3D extension of Steiner chains problem

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Abstract

A natural 3D extension of the Steiner chains problem, original to the authors of this article, where circles are substituted by spheres, is presented. Given three spheres such that either two of them are contained in (or intersect) the third one, chains of spheres, each one externally tangent to its two neighbors in the chain and to the first and second given spheres, and internally tangent to the third given sphere, are considered. A condition for these chains to be closed has been stated and the Steiner alternative or Steiner porism has been extended to 3D. Remarkably, the process is of symbolic-numeric nature. Using a computer algebra system is almost a must, because a theorem in the constructive theory in the background requires using the explicit general solution of a non-linear algebraic system. However, obtaining a particular solution requires computing concatenated processes involving trigonometric expressions. In this case, it is recommended to use approximated calculations to avoid obtaining huge expressions.

Keywords: Steiner chains of spheres; 3D Steiner alternative; Steiner porism; Soddy’s Hexlet; Computer algebra

1. Introduction

As our goal is to extend the Steiner classic chains problem to 3D, we shall begin with a summary of the well-known 2D case.

1.1. Steiner classic chains problem in 2D

Suppose we are given two non-intersecting circles, \( \alpha \) and \( \gamma \), one inside the other. We can then insert a finite chain of circles that are tangent to the given circles \( \alpha \) and \( \gamma \), and such that each circle of the chain is tangent to its two neighbors. Depending on the relative size and position of \( \alpha \) and \( \gamma \), the last circle of the chain can be tangent to the first one. Then the chain is said to be closed (Fig. 1).

The great nineteenth-century geometer Jakob Steiner discovered that, for two circles \( \alpha \) and \( \gamma \), if one such closed chain of \( n \) circles exists, then any other chain of \( n \) circles for \( \alpha \) and \( \gamma \) is also closed, that is, it does not depend on the place where the chain is started. This property is called a Steiner alternative or Steiner porism.

This result follows from the fact that \( \alpha \) and \( \gamma \) can be transformed into two concentric circles (Fig. 2), using an appropriate transformation: the “inversion”, discovered by Steiner. That is why the configuration considered above
1.2. Soddy’s Hexlet and other related 3D problems

For the sake of brevity, we shall hereinafter use the word “spheres” to denote what should be denoted as “spherical surfaces”.

A related work is the extension to 3D of the Apollonius problem. The extension allows one to determine the spheres that are tangent to four given spheres. We solved this problem in general (see [5]). Another 3D extension of the Apollonius problem, that considers ellipsoids instead of spheres and uses Dixon’s resultant, can be found in [6].

Soddy’s Hexlet is a very particular extension to 3D of the Steiner chains problem. Three spheres are given, two of them contained in the third one and each one tangent to the two other spheres. With this configuration, chains of tangent spheres that are externally tangent to the first and second given spheres and internally tangent to the third one are to be found. This problem is named after Sir Frederick Soddy, who solved it in the early twentieth century [7]. Curiously, all the closed chains consist exactly of six spheres, as explained in [3]. Amusing implementations of this configuration can be found on the web pages [8,9].

Descartes’ circle theorem [10] is the first work that relates the radii of mutually tangent circles. Soddy extended it to 3D [7] and Gosset extended it to the $n$-dimensional case [11]. Coxeter later extended it to spiral sequences of spheres [12].

We can also reference yet another work by Coxeter [13]. A ring of spheres consists of a sequence of spheres, such that every one touches the next one, and it may happen to be finite, i.e., cyclic. The study focuses on the existence of two finite interlocked rings of spheres, such that each sphere in a ring touches every sphere in the other ring. A particular case is Soddy’s Hexlet, consisting of two rings of 3 and 6 spheres, respectively. The recent work [14] follows this line of research.

1.3. Extending Steiner classic chains problem to 3D

We think the natural extension to 3D of the Steiner classic chains problem is the following: given three spheres, obtain chains of tangent spheres that are externally tangent to the first and second given spheres and internally tangent to the third given sphere.
Observe that in the 2D case described in Section 1.1, one circle must be interior to the other because no Steiner closed chain would exist otherwise. Therefore, the conditions for the 3D case are somehow more general than those of the 2D case.

The main goals of this work are to obtain hypotheses under which the existence of solutions can be assured and to design and implement a general constructive process to find such solutions.

In our extension, unlike in Soddy’s Hexlet and Coxeter’s interlocked rings, the two inner spheres can be not only tangent to each other, but one can also lie strictly outside the other or they can be secant. Moreover, these two spheres can be tangent, secant, or they can lie strictly inside the third sphere (see Fig. 3). Let us remember that Soddy’s Hexlet had a chemical (molecular) origin and, therefore, secant spheres were not considered. Finally, in our extension, the closed chains do not necessarily consist of six spheres, as occurs in Soddy’s Hexlet.

Summarizing, we think that the problem we have studied is the natural extension to 3D of the Steiner chains problem in 2D, whereas Soddy’s Hexlet and Coxeter’s interlocked rings consider the relation between two dual chains of spheres. That is why the tangency conditions in our extension can be much weaker.

2. In depth details about the problem in 3D

Three spheres must be given and two of them are either contained in the third one or intersect the third one. Now, chains of tangent spheres that are externally tangent to the first and second given spheres and internally tangent to the third sphere are to be found (see Fig. 3).

2.1. Conditions for the three given spheres

Let us denote the three given spheres by $\alpha$, $\beta$, $\gamma$, their centers by $A$, $B$, $C$ (respectively) and their radii by $r_a$, $r_b$, $r_c$ (respectively).

The following conditions are imposed in order to assure that the Steiner chains are closed (after one or more cycles) or that, at the very least, they do overlap (see Fig. 3):

(i) Each of the two first spheres, $\alpha$ and $\beta$, has to lie inside, be internally tangent or secant to the third sphere $\gamma$ (that is, they can neither lie outside nor be externally tangent to $\gamma$).

(ii) The two first spheres, $\alpha$ and $\beta$, have to lie outside, be externally tangent or be secant to each other (that is, they can neither be internally tangent to each other nor can one lie inside the other).

(iii) If $\alpha$ and $\beta$ share points, these points must be the interior of $\gamma$.

(iv) If $\alpha$ and $\beta$ lie strictly outside each other, then their intersection points with segment $\overline{AB}$ (whose endpoints are their centers $A$ and $B$) must be in the interior of $\gamma$.

These four conditions can be summarized as follows (where $\hat{\gamma}$ denotes the interior of $\gamma$):

\[
\begin{align*}
\text{dist}(A, C) &< r_a + r_c \\
\text{dist}(B, C) &< r_b + r_c \\
\text{dist}(A, B) &> |r_a - r_b| \\
\alpha \cap \beta &\neq \emptyset \implies \alpha \cap \beta \subset \hat{\gamma} \\
\alpha \cap \beta &= \emptyset \implies \alpha \cap \overline{AB}, \beta \cap \overline{AB} \in \hat{\gamma}.
\end{align*}
\]
2.2. Defining 3D Steiner chains

Let us define the concept of a Steiner chain in the 3D case.

**Definition 1.** Let $\alpha$, $\beta$ and $\gamma$ be three spheres, which satisfy the conditions in (1). Then a Steiner chain of spheres for $\alpha$, $\beta$, $\gamma$ is a finite sequence of spheres, $\sigma_1, \sigma_2, \ldots, \sigma_n$ ($n > 1$), that satisfies the following four conditions:

1. $\forall i \in \{1, 2, \ldots, n\}, \sigma_i$ is externally tangent to $\alpha$ and to $\beta$
2. $\forall i \in \{1, 2, \ldots, n\}, \sigma_i$ is internally tangent to $\gamma$
3. $\forall i \in \{1, 2, \ldots, n - 1\}, \sigma_{i+1}$ is externally tangent to $\sigma_i$
4. $\forall i, j \in \{1, 2, \ldots, n\}, i \neq j$: $\sigma_i \neq \sigma_j$.

In the particular case that $\sigma_n$ is externally tangent to $\sigma_1$, then it is said that this is a closed chain of $n$ spheres ($n$ is defined to be the smallest integer for which these conditions hold).

As a consequence, if $S$ is the center and $r_s$ is the radius of any of the spheres $\sigma$ in the chain, the following three conditions hold (Fig. 4):

$$\begin{cases} 
\text{dist}(S, A) = r_a + r_s \\
\text{dist}(S, B) = r_b + r_s \\
\text{dist}(S, C) = r_c - r_s.
\end{cases}$$

(2)

In order to determine the relative positions of the centers of the three given spheres, we consider point $C'$, the orthogonal projection of $C$ on line $AB$ (Fig. 5). The relative positions are intrinsically determined by the following...
In order to calculate the coordinates in 3D Euclidean real space, it is recommended to choose an appropriate reference system with origin \( A \), whose first coordinate axis passes through \( B \), whose second coordinate axis is parallel to the ray \( AB \) and whose third coordinate axis is perpendicular to the two first coordinate axes. If \( C \) lies on line \( AB \), then any perpendicular line to \( AB \) passing through \( A \) can be chosen as the second coordinate axis.

Denoting \( \text{dist}(A, B) = b \), \( \text{dist}_0(A, C') = c \), \( \text{dist}(C, C') = f \), the coordinates of the centers of the three given spheres are: \( A = (0, 0, 0) \), \( B = (b, 0, 0) \), \( C = (c, f, 0) \).

Each sphere will be codified by a list of two elements. The first one is the list of the coordinates of its center and the second one is the length of its radius. This way, the three given spheres will be codified as follows: \( \alpha \) by list \([0, 0, 0], r_\alpha \), \( \beta \) by list \([b, 0, 0], r_\beta \) and \( \gamma \) by list \([c, f, 0], r_\gamma \).

3. Generating 3D Steiner chains

To solve the problem, we first reduce the problem to the simpler case where the centers of the three given spheres are collinear, using an appropriate 3D inversion (spherical inversion).

3.1. Case I: Collinear centers

If \( A, B, C \) are collinear points then \( f = 0 \) (\( C' = C \)). The configuration of the three given spheres has radial symmetry (whose axis is the line through the centers \( AB \)). Consequently, all spheres that satisfy the first two conditions of Definition 1 will have their centers on the same plane, \( \omega \), perpendicular to line \( AB \). More precisely, the centers of such spheres will be on the same circumference \( \delta \) with center at the point \( O \) at the intersection of line \( AB \) and its perpendicular plane \( \omega \). All spheres satisfying these two conditions will have the same radii, \( r_\delta \). Therefore, this will hold for all spheres in the chain.

Let us denote by \( R \) the radius of circumference \( \delta \). There are infinite spheres that satisfy the tangency conditions in (2). They all have their centers on the circumference \( \delta \). Let us consider among them, for the sake of simplicity, circumference \( \sigma \), whose center, \( S \), is on the plane containing the first two coordinate axes of the reference system mentioned in Section 2.2. This way, the coordinates of points \( O \) and \( S \) with respect to this reference system will be \( O = (d, 0, 0), S = (d, R, 0) \), where \( d = \text{dist}_0(A, \omega) \). Consequently, the sphere \( \sigma \) will be codified by the list \([d, R, 0], r_\delta \).

Initially, what we have to do is to obtain the values of \( d, R \), and \( r_\delta \) (that determine the sphere \( \sigma \)) as a function of the data that determine the three given spheres, that is, as a function of the parameters \( b, c, r_\alpha, r_\beta, \) and \( r_\gamma \). To achieve this, it is enough to consider the tangency conditions (2), that can be algebraically translated into the three following conditions:

\[
\begin{align*}
\frac{d^2 + R^2 - (r_\alpha + r_\delta)^2}{2(br_\alpha + br_\gamma - r_\alpha c + r_\gamma c)} & = 0 \\
\frac{b^2 - 2bd + d^2 + R^2 - (r_\beta + r_\delta)^2}{2bc + c^2 + 2R^2 - (c - r_\delta)^2} & = 0 \\
\frac{c^2 - 2cd + d^2 + R^2 - (c - r_\delta)^2}{2bc + c^2 + 2R^2 - (c - r_\delta)^2} & = 0.
\end{align*}
\]

Solving symbolically this non-linear algebraic system (with the help of a computer algebra system that can compute Gröbner bases [15]) and factorizing the solutions (again with the help of a computer algebra system), the following expressions are obtained (Maple [16,17] was the computer algebra system used):

\[
\begin{align*}
\text{dist}(A, B) & = d = \frac{-c^2r_\beta + c^2r_\gamma + r_\alpha^2r_\beta + r_\alpha^2r_\gamma - r_\alpha^2b^2 + r_\alpha^2c^2 - r_\alpha^2r_\beta + r_\gamma b^2 + r_\gamma c^2 - r_\gamma r_\beta^2 + r_\gamma c^2 - r_\gamma r_\beta^2}{2(br_\alpha + br_\gamma - r_\alpha c + r_\gamma c)} \\
\text{dist}(C, C') / \text{dist}_0(A, C') & = R = \frac{\sqrt{(c^2 - (r_\gamma + r_\alpha)^2)(b^2 - (r_\beta - r_\alpha)^2)((b - c)^2 - (r\beta + r_\gamma)^2)}}{2(br_\alpha + br_\gamma - r_\alpha c + r_\gamma c)} \\
\text{dist}(C, C') / \text{dist}_0(A, C') & = r_\delta = \frac{-bc^2 - br_\alpha^2 + br_\alpha^2 + b^2c + r_\alpha^2c - r_\beta^2c}{2(br_\alpha + br_\gamma - r_\alpha c + r_\gamma c)}.
\end{align*}
\]

Observe that \( R \) is given by a rational expression whose numerator is the square root of a degree 4 polynomial in each one of the five parameters, while its denominator is the same as in the rational expressions for \( r_\delta \) and \( d \).
Fig. 6. Oriented dihedron in order to determine the center S₁.

These results can be summarized in Theorem 1.

**Theorem 1.** Let α, β, γ be three spheres, rₐ, rₐ, rₖ their radii (respectively), A, B, C their collinear centers (respectively) such that dist(A, B) = b ≠ 0 and the oriented distance from A to C (in the direction from A towards B) is c. Their Steiner chains are formed by spheres whose centers lie on a circumference with the center point O lying on line AB, such that the oriented distance, d, from A towards O (in the direction from A towards B) is given by expression (3). This circumference is contained in the plane perpendicular to AB through O with its radius R given by expression (4). All spheres in the chain have the same radius, rₖ, given by expression (5).

The first sphere in the chain, σ₁, has, as do all the spheres in the chain, radius rₖ, therefore being determined by its center, S₁, that lies on the circumference δ. In order to determine S₁ in δ, an oriented convex dihedron will be considered (Fig. 6). Its faces are two halfplanes bounded by the first axis of the reference system (that is, the axis forms the edge of the dihedron). Its initial face contains the second axis of the reference system and its end face contains the center S₁. We shall denote by φ₁ the measure of this dihedron (the orientation from the second axis towards the third axis of the reference system is considered to be in the positive direction). According to this agreement, S₁ is the sphere with center point S₁ (of coordinates [d, R cos(φ₁), R sin(φ₁)]) and radius rₖ, that is, the sphere codified as [[d, R cos(φ₁), R sin(φ₁)], rₖ].

Once a sphere of the chain, σ (with center S), is determined, to calculate the following sphere in the chain, σ', whose center will be denoted S', it is enough to observe that OSS' is an isosceles triangle with two sides of length R and the third side of length 2rₖ (Fig. 7). Therefore, the measure, φ, of angle SOS' will be given by

\[ φ = 2 \arcsin \left( \frac{rₖ}{R} \right). \tag{6} \]

So, S' will be the image of S in the rotation of the axis line AB and angle φ. Consequently, supposing that the center S has coordinates (d, e, h), the coordinates of the center S' will be (d, e cos(φ) − h sin(φ), e sin(φ) + h cos(φ)).

Finally, a chain of n spheres will be closed if and only if \( \frac{nπ}{2} \) is an integer. Such an integer, N, will be the number of cycles (full turns the chain gives until it is closed). Therefore, taking (6) into account, the condition for a chain of n spheres and N cycles to be closed is \( \arcsin \left( \frac{rₖ}{R} \right) n = π N \) or, what is equivalent \( \frac{rₖ}{R} = \sin \left( \frac{π N}{n} \right) \) which, taking (4) and (5) into account, leads to Theorem 2.

**Theorem 2.** Let α, β, γ be three spheres, rₐ, rₐ, rₖ their radii (respectively), A, B, C their collinear centers (respectively) such that dist(A, B) = b ≠ 0 and the oriented distance from A to C (in the direction from A towards B) is c. A Steiner chain of n spheres for α, β, γ is closed, after going over N cycles, if and only if their parameters satisfy:

\[ \sqrt{(c^2 - (rₖ + rₐ)^2)(b^2 - (rₖ - rₐ)^2)((b - c)^2 - (rₖ + rₖ)^2)} \cdot \sin \left( \frac{π N}{n} \right) \]

\[ = -bc^2 - bₐ^2 + bₖ^2 + b₂c + rₐ^2c - rₖ^2c. \tag{7} \]
For a chain with 11 spheres, closed after one cycle, determined by the spheres
Let us underline that the values assigned to the parameters
returns an imaginary value for angle
Fig. 8
Definition 1
leads to an equation of degree 4 in $r_c$, whose solutions are given by very long symbolic expressions that do not contribute any interesting information and thus are omitted for the sake of brevity.

The process is implemented in Maple and can be solved in exact arithmetic. Nevertheless, it is normally more convenient to approximate the solutions or to work in floating point arithmetic in order to avoid the huge expressions that are usually obtained (the reason that these huge expressions appear, is because trigonometric expressions do not normally simplify and their computations are carried along).

Example 1. For a chain with 11 spheres, closed after one cycle, determined by the spheres $\alpha$, $\beta$, $\gamma$ with collinear centers $A$, $B$, $C$, such that $\text{dist}(A, B) = 10$, $\text{dist}(A, C) = 6$, radius($\alpha$) = 8, radius($\beta$) = 9, that is, for $b = 10$, $c = 6$, $r_a = 8$, $r_b = 9$, $n = 11$, $N = 1$, the previous equation is reduced to
\[
\sqrt{-99(14 + r_c)(2 + r_c)(r_c + 13)(-r_c - 5)} \sin(\pi/11) = -502 + 10r_c^2.
\]

The positive solution to this equation, approximated with 5 digits, is $r_c = 13.230$. For this value of $r_c$ and for an initial oriented angle $\phi_1 = 0$, the following closed Steiner chain is obtained (it is codified as mentioned in Section 2.2):

\[
\begin{align*}
\begin{cases}
[3.8641, 10.148, 0.1], 2.8592, \\
[3.8641, 8.5368, 5.4868], 2.8592, \\
[3.8641, 4.2148, 9.2314], 2.8592, \\
[3.8641, -1.4456, 10.045], 2.8592, \\
[3.8641, -6.6472, 7.6686], 2.8592, \\
[3.8641, -9.7381, 2.8571], 2.8592, \\
[3.8641, -9.7368, -2.8617], 2.8592, \\
[3.8641, -6.6436, -7.6718], 2.8592, \\
[3.8641, -1.4408, -10.046], 2.8592, \\
[3.8641, 4.2197, -9.2300], 2.8592, \\
[3.8641, 8.5402, -5.4831], 2.8592.
\end{cases}
\end{align*}
\]

The 11 spheres in this chain, together with the three given spheres $\alpha$, $\beta$, $\gamma$, are drawn in Fig. 8.

Remark 1. Let us underline that the values assigned to the parameters $b$, $c$, $r_a$, $r_b$, and $r_c$ could make $r_s > R$ (see Fig. 9), and in such a case the spheres $\sigma_i$ of the chain would be secant, contradicting condition 3 of Definition 1. Such an anomalous case can be easily detected because if $r_s > R$, then (6) returns an imaginary value for angle $\phi$.

3.2. Case II: Non-collinear centers

If the centers are non-collinear, the goal is to reduce to the previous case of collinear centers (as in the classic 2D case, the goal was to reduce to the case of concentric centers). In the 3D case, a transformation that converts spheres into spheres and preserves tangency, in order to transform $\alpha$, $\beta$ and $\gamma$ into three spheres of collinear centers, will
be used. An adequate 3D inversion does it. Unfortunately, the center of the inverse sphere is not the inverse of the center of the original sphere, which makes determining the right inversion more difficult. On the other hand, in order to simplify the number of spheres to be inverted, we can consider as the power of inversion the geometric power of the pole $P$ with respect to $\alpha$ (therefore $\alpha$ would be invariant or double in the inversion). Even better, if $P$ is chosen in the radical plane $\tau$ of spheres $\alpha$ and $\beta$ (that is, the locus of the points which have the same power with respect to $\alpha$ and $\beta$), then $\beta$ would also be double in the same inversion. Therefore, we have to determine such an inversion which allows us to reduce the problem to calculating the inverse of the third given sphere $\gamma$.

Theorem 3 gives conditions in which the existence and uniqueness of such an inversion can be assured.

**Theorem 3.** Let $\alpha$, $\beta$, $\gamma$ be three spheres verifying the conditions in (1). Then one and only one inversion $I$ exists such that: $I(\alpha) = \alpha$, $I(\beta) = \beta$, the center of $I(\gamma)$ lies on line $AB$ and the three spheres $I(\alpha)$, $I(\beta)$, $I(\gamma)$ also verify the conditions in (1).

The proof is completely geometrical and is omitted for the sake of brevity (it does not provide new information about the way the pole $P$ is determined below).

Let $e$ be the intersection line of plane $ABC$ and the radical plane $\tau$ of spheres $\alpha$ and $\beta$. Let $G$ be the intersection point of $e$ with line $AB$ and let $M$, $N$ be the intersection points of $\gamma$ with $e$ (Fig. 10). To simplify the configuration, the pole, $P$, of the inversion mentioned above will be chosen on line $e$. The coordinates of these new points, with respect to the coordinate system previously considered can be supposed to be: $G = (g, 0, 0)$, $P = (g, k, 0)$, $M = (g, m, 0)$, $N = (g, n, 0)$. According to this notation, the powers of $P$ with respect to $\alpha$ and $\beta$:

\[
\text{Power}(P, \alpha) = PA^2 - ra^2 = k^2 + g^2 - ra^2
\]
\[
\text{Power}(P, \beta) = PB^2 - rb^2 = k^2 + (b - g)^2 - rb^2
\]
are equal, as \( P \) is a point on the radical plane, \( \tau \), of both spheres, and therefore \( k^2 + g^2 - r_a^2 = k^2 + (b - g)^2 - r_b^2 \), from which:

\[
g = \frac{r_a^2 + b^2 - r_b^2}{2b}.
\]  \hspace{1cm} (8)

The value of the coordinate \( k \), of point \( P \) (\( P \in e \)), such that the sphere inverse of \( \gamma \) has its center on line \( AB \) is not yet determined. Let us consider the involutive transformation induced by inversion \( I \) on line \( e \). Let us denote by \( M' \) and \( N' \) the respective inverses of \( M \) and \( N \) in such an involution. On the other hand, \( I(\gamma) \) will have its center on line \( AB \) if and only if the two points \( M', N' \) are reflected points in the line \( AB \) (Fig. 10). In accordance with what was said above, we can suppose that: \( M' = (g, m', 0) \), \( N' = (g, n', 0) \). Summarizing, as \( M, M' \) are inverse points; \( N, N' \) are inverse points and \( M', N' \) are reflected points in the line \( AB \), the following three equalities must hold:

\[
\begin{align*}
(m - k)(m' - k) &= k^2 + g^2 - r_a^2 \\
(n - k)(n' - k) &= k^2 + g^2 - r_a^2 \\
m' + n' &= 0.
\end{align*}
\]  \hspace{1cm} (9)

If \( n' \) is substituted by \(-m'\) in the first two equations of (9), this system is reduced to a quadratic algebraic system in the unknowns \( k, m' \). Theorem 3 allows us to assure the existence of real solutions. Such solutions give the values of the unknowns \( k, m' \) (two values for each unknown). These values for \( k \) and \( m' \) determine the points \( P \) and \( M' \). Because of their length, the expressions for \( k \) and \( m' \) as a function of the parameters \( b, c, f, r_a, r_b \), and \( r_c \) are omitted.

The center, \( Q \), of the sphere \( I(\gamma) \) must lie on line \( AB \) (because \( M' \) and \( N' \) are reflected points in \( AB \) and must be aligned with the pole of inversion, \( P \), and the center \( C \) of \( \gamma \), and so \( Q \) is the intersection point of lines \( PC \) and \( AB \). Moreover, as \( Q \) is on the line \( AB \) and this line is the first coordinate axis in the reference system mentioned above, the coordinates of \( Q \) with respect to this line are of the form \((q, 0, 0)\).

This is all summarized in the following lemma.

**Lemma 1.** Let \( \alpha, \beta, \gamma \) be three spheres, \( r_a, r_b, r_c \) their radii (respectively), and \( A, B, C \) their non-collinear centers (respectively). Let us denote by \( C' \) the orthogonal projection of \( C \) on line \( AB \), \( \text{dist}(A, B) \) by \( h \), the oriented distance from \( A \) to \( C' \) (in the direction from \( A \) towards \( B \)) by \( c \) and \( \text{dist}(C, C') \) by \( f \). Let \( e \) be the intersection line of the radical plane of \( \alpha \) and \( \beta \) with plane \( ABC \), \( G \) be the intersection point of lines \( e \) and \( AB \) and \( M, N \) be the intersection points of \( e \) with the sphere \( \gamma \). Then a point, \( P \), on line \( e \), such that the images of \( M \) and \( N \) in the inversion, \( I \), of pole \( P \), for which the power of inversion is the geometric power of \( P \) with respect to sphere \( \alpha \), are two points, \( M' \) and \( N' \), whose midpoint \( G \) exists and can be calculated. The coordinates of \( P = (g, k, 0) \) and \( M' = (g, m', 0) \) are determined by (8)
and (9). The spheres $\alpha$ and $\beta$ are invariant by $I$. The sphere $I(\gamma)$ passes through $M'$ and its center is the intersection point of lines $PC$ and $AB$.

Once the inversion $I$ is determined and $I(\gamma)$ is calculated, a Steiner chain for $\alpha$, $\beta$, $I(\gamma)$ can be calculated as in Section 3.1. As an inversion transforms spheres into spheres and preserves dihedron angles, and, consequently, preserves tangency of spheres, the inverses of the spheres in a Steiner chain for $\alpha$, $\beta$, $I(\gamma)$ will form a Steiner chain for $\alpha$, $\beta$, $\gamma$. It is all summarized in the following theorem, that together with the previous lemma, allow us to build Steiner chains for three spheres with non-collinear centers.

Theorem 4. Let $\alpha$, $\beta$, $\gamma$ and $I$ be those of the previous lemma, and let $\sigma_1$, $\sigma_2$, $\ldots$, $\sigma_n$ be a Steiner chain for $\alpha$, $\beta$, $I(\gamma)$. Then $I(\sigma_1)$, $I(\sigma_2)$, $\ldots$, $I(\sigma_n)$ is a Steiner chain for $\alpha$, $\beta$, $\gamma$.

The process is also implemented in Maple. Again, it is normally more convenient to approximate the solutions or to work in floating point arithmetic in order to obtain a particular solution.

Example 2. From the three spheres of non-collinear centers where parametric values are $b = 6$, $c = 4.054$, $f = 1.295$, $r_a = 4$, $r_b = 3$, $r_c = 4.292$, $\phi_1 = 0$, the following closed Steiner 12-spheres chain is obtained with 4 digits of approximation and it is represented in Fig. 11

[[3.657, -2.530, 0], .4480], [[3.661, -2.397, .9078], .4764],
[[3.674, -1.953, 1.824], .5465], [[3.700, -1.072, 2.688], .6955],
[[3.743, 4995, 3.220], .9635], [[3.808, 2.715, 2.581], 1.342],
[[3.844, 4.020, -.01203], 1.560], [[3.806, 2.695, -2.592], 1.342],
[[3.743, .4850, -3.218], .9575], [[3.699, -1.074, -2.684], .6920],
[[3.674, -1.960, -1.816], .5460], [[3.661, -2.395, -.9007], .4677].

4. Alternative and envelope of closed chains

Returning to the case of collinear centers treated in Section 3.1, let us consider fixed values for the five parameters $b$, $c$, $r_a$, $r_b$, and $r_c$. If a chain is closed for a certain initial oriented angle, $\phi_1$, then, for any other value of $\phi_1$, the chain is also closed, as a consequence of the radial symmetry with respect to the line of centers $AB$.

For the case of non-collinear centers treated in Section 3.2, let us consider fixed values for the six parameters $b$, $c$, $f$, $r_a$, $r_b$, and $r_c$. If a chain is closed for a certain initial oriented angle, $\phi_1$, then for any other value of $\phi_1$ the chain is also closed, as a consequence that the inversion $I$ considered in Section 3 allows us to reduce this case to the previous one of collinear centers.

This result, well known for Steiner classic chains in 2D, where it is known as the Steiner alternative or Steiner porism, is therefore also true when extending to 3D. The result can be written as follows.
Fig. 12. Envelope surface of all Steiner chains.

**Theorem 5.** If for the given spheres $\alpha, \beta, \gamma$ (of collinear or non-collinear centers) a Steiner chain of $n$ spheres is closed, then all Steiner chains of $n$ spheres for $\alpha, \beta, \gamma$ are also closed.

Therefore, for the collinear centers case, the spheres of all closed chains will have the same radius, $r_s$, given by expression (5). As a consequence, the envelope surface of all closed chains is the surface of revolution obtained by rotating around the line of centers, $AB$, the circumference intersection of the sphere coded $[[d, R, 0], r_s]$ with the plane that, containing the line $AB$, passes through the center of this sphere. The equation of such a toric envelope surface can be obtained from the equation of the first sphere of the chain $\sigma_1$, whose initial oriented angle is a new parameter, $t$, by differentiating the equation of $\sigma_1$ with respect to $t$ and then eliminating $t$ from the equation of $\sigma_1$ and its derivative.

For the non-collinear case, the envelope of all possible chains for $\alpha, \beta, \gamma$ can be obtained by taking into account that the inversion is an isogonal transformation, through the following steps:

(i) reduce to the collinear case, using inversion $I$, of Section 3.2
(ii) calculate the toric revolution surface for the reduced (collinear) case
(iii) calculate the image of the toric surface of revolution by $I$.

**Example 3.** For the three spheres of non-collinear centers of the previous example, the envelope of all their Steiner chains is represented in Fig. 12. Its equation is expressed through a degree 8 polynomial with 85 terms (that is omitted for the sake of brevity).

5. Conclusions

In this work we have extended to 3D the Steiner classic chains problem, considering chains of tangent spheres that are tangent to three given spheres, $\alpha, \beta, \gamma$, in such a way that the spheres of the chain are inside $\gamma$ and outside $\alpha$ and $\beta$. We have stated the condition for a chain to be closed, and we have extended to 3D the alternative or porism due to Steiner. This analysis is, as far as we know, new.

The computer algebra system Maple has been used to compute some long algebraic calculations in the mathematical proof of the constructive processes. Moreover, the processes developed have been implemented in Maple and used (together with the 3D drawing system DPGraph) to obtain the 3D figures that illustrate the article. The corresponding Maple programs can be freely obtained from the authors.

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**References**