Extensive Measurement: Continuous Additive Utility Functions on Semigroups

Juan Carlos Candeal
Universidad de Zaragoza, Spain

and

Juan Ramón De Miguel and Esteban Indurain*
Universidad Pública de Navarra, Pamplona, Spain

We study continuity in extensive measurement which is related to the existence of continuous additive utility functions on semigroups. In the particular case of groups we obtain a continuous version of the classical Hölder’s theorem on Archimedeaness, and in the general case we introduce some conditions that guarantee additive representability, and then we analyze when they are good enough to get continuity in additive representations. Positive results are obtained when the structure considered is a topological totally ordered semigroup.

1. INTRODUCTION

One of the specially relevant problems in utility and measurement theory deals with the representability of a given ordered structure, say a total ordering or a complete preorder, through a real-valued numerical function called utility function.

For a particular case of total orderings, the above problem was posed and solved by Cantor (1895, 1897). Successive generalizations and solutions were given by Milgram (1939), Birkhoff (1940), Eilenberg (1941), Debreu (1954, 1964), Bowen (1968), Fishburn (1970), Jaffray (1975), and Wakker (1988), among others.

Concerning those generalizations, a prominent approach is given in the contribution due to Debreu (1964), who analyzed the question of representability of a total ordered set, endowed with the order topology, by means of a continuous utility function (considering the Euclidean induced topology on the real line \(\mathbb{R}\), that contains the range of the utility function). As a matter of fact, Debreu should be considered as one of the pioneers in dealing with topological questions concerning utility functions. (Other important papers within this topological approach are those by Eilenberg, 1941, and Nachbin, 1950, 1965.) A key result to solve the question posed by Debreu is his famous Debreu’s open gap lemma.

In a different framework searching for applications to physics, Hölder (1901) studied the question of representability of a totally ordered group by means of an order-preserving group homomorphism into the additive group of the reals \((\mathbb{R}, +, \leq)\). In other words: Hölder was looking for additive utility functions defined on a totally ordered group. The condition that characterizes this kind of representation is named the Archimedean property, a crucial property in expected utility theory (see, e.g., Fishburn, 1982) and measurement theory (see Falmagne, 1975; Roberts, 1979; Narens, 1985; or Luce & Narens 1992), where a very important particular case is the study of the measurability of utility. (See Chipman, 1960, pp. 215–221.) (Also, an interesting set of historical notes concerning the original Hölder’s ideas and applications to physics may be seen in Krull, 1960, Section 3.)

It is important to say that, in the aforementioned works, no explicit mention is made about the topology of Hölder’s theorem, in particular, about the continuity of the additive utility representations obtained via Hölder’s theorem. One may expect that the mere condition of totally ordered group should give rise to good topological properties when dealing with representation through utility functions.

In groups, this is indeed the case, and the confirmation of this fact is one of the objectives of the present paper.

First we obtain a refinement of the classical Hölder’s theorem by using the topological ideas of Debreu’s approach. In other words, we prove that the Archimedean property characterizes the representability through an additive and continuous utility function on groups.

In addition, this property of Archimedeaness, characteristic for totally ordered groups, is not sufficient to
guarantee the additive representability of totally ordered semigroups, even by means of discontinuous utility functions.

Fortunately, they are some other “Archimedean-like” conditions that characterize such an existence of additive representations. (See Alimov, 1950; Fuchs, 1963; Holman, 1969; Skala, 1974; Roberts, 1979; or De Miguel, Candelo, & Indurain, 1996.) Nevertheless, it may happen that even in the case of the existence of additive representations, these representations always fail to be continuous.

We will prove that a necessary condition to get continuity in additive representations is that the binary operation of the algebraic structure of a semigroup be continuous as regards the order topology. Finally, in this case of topological totally ordered semigroups we will show that several Archimedean-like conditions are equivalent to the existence of continuous additive representations.

2. INTRODUCTORY CONCEPTS AND RESULTS

Let \((X, \leq)\) be a totally ordered set (i.e., \(\leq\) is a reflexive, antisymmetric, transitive and complete binary relation defined on \(X\)). The notation \(x < y\) will mean “it is not true that \(y \leq x\).” We say that \((X, \leq)\) is representable (respectively, pseudo-representable) if there exists a map \(u: X \rightarrow \mathbb{R}\) such that \(x \leq y\) if and only if \(u(x) \leq u(y)\) (respectively, \(x \leq y \Rightarrow u(x) \leq u(y)\)), for every \(x, y \in X\). The map \(u\) is said to be a utility (respectively, pseudo-utility) function for \(\leq\). The relation \(<\) allows us to define a topology on \(X\), called order topology, that we shall denote by \(\mathcal{O}\), a subsbasis of which is given by the family of sets \(\{\langle a, b \rangle \mid a < b\}\} \cup \{(a, a) = \{x \in X \mid a < x\}\} \cup \{(a, b) \mid a, b \in X\}\).

A semigroup \((S, +)\) is a set endowed with a binary operation \(+\) that is associative. A semigroup \(S\) having a null element \(e\) such that \(x + e = x = e + x\) for every \(x \in S\) is said to be a monoid. If every element \(x\) of a monoid \(S\) has a converse \(-x\) such that \(x + (-x) = (-x) + x = e\) then \(S\) is said to be a group. A semigroup \((S, +)\) endowed with a total ordering \(\leq\) is said to be a totally ordered semigroup if the ordering \(\leq\) is translation-invariant (i.e., \(x + y = z\) \(\Rightarrow\) \(x \leq z\)) \(\leq\) \(x + z = y + z\) \(\Rightarrow\) \(y + z = x + z\) for every \(x, y, z \in S\). In particular, a totally ordered semigroup \(S\) is always cancellative, i.e., \(x + z = y + z\) \(\Rightarrow\) \(x = y\) \(\Rightarrow\) \(x + z = z + x\) for every \(x, y, z \in S\).

Given a totally ordered semigroup \((S, +, \leq)\), an element \(x \in S\) is said to be positive (respectively, negative) when \(y + x = y\) and also \(y = y < x\) (respectively, when \(x + y < y\) and also \(y < y < x\) for every \(y \in S\). Notice that an element \(x \in S\) is positive (respectively, negative) if and only \(x < x + x\) (respectively, \(x + x < x\)). The set of positive (respectively, negative) elements of \(S\) is said to be the positive cone of \(S\), denoted \(S^+\) (respectively, \(S^-\)). A simple exercise shows that these cones are stable in the following sense: If \(x, y \in S^+\) (respectively, \(S^-\)) then \(x + y, y + x \in S^+\) (respectively, \(S^-\)). Notice also that \(S\) can only have an element \(e\) that is neither positive nor negative. In this case \(e\) must be the null element for the operation \(+\), and \(S\) is, a fortiori, a monoid. Moreover, in this case it is clear that an element \(x\) is positive (respectively, negative) if and only if \(e < x\) (respectively, \(x < e\)).

A totally ordered semigroup \((S, +, \leq)\) is said to be:

(i) positive (respectively, negative) if it consists only of positive (respectively, negative) elements,

(ii) additively representable (respectively, pseudo-representable) if there exists a utility (respectively, pseudo-utility) function \(u\) for \(\leq\) that is an homomorphism (i.e.: \(u(x + y) = u(x) + u(y)\), for every \(x, y \in S\). The associated function \(u\) is said to be an additive utility (respectively: pseudo-utility) function.

A positive semigroup \((S, +, \leq)\) is said to be:

(i) Archimedean if for every \(x, y \in S\) with \(x < y\), there exists \(n \in \mathbb{N}\) such that \(y < n \cdot x\), (ii) super-Archimedean if for every \(x, y \in S\) such that \(x < y\) there exists \(n \in \mathbb{N}\) such that \((n + 1) \cdot x < n \cdot y\).

A totally ordered group is said to be Archimedean if its positive cone is Archimedean.

In the case of totally ordered groups, the existence of an additive representation was characterized by Hölder, early in 1901. This key result can be stated as follows.

**Lemma 1.** (Hölder, 1901). A totally ordered group \((G, +, \leq)\) is additively representable if and only if it is Archimedean.

**Proof.** See the original paper (Hölder, 1901) or (Birkhoff 1967, p 300).

**Remark 1.** In the case of positive semigroups Archimedeaness is not good enough to guarantee the additive representability. An example is the strictly positive cone \((0, \infty) \times (0, \infty)\) of the lexicographic plane \((\mathbb{R}^2, +, \leq_L)\), where the sum \(+\) is defined coordinatewise and the ordering \(\leq_L\), is given by \((a, b) \leq_L (c, d)\) if \(a < c\) or else \(a = c, b \leq d\). It is well known that this ordered set does not admit a utility representation, even nonadditive. (See Birkhoff, 1967, pp 200–201.)

In this framework of semigroups, there is also a characterization of additive representability:

**Lemma 2.** (a) The following statements are equivalent for a positive totally ordered semigroup \((S, +, \leq)\):

(i) \((S, +, \leq)\) is additively representable,

(ii) \((S, +, \leq)\) is super-Archimedean.

(b) A semigroup \((S, +, \leq)\) is additively representable if and only if its positive and negative cones are additively representable.
Proof. (a) was proved, essentially, in Alimov (1950). See also Fuchs (1963, pp. 167 and ff.), Skala (1974, pp. 88 and ff.), or Roberts (1979, Exercise 12 in Section 3.2). A complete proof of Lemma 2 may be seen in De Miguel et al. (1996).

For the sake of completeness let us see the main ideas involved along the proof. In order to prove the key implication (ii) \( \Rightarrow \) (i) of part (a), fix an element \( x_0 \in S \) and, given \( x \in S \), set \( u(x) = \text{sup\{}\{\min\{m, n \in \mathbb{N}, m \cdot x_0 < n \cdot x\}\} \). Following the proof of Hölder's theorem that appears in pages 300–301 of Birkhoff (1967), we obtain that \( u \) is an additive pseudo-utility. So, it only remains to check the injectivity of \( u \), and this comes from the fact of \( S \) being super-Archimedean: Observe that as \( x, y \in S \) such that \( x < y \), there exists \( n \in \mathbb{N} \) for which \( (n + 1) \cdot x < n \cdot y \Rightarrow (n + 1) \cdot u(x) \leq n \cdot u(y) \Rightarrow u(x) \leq u(y) \). Since that test \( u \) is the required additive utility function.

A totally ordered semifield \((S, +, \leq)\) is said to be super-Archimedean and also the negative cone \((S^-, +, \leq_{\text{op}})\) endowed with the converse ordering \(\leq_{\text{op}}\) defined by \(x \leq_{\text{op}} y \iff y \leq x \quad (x, y \in S)\), is super-Archimedean.

### 3. CONTINUITY IN EXTENSIVE MEASUREMENT ON TOTALLY ORDERED SEMIGROUPS

Following Narens (1985, pp. 73–102), a type of relational structures of fundamental importance to the theory of measurement are the positive concatenation structures. A particular case of such structures are those whose concatenation operations are representable by addition. Such structures are usually called “extensive” and they were the first ones to be systematically investigated by measurement theorists. A typical class of extensive structures, widely studied in the literature corresponds to the structures that have representations into \((0, + \infty), +, \leq)\). Some characterization of this class may be found in Falmagne (1971, 1975). However, since we usually want to have representations of the concatenation structures considered into the additive semifield of the positive real numbers \((0, + \infty), +, \leq)\), endowed with the usual order and Euclidean topology, it seems reasonable to start directly with a structure \((S, +, \leq)\), that is, a totally ordered positive semifield, and then to look for additive utility functions that represent such a structure. (See Chap. 2 in Narens, 1985, or Section 3.2 in Roberts, 1979, for further motivation.

**Remark 2.** It is natural to consider the particular case in which \( S \) is the positive cone of a group \( G \). For instance, Falmagne (1975) deals with the concept of an extensive system considered as being a structure \((S, +, \leq)\) representable by additive utility functions “\( u \)” that are regular, that is, \((u(S)\) must be the intersection of some real interval \( I \) and some subgroup \( G \) of the reals. This would motivate the study of additive utility functions on totally ordered groups. In this particular case of dealing with a group structure, the key for the existence of an additive utility representation is given by the well-known Hölder’s theorem. Unfortunately, not always can a given totally ordered semigroup be embedded in a totally ordered group. See Chehata (1953) or Fuchs (1963, p. 16). This definitively forces us to study the general case of totally ordered semigroups.

**Remark 3.** Working with semigroups we will not ask the additive utility functions to be “regular” in the sense of Falmagne (1975). This would be a restriction and it is not always used by measurement theorists. (See, for instance, Section 8 in Narens, 1985, or else Sections 2.2 and 3.2 in Roberts, 1979, where a less restrictive concept of regularity appears.) We will only deal with additive utility functions (not necessarily regular) on totally ordered semigroups. As an example, the set of real numbers \((A, +, \leq)\), where \( A = \{4, 6\} \cup [8, + \infty)\) is a semigroup without a regular (in the sense of Falmagne, 1975) additive utility representation. However, it admits additive utility functions that are continuous: The natural embedding of \( A \) in \((0, + \infty)\) is one of them.

In the general case of semigroups, there are several generalizations of Hölder’s theorem that provide characterizations for the existence of extensive measurement. (See, for instance, Roberts, 1979, problem 12 on p. 133.)

One of the main results in this paper deals with the continuity of the extensive measurement that comes from Hölder’s theorem, and will be obtained by matching two crucial results, namely, Debreu’s open gap lemma and of course Hölder’s theorem.

Let us now recall the statement of the “Open gap lemma.” (See Debreu, 1964; Bowen, 1968; Beardon, 1992; or Beardon & Mehta, 1994a, 1994b.)

**Definition.** Let \( \mathbb{R} \setminus \{0\} \cup \{\infty\} \) denote the extended real line. A degenerate set in \( \mathbb{R} \) is one having at most one element. A gap of a subset \( S \) of \( \mathbb{R} \) is a maximal nondegenerate interval disjoint from \( S \) and with a lower bound and an upper bound in \( S \). An interval of \( \mathbb{R} \) of the form \((a, b)\) or \([a, b)\) is said to be half-open half-closed.

**Debreu’s Open Gap Lemma.**

1. If \( S \) is a subset of \( \mathbb{R} \), there is an increasing function \( g: S \rightarrow \mathbb{R} \) such that all the gaps of \( g(S) \) are open.

2. Let \((a, b)\) be a totally ordered set representable by a utility function \( u: X \rightarrow \mathbb{R} \). If \( u(X) \) has no half-open half-closed gaps, then \( u \) is continuous as a map from \((X, \text{order topology})\) to \((\mathbb{R}, \text{Euclidean topology})\).
We are ready to introduce the main result on groups. First we need a preparatory lemma.

Chocquet’s Lemma. Let \((S, +, \leq)\) be a subgroup of the additive set of real numbers \((\mathbb{R}, +, \leq)\). Then one of the three excluding situations must hold:

(i) \(S = \{0\}\),

(ii) There exists some \(a \in (0, \infty)\) such that \(S = a \cdot \mathbb{Z} = \{a \cdot z : z \in \mathbb{Z}\}\),

(iii) \(S\) is dense in \(\mathbb{R}\); i.e., \(S\) meets every nonempty Euclidean open subset of \(\mathbb{R}\).


Remark 4. Observe that the only subgroups \((\mathbb{R}, +, \leq)\) of \((\mathbb{R}, +, \leq)\) that may have a gap are those that correspond to the second situation in Choquet’s lemma; that is, \(a \cdot \mathbb{Z} = \{a \cdot z : z \in \mathbb{Z}\}\) (for some \(a \in (0, \infty)\)). Moreover, in this case all the gaps are open subsets of \(\mathbb{R}\).

We obtain a topological extension of Hölder’s theorem on groups as an immediate consequence of Hölder’s theorem, Debreu’s open lemma, Choquet’s lemma, and Remark 4.

Theorem 1. Let \((G, +, \leq)\) be a totally ordered group. Then \((G, +, \leq)\) is representable through a continuous and additive utility function if and only if \((G, +, \leq)\) is Archimedean.

Remark 5. One may expect that the key property of Archimedeaness established in Theorem 1 above that in the case of totally ordered groups guarantees the existence of a continuous additive utility function, will be maintained for semigroups. Unfortunately things are no longer the same in this case. It follows from the discussion previous to the statement of Lemma 2 that Archimedeaness is not good enough to obtain additive utility representations (continuous or not) for totally ordered semigroups. A first condition that characterizes the additive representability of a totally ordered semigroup appeared in Alimov (1950). (See also Fuchs, 1963, Theorem 4, p. 167.) A similar condition was introduced by Holman (1969). Further studies about “Archimedean-like” conditions that characterize the additive representability of totally ordered semigroups have been made in De Miguel (1995), where it is shown that: \(Even\ being\ representable\ by\ an\ additive\ utility\ function,\ a\ semigroup\ of\ the\ reals\ could\ not\ admit\ a\ continuous\ additive\ utility\ representation.\ An\ example\ is\ the\ semigroup\ \(S = [2, 3) \cup [4, \infty)\) with the usual addition and ordering of the reals. The nub for the nonexistence of a continuous and additive utility function in this example is the discontinuity as regards the order topology of the algebraic operation +. In other words, \(S\) is not a “topological” totally ordered semigroup in the sense that the algebraic operation is not continuous as regards the order topology.

Let \((S, +, \leq)\) be a totally ordered semigroup. First we might notice that there is no topology given a priori on \(S\), except maybe the order topology. But, even endowed with the order topology, we do not know whether \((S, +, \leq)\) is a topological semigroup or not, in the sense of the following definition.

Definition. A topological semigroup \((S, +, \tau)\) is a semigroup \((S, +)\) endowed with a topology “\(\tau\)” that makes continuous the binary operation “\(+\)” \((x, y) \in S \times S \mapsto x + y \in S\). A totally ordered semigroup \((S, +, \leq)\) is said to be a topologically totally ordered semigroup if the binary operation “\(+\)” is continuous with respect to the order topology \(\tau\) on \(S\). Similarly, a topological group \((G, +, \tau)\) is a group \((G, +)\) endowed with a topology “\(\tau\)” that makes continuous the binary operations “\(+\)” \(G \times G \rightarrow G\), and “\(\cdot\)” \(G \rightarrow G\), given by \(\text{inv}(x) = -x\), for every \(x \in G\). So a topological totally ordered group is a totally ordered group \((G, +, \leq)\) such that + and inv are both \(\theta\)-continuous.

Remark 6. In the sense of the above definition, it seems natural, given a totally ordered semigroup \((S, +, \leq)\), to have a connection between the algebraic binary relation + and the topology \(\theta\) given by the ordering \(\leq\). Otherwise the structures \((S, +)\) and \((S, \theta)\) would have a disconnected meaning, and should be studied separately. Moreover, it is also natural to study the continuity of + as regards the order topology \(\theta\). This choice of a concrete topology, \(\theta\), is not whimsical, because the standard topologies to deal with when studying ordered structures use to be finer than the order topology. Indeed, the term “natural topology” was introduced in Debreu (1954) to refer to such topologies.

Remark 7. It is known that totally ordered groups are topological as regards the order topology. (See Nyikos & Reichel, 1975; or Fuchs, 1963, p. 33.) As was pointed out in Remark 6, the above property cannot be extended to totally ordered semigroups. In fact, a result by Clifford (1959) (see also Fuchs, 1963, pp. 176–177) states that if \((S, +, \leq)\) is a topological totally ordered semigroup then if \(A, B\) are nonempty subsets of \(S\) for which \(\sup A\) and \(\sup B\) (suprema of, respectively, \(A\) and \(B\)) exist, then \(\sup(A + B)\) also exists and \(\sup(A + B) = \sup A + \sup B\). (Here \(A + B = \{a + b : a \in A, b \in B\}\).) This does not happen in the semigroup provided as an example in Remark 5; observe that \(\sup(2 + (2, 3)) = \sup(4, 5) = 5 < 2 \sup(2, 3) = 2 + 4 = 6\).

In what follows, unless otherwise stated, the totally ordered semigroups we shall deal with will be topological totally ordered semigroups.

Let us prove that “to be topological” is a necessary condition for the existence of a continuous additive representation on totally ordered semigroups.

Proposition 1. Let \((S, +, \leq)\) be a totally ordered semigroup, additively representable by a continuous utility
function \( u : S \to \mathbb{R} \). Then \((S, + , \leq)\) is a topological semigroup as regards the order topology \(0\).

**Proof.** Since \(u\) is a continuous utility function it can be easily seen that the order and the Euclidean topologies coincide on \(u(S)\). Thus the result follows from the obvious fact that \((u(S), + )\) is a topological semigroup as regards the Euclidean topology. 

The following main question arises now: Let \((S, + , \leq)\) be a super-Archimedean topological totally ordered semigroup. Is \(S\) representable by a continuous utility function?

Using again Debreu’s open gap lemma, the strategy to deal with additive utility functions on totally ordered semigroups is now clear: If we are able to prove that an additive utility function never gives rise to half-open half-closed gaps, then that additive utility function will be continuous. With this idea in mind we introduce next Theorem 2.

**Theorem 2.** Let \((S, + , \leq)\) be a super-Archimedean topological totally ordered semigroup. Then \(S\) is representable by a continuous additive utility function.

**Proof.** We know, by Lemma 2, that there exists an additive utility function \(u : S \to \mathbb{R}\) that represents \((S, + , \leq)\). Let us prove that \(u\) must be continuous, showing that it does not give rise to half-open half-closed gaps in \(u(S)\): Assume, by contradiction, that there exists a gap in \((u(S), 0)\) of one of the following types:

(i) \( [a, b), a > 0, a < b, \)
(ii) \( (a, b], a > 0, a < b, \)
(iii) \( [a, b), b < 0, a < b, \)
(iv) \( (a, b], b < 0, a < b, \)
(v) \( [a, b), a \leq 0, b > 0, a < b, \)
(vi) \( (a, b], a \leq 0, b > 0, a < b, \)
(vii) \( [a, b), a < 0, b \geq 0, a < b, \)
(viii) \( (a, b], a < 0, b \geq 0, a < b, \)

In case (i) observe that \(a \not\in u(S), b \in u(S)\). Moreover, for every \(s \in u(S)\) with \(s < b\) it holds that \(s < a\). Also, every neighborhood of \(b\) in \((u(S), 0)\) must contain some element \(z\) such that \(z < a, z \in u(S)\). Let \(n_0(\cdot)\) be such that \((n_0 + 1) \cdot a < n_0 \cdot b\). Observe that there must exist \(s \in u(S)\) such that \(n_0 \cdot a < s < n_0 \cdot b\). Actually, if it were \((n_0 + 1) \cdot s \leq n_0 \cdot a\) for every \(s \in u(S)\) with \(s < a\), then we would have that \(s \leq (n_0/(n_0 + 1)) \cdot a < a\) for every \(s \in u(S)\) such that \(s < a\), in contradiction with the fact that \(a = \sup \{s \in u(S) ; s < a\}\). Thus \(n_0 \cdot a < (n_0 + 1) \cdot s < (n_0 + 1) \cdot a < n_0 \cdot b < (n_0 + 1) \cdot b\).

In particular \((n_0 + 1) \cdot s < n_0 \cdot b < (n_0 + 1) \cdot b\), with \((n_0 + 1) \cdot s, n_0 \cdot a, (n_0 + 1) \cdot b \in u(S)\). So \(\{(n_0 + 1) \cdot s, (n_0 + 1) \cdot b\} \) is a neighborhood of \(n_0 \cdot b\) in the order topology \((u(S), 0)\). Since \(S\) is a topological semigroup and \(u : (S, \theta) \to (u(S), 0)\) is a homeomorphism, it follows that \(+\) is continuous on \((u(S), 0)\). Then there exists a neighborhood \(E_0\) of \(b\) in \((u(S), 0)\) such that \(n_0 \cdot E_0 = \{n_0 \cdot z ; z \in E_0\} \subseteq ((n_0 + 1) \cdot s, (n_0 + 1) \cdot b)\). Therefore, \((n_0 + 1) \cdot s < n_0 \cdot b < (n_0 + 1) \cdot b\Rightarrow n_0 \cdot a < n_0 \cdot z < b < n_0 \cdot z < a < z < b\), for every \(z \in E_0\). This contradicts the fact that every neighbourhood of \(b\) must contain elements that are strictly smaller than \(a\).

Let us analyze case (ii) of the existence of a gap \([a, b)\), \(a > 0, b > 0\) in \((u(S), 0)\). It follows that \(a \in u(S), b \not\in u(S)\). Moreover, for every \(s \in u(S)\) with \(a < s\) it holds that \(b < s\). Since \(b \not\in u(S)\) and \(b = \inf \{x \in u(S) ; a < x\}\) there exists, in addition, an element \(s' \in u(S)\) such that \(b < s' < s\). Also, it is easy to see that every neighbourhood of \(a\) in \((u(S), 0)\) must contain some element \(z\) such that \(b < z, z \in u(S)\). In particular, \(a < 2 \cdot a \in u(S)\), so there exists \(s \in u(S)\) such that \(b < s < 2 \cdot a\). Let \(n_0 \in \mathbb{N}\) be such that \((n_0 + 1) \cdot a < n_0 \cdot b\). A fortiori \(n_0 \geq 1\) because \(2 \cdot a = (1 + 1) \cdot a\). Thus \(b < s < 2 \cdot a \Leftrightarrow b < n_0 \cdot a < (n_0 + 1) \cdot a < n_0 \cdot b\). In particular \(s < 2 \cdot a \leq n_0 \cdot a < (n_0 + 1) \cdot a\) with \(s, n_0 \cdot a, (n_0 + 1) \cdot a \in u(S)\). In other words, \((s, (n_0 + 1) \cdot a)\) is a neighbourhood of \(n_0 \cdot a\) in \((u(S), 0)\). Since + is continuous on \((u(S), 0)\) there exists a neighbourhood \(E_0\) of \(a\) in \((u(S), 0)\) such that \(n_0 \cdot E_0 \subseteq (s, (n_0 + 1) \cdot a)\). Therefore \(s < n_0 \cdot a < z < (n_0 + 1) \cdot a < a = n_0 \cdot b \Rightarrow n_0 \cdot b < z < b\), for every \(z \in E_0\). This contradicts the fact that every neighbourhood of \(a\) must contain elements that are strictly bigger than \(b\).

Cases (iii) and (iv) are similar to, respectively, cases (ii) and (i).

Let us analyze case (v) of the existence of a gap \([a, b)\), \(a \leq 0, b > 0\) in \((u(S), 0)\). In this case \(a \not\in u(S)\) but \(b \in u(S)\). Notice that \(b > 0 \Rightarrow a < b < a\), and \(a = \sup \{s \in u(S) ; x < a\}\). So there exists \(s \in u(S)\) such that \(a < b < b \Rightarrow a < b < a < b + a < b \Rightarrow a + b < b\) (because \(a \leq 0\)). Thus \(s + b \in [0, b)\). But this is a contradiction because \(s + b \in u(S)\). In consequence, the gaps corresponding to this case (v) can never appear.

Let us study now case (vi) of the existence of a gap \([a, b)\), \(a \leq 0, b > 0\) in \((u(S), 0)\). In this case \(a \in u(S)\) but \(b \not\in u(S)\). Observe that in every \(\theta\)-neighbourhood of \(a\) there exists some \(z \in u(S)\) such that \(b < z\). But \(b > 0 \Rightarrow b < 2b\), and \(b = \inf \{x \in u(S) ; b < x\}\). Hence there exists \(s \in u(S)\) such that \(b < s < 2b\). Now observe that \(2a \leq a < a = (-\infty, s)\) is \(\theta\)-neighbourhood of \(2a\). Thus, there exists a \(\theta\)-neighbourhood \(E\) of \(a\) such that \(E + E \subseteq (-\infty, s)\). But this implies \(2a \leq s < 2b\) for every \(z \in E\). Hence \(2a \leq 2b \Rightarrow z < b\), for every \(z \in E\), and we arrive to a contradiction with the definition of \(b\).

Cases (vii) and (viii) are similar to, respectively, cases (vi) and (v).

This concludes the proof. 

Matching Lemma 2, Proposition 1, and Theorem 2, we obtain the following characterization of the existence of continuous additive representations on topological totally ordered semigroups.

**Corollary.** A topological totally ordered semigroup is representable by a continuous additive utility function if and only if it is super-Archimedean.
Proof. It is an immediate consequence of Lemma 2 and Theorem 2.

Remark 8. The key for the continuity of the additive utility functions that appear in Theorem 2 above is the coincidence of the topologies $(u(S), 0)$ and $(u(S), \mathbb{E}uclidean)$. In Beardon (1994) this kind of issue has been studied in great depth in the context of continuously representable totally ordered subsets of Euclidean space.

ACKNOWLEDGMENT

Thanks are given to Professor Jean Claude Falmagne for his valuable suggestions and encouragement.

REFERENCES


Received: November 29, 1995