in the full input energy domain.

Appendix C: Eigenvectors and Eigenvalues of Toeplitz matrices

We remark that all Toeplitz matrices that belong to the Wiener class commute asymptotically, because in this limit they commute with circulant matrices which all commute between each other (using \cite{28} and \cite{29}). We recall that a circulant matrix $A$ with dimension $n \times n$ is defined as

$$A_{ij} = a_{i-j \mod n}.$$ 

Therefore, we can introduce the notation $k = i - j \mod n$ which indicates the $k$th diagonal of $A$. From \cite{28} we state, that the eigenvalues of $A$ are

$$\gamma_{in,k}^{q,p}(x) = \sum_{k=0}^{n-1} a_k e^{-i 2\pi mk/n}, \quad m = 1, 2, \ldots, n,$$

If we take the limit $n \to \infty$ the latter becomes the valid solution for the eigenvalue spectrum of all Toeplitz matrices (that belong to the Wiener class).

This means that the optimal input spectrum is

$$\gamma_{in,k}^{q,p}(x) = \sum_{k=0}^{\infty} \gamma_{in,k} e^{-i kx}, \quad x \in [0, 2\pi],$$

where $\gamma_{in,k}$ is the $k$th diagonal of the optimal input covariance matrix (and the Fourier coefficient of a Fourier series) in the original basis. Since $\gamma_{in,k}^{q,p}(x)$ is Riemann integrable we conclude that

$$\gamma_{in,k} = \frac{1}{2\pi} \int_{0}^{2\pi} dx e^{i kx} \gamma_{in,k}^{q,p}(x), \quad k = 0, 1, 2, \ldots, \infty.$$ \hspace{1cm} (C1)