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JENKINS-SERRIN GRAPHS IN $M \times \mathbb{R}$

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ABSTRACT. The so called Jenkins-Serrin problem is a kind of Dirichlet problem for graphs with prescribed mean curvature that combines, at the same time, continuous boundary data with regions of the boundary where the boundary values explodes either to $+\infty$ or to $-\infty$. We give a survey on the development of Jenkins-Serrin type problems over domains on Riemannian manifolds. The existence of this type of graphs imposes restrictions on the geometry of the boundary of these domains. We also improve some earlier results by proving Theorem 1.8, and prove the existence of translating Jenkins-Serrin graphs (Theorem 1.9).

1. INTRODUCTION

Let (M, σ) be a Riemannian n -manifold (not necessarily complete) and let $\Omega \subset M$ be a domain whose boundary is piecewise smooth. The graph $\Sigma = \text{Graph}(u)$ of a smooth function $u: \Omega \rightarrow \mathbb{R}$ is a smooth hypersurface in the product manifold $M \times \mathbb{R}$ endowed with the product metric $g_0 := \sigma + dt^2$. Moreover, it is standard to check that if $H: M \rightarrow \mathbb{R}$ is a smooth function, then Σ has prescribed mean curvature H if and only if

$$\operatorname{div} \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = H, \quad (1.1)$$

for all $x \in \Omega$, where ∇ and div are the gradient and divergence with respect to σ .

In this paper we will work in the cases when either Σ is minimal, or Σ has constant mean curvature, or Σ is a translating soliton of the mean curvature flow. Moreover, we will adopt the convention that *the mean curvature is just the trace of the second fundamental form*.

Date: July 5, 2019.

E. S. Gama was supported by Capes/PDSE/88881.132464/2016-01. E. Heinonen was supported by the Finnish Academy of Science and Letters and by the Centre of Excellence in Analysis and Dynamics Research (Academy of Finland, decision 271983). J. H. de Lira was supported by PRONEX/FUNCAP/CNPq PR2-0101-00089.01.00-15. and CNPq/Edital Universal 409689/2016-5. F. Martín was partially supported by the MINECO/FEDER grant MTM2017-89677-P and by the Leverhulme Trust grant IN-2016-019.

First of all, let us explain what we mean by a Jenkins-Serrin solution of (1.1). Given any Riemannian manifold M of dimension $n \geq 2$, let $\Omega \subset M$ be a bounded Lipschitz domain whose boundary can be written as

$$\partial\Omega = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2 \cup N$$

where $\Gamma_0, \Gamma_1, \Gamma_2$ are open disjoint subsets of class C^1 and N is a closed subset such that the $(n-1)$ -dimensional Hausdorff measure $\mathcal{H}_{n-1}(N) = 0$. The Jenkins-Serrin solution of (1.1) with continuous data $f_0: \Gamma_0 \rightarrow \mathbb{R}$ is a smooth function $u: \Omega \rightarrow \mathbb{R}$ that satisfies (1.1), $u|_{\Gamma_0} = f_0$ and

- i. $u \rightarrow +\infty$ as $x \rightarrow \Gamma_1$;
- ii. $u \rightarrow -\infty$ as $x \rightarrow \Gamma_2$.

As we will see later, the conditions i and ii impose important restrictions on the extrinsic geometry of Γ_1 and Γ_2 , in general.

Probably, the best known example of Jenkins-Serrin solutions is given by the classical Scherk's minimal surface, that is the graph of the function $f(x, y) = \ln(\cos(x)/\cos(y))$ over the domain $\Omega = (-\pi/2, \pi/2) \times (-\pi/2, \pi/2)$. This idea of construction was generalized by H. Jenkins and J. Serrin in [14], but before stating their theorem, we need some notation. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain whose boundary consists of a finite number of open straight arcs A_1, \dots, A_r (the set Γ_1), B_1, \dots, B_s (the set Γ_2), and a finite number of convex arcs C_1, \dots, C_t (the set Γ_0) with respect to Ω and the end points of these arcs (the set N). Let \mathcal{P} be a polygon whose vertices are chosen among the vertices of Ω ; this polygon will be called *an admissible polygon*. We are going to use also the following notation:

$$\alpha(\mathcal{P}) = \sum_{A_i \subset \mathcal{P}} \|A_i\|, \quad \beta(\mathcal{P}) = \sum_{B_i \subset \mathcal{P}} \|B_i\| \quad \text{and} \quad \ell(\mathcal{P}) = \text{perimeter}(\mathcal{P}),$$

where $\|A\|$ denotes the arc length of A .

Theorem 1.1 (Jenkins-Serrin). *Let $\Omega \subset \mathbb{R}^2$ be as above. Assume also that no two arcs A_i and no two arcs B_i have a common endpoint. Given any continuous data $f_i: C_i \rightarrow \mathbb{R}$, then there exists a Jenkins-Serrin solution $u: \Omega \rightarrow \mathbb{R}$ for the minimal graph equation with continuous data $u|_{C_i} = f_i$ if, and only if, for any admissible polygon \mathcal{P} we have*

$$2\alpha(\mathcal{P}) < \ell(\mathcal{P}) \quad \text{and} \quad 2\beta(\mathcal{P}) < \ell(\mathcal{P}). \quad (1.2)$$

If $\{C_i\} = \emptyset$, we also require that $\alpha(\partial\Omega) = \beta(\partial\Omega)$ for $\mathcal{P} = \partial\Omega$. Moreover, if $\{C_i\} \neq \emptyset$, then u is unique, and if $\{C_i\} = \emptyset$, then u is unique up to adding a constant.

In [14], Jenkins and Serrin proved also that the existence of minimal Jenkins-Serrin solutions imposes several restrictions over the domain.

Theorem 1.2 (Jenkins-Serrin). *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain. Suppose that $u: \Omega \rightarrow \mathbb{R}$ is a solution of the minimal graph equation satisfying $u \rightarrow \pm\infty$ as $x \rightarrow \gamma$, where γ is a smooth connected component of $\partial\Omega$. Then γ is a geodesic.*

J. Spruck [26] extended Jenkins' and Serrin's work for the CMC case. He proved the existence of CMC Jenkins-Serrin solutions for very specific domains in \mathbb{R}^2 . The setting in [26] was the following:

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain whose boundary consists of finite number of open arcs $A_1, \dots, A_r, B_1, \dots, B_s, C_1, \dots, C_t$ and the endpoints of these arcs. Suppose that the curvatures of these arcs satisfy

$$\kappa(A_i) = H, \quad \kappa(B_i) = -H, \quad \kappa(C_i) \geq H. \quad (1.3)$$

Now, we say that \mathcal{P} is *admissible* if \mathcal{P} is a simple closed curvilinear polygon whose sides consists of circular arcs of curvature $\kappa = \pm H$ and whose vertices belong to the set of end points of the arcs $\{A_i\}$ and $\{B_i\}$. As before, α and β , respectively, denotes the total length of the arcs A_i and B_i , and ℓ denotes the perimeter of \mathcal{P} .

Theorem 1.3 (Spruck). *Let $\Omega \subset \mathbb{R}^2$ be a domain as above satisfying (1.3) with $\{C_i\} \neq \emptyset$ and $|B_i| < \pi/H$. Assume also that no two arcs A_i and no two arcs B_i have a common endpoint and that the domain Ω^* formed by reflection¹ of the arcs B_i is contained in a disk of radius $1/H$. Given any continuous data $f_i: C_i \rightarrow \mathbb{R}$, there exists a unique Jenkins-Serrin solution $u: \Omega \rightarrow \mathbb{R}$ with mean curvature H and satisfying $u|_{C_i} = f_i$, if and only if, for any admissible polygon \mathcal{P} we have*

$$2\alpha(\mathcal{P}) < \ell(\mathcal{P}) + H \text{Area}(\mathcal{P}) \quad \text{and} \quad 2\beta(\mathcal{P}) < \ell(\mathcal{P}) - H \text{Area}(\mathcal{P}).$$

In the case $\{C_i\} = \emptyset$, he obtained that the necessary and sufficient conditions for the existence of the solution are

$$2\alpha(\mathcal{P}) < \ell(\mathcal{P}) + H \text{Area}(\mathcal{P}) \quad \text{and} \quad 2\beta(\mathcal{P}) < \ell(\mathcal{P}) - H \text{Area}(\mathcal{P})$$

and

$$\alpha(\mathcal{P}) = \beta(\mathcal{P}) + H \text{Area}(\Omega).$$

In higher dimensions the question is more difficult but Spruck proved the following local existence of CMC Jenkins-Serrin solution.

Theorem 1.4 (Spruck). *Let S be a C^2 hypersurface in \mathbb{R}^n with mean curvature $H(>0)$. Let p be an interior point of S , and $B_\varepsilon(p)$ a ball with center p and radius ε . For small ε , let D^+ and D^- be the domains bounded by S and $\partial B_\varepsilon(p)$ so that the mean curvature $H_S = H$ with respect to D^+ and $H_S = -H$ with respect to D^- . Then for ε small enough, there exists a Jenkins-Serrin solution u in D^+ (respectively*

¹ We reflect B_i with respect to the line passing through the end points. In this way we obtain an arc B_i^* with curvature $\kappa(B_i^*) = H$.

$D^-)$ with boundary values $+\infty$ (respectively $-\infty$) on S and prescribed continuous data f on the remainder of the boundary.

Moreover, he obtained an analogous result to Theorem 1.2 making a clever use of suitable barriers.

Theorem 1.5 (Spruck). *Let $\Omega \subset \mathbb{R}^n$ be a domain that is bounded in part by a C^2 hypersurface S . Suppose that $u: \Omega \rightarrow \mathbb{R}$ is a solution of the CMC equation with constant mean curvature $H(> 0)$ satisfying $u \rightarrow \infty$ (respectively $u \rightarrow -\infty$) as $x \rightarrow S$. Then $H_S = H$ (respectively $-H$), where H_S denotes the mean curvature of S in \mathbb{R}^n .*

Further results about Jenkins-Serrin solutions in higher dimensions can be found in [10, 16].

B. Nelli and H. Rosenberg [21] extended the Jenkins-Serrin theory for domains in the hyperbolic plane \mathbb{H}^2 , and after that, A. L. Pinheiro [22] to more general Riemannian surfaces.

Although we are only considering the case of bounded domains, we would like to mention that L. Mazet, M. Rodríguez and H. Rosenberg [17] proved that, in the minimal case, it is possible to get Jenkins-Serrin solutions over non-compact domains in the hyperbolic plane \mathbb{H}^2 . Their result generalizes previous results obtained by P. Collin and H. Rosenberg in [6]. To state the result, we use the following notation.

We say that a domain $\Omega \subset \mathbb{H}^2$ is an ideal Scherk domain if $\partial_\infty \Omega$ consists of a finite number of geodesic arcs A_i, B_i , a finite number of convex arcs C_i (with respect to Ω), a finite number of open arcs D_i at $\partial_\infty \mathbb{H}^2$, together with their endpoints (called the vertices of Ω) and no two A_i and no two B_i have a common end point. Now let Ω be an ideal Scherk domain; a polygon $\mathcal{P} \subset \Omega$ is said to be an admissible polygon if its vertices are among the vertices of Ω . For any ideal vertex p_i of Ω at $\partial_\infty \mathbb{H}^2$, let H_i be a horocycle at p_i such that for any $i \neq j$ we have $H_i \cap H_j = \emptyset$, and H_i does not intersect the bounded edges of $\partial\Omega$. Finally, given any admissible polygon $\mathcal{P} \subset \Omega$, denote by $\Gamma(\mathcal{P})$ the part of $\partial\mathcal{P}$ outside the horocycles H_i and call

$$\alpha(\mathcal{P}) = \sum_{A_i \subset \mathcal{P}} \|A_i \cap \Gamma(\mathcal{P})\| \text{ and } \beta(\mathcal{P}) = \sum_{B_i \subset \mathcal{P}} \|B_i \cap \Gamma(\mathcal{P})\|$$

With this notation, we can state their theorem.

Theorem 1.6 (Mazet-Rodríguez-Rosenberg). *Suppose that there is at least one edge C_i or D_i in $\partial_\infty \Omega$. Then there exists a Jenkins-Serrin solution with continuous data $f_i: C_i \rightarrow \mathbb{R}$ and $g_i: D_i \rightarrow \mathbb{R}$ if, and only if, the horocycles H_i can be chosen so that*

$$2\alpha(\mathcal{P}) < \|\Gamma(\mathcal{P})\| \text{ and } 2\beta(\mathcal{P}) < \|\Gamma(\mathcal{P})\|.$$

for any admissible polygon \mathcal{P} in Ω .

Around the same time, J. Gálvez and H. Rosenberg [9] proved similar result for simply connected and complete Riemannian surfaces whose Gaussian curvature is bounded from above by a negative constant.

Concerning the CMC case, L. Hauswirth, H. Rosenberg and J. Spruck [11] proved the existence of Jenkins-Serrin solutions in \mathbb{H}^2 and \mathbb{S}^2 . Later A. Folha and H. Rosenberg [8] generalized these results to the case of Hadamard surfaces.

We would like to point out that Folha and Rosenberg also proved the following theorem that provides a relationship between the mean curvature of the graph and the mean curvature of the boundary of the domain.

Theorem 1.7 (Folha-Rosenberg). *Let M be a Hadamard surface and $\Omega \subset M$ be a bounded domain. Suppose that $u: \Omega \rightarrow \mathbb{R}$ is a solution of CMC equation with constant mean curvature $H(> 0)$ satisfying $u \rightarrow \infty$ (respectively $u \rightarrow -\infty$) as $x \rightarrow \gamma$, where γ is a smooth connected component of $\partial\Omega$. Then $H_\gamma = H$ (respectively $-H$), where H_γ denotes the mean curvature of γ in M .*

Notice that a fact repeatedly appears in Theorems 1.2, 1.5 and 1.7. Namely, they showed that if we have a Jenkins-Serrin solution over a domain Ω , then the mean curvature of $\partial\Omega$ and the mean curvature of the surface given by the graph of the Jenkins-Serrin solution are equal, up to a sign. This suggests the following question:

Question. How does the existence of complete graphs with prescribed mean curvature over a regular domain affect the geometrical structure of its boundary?

In relation to the translating graphs, L. Shahriyari [24] obtained a complete result for translating graphs over regular domains in \mathbb{R}^2 . The main idea in Shahriyari's proofs is to use Schoen's type estimates for stable minimal surfaces in 3-dimensional manifolds, an important tool in Colding & Minicozzi theory, to get information about the behaviour of the graph as we approach a component γ of the boundary, like Folha and Rosenberg do in [8]. Finally, she concluded that γ is a geodesic or has constant principal curvature in a CMC case. The main drawback of their method is that their proofs do not work in higher dimension.

Using a quite different strategy M. Eichmair and J. Metzger [7] were able to improve earlier result by Spruck (Theorem 1.3), in the case where $\{C_i\} = \emptyset$, into Riemannian manifolds of dimension $2 \leq n \leq 7$. Namely, they didn't need the existence of an auxiliary domain Ω^* nor a priori assumptions about existence of subsolutions, see [7, Section 6.1]. On the other hand, inspired by a classical compactness results of stable hypersurfaces by R. Schoen and L. Simon [23], Eichmair and Metzger gave a very simple proof that makes it possible to get an extension of Theorem 1.2, Theorem 1.5 and Theorem 1.7 for any n -dimensional Riemannian manifold for $2 \leq n \leq 7$. We will show that the same method can be applied for any Riemannian manifold without any restriction on the dimension. More precisely, we prove the following theorem.

Theorem 1.8. *Let M be a complete Riemannian manifold and $\Omega \subset M$ be a domain with piecewise smooth boundary. Let $\Lambda \subset \partial\Omega$ be a smooth connected region and let Σ be the graph of a smooth function u on Ω in $M \times \mathbb{R}$. Suppose Σ is complete as we approach Λ . Then, we have*

- a. *If Σ is a translating soliton, then $H_\Lambda = 0$;*
- b. *If Σ has constant mean curvature H_0 (with respect to the upward pointing normal vector field), then $H_\Lambda = H_0$, up to the sign.*

The crucial ingredient in our proof is a local area estimate for the components of the graph in closed balls around the cylinder over the boundary. These area estimates, combined with a suitable compactness theorems (Theorem 2.3 in the minimal case and Theorem 2.5 for CMC hypersurfaces), yield precise information of the geometric structure of the domain through a surprisingly simple local analysis.

In the last section we consider the existence of Jenkins-Serrin type solutions to the translating soliton equation and prove the following result.

Theorem 1.9. *Let $\Omega \subset M$ be an admissible domain with $\{B_i\} = \emptyset$. Given any continuous data $f_i: C_i \rightarrow \mathbb{R}$, there exists a Jenkins-Serrin solution $u: \Omega \rightarrow \mathbb{R}$ for translating equation with continuous data $u|_{C_i} = f_i$ if for any admissible polygon \mathcal{P} we have*

$$2\alpha(\mathcal{P}) < \ell(\mathcal{P}).$$

Acknowledgements. We would like to thank Brian White for his valuable comments, in particular concerning Theorem 3.12. E. S. Gama is very grateful to the Institute of Mathematics at the University of Granada for its hospitality during the time the research and preparation of this article were conducted.

2. COMPACTNESS THEOREM FOR STABLE INTEGRAL VARIFOLDS

In this section, we give a brief review of the results that we need about varifolds. We refer the reader to [25] for an elegant introduction on the subject. Before we state the compactness theorem for varifolds, recall the following concepts of weak and strong convergence.

Definition 2.1. *Let $\{S_i\}$ be a sequence of varifolds in a smooth manifold \bar{M} . We say that $\{S_i\}$ converges weakly to a varifold S , if for every smooth function $\varphi: \bar{M} \rightarrow \mathbb{R}$ with compact support we have*

$$\lim_{i \rightarrow \infty} \int_{\bar{M}} \varphi \, d\mu_{S_i} = \int_{\bar{M}} \varphi \, d\mu_S,$$

where μ_{S_i} is the Radon measure in \bar{M} associated to varifold S_i . If the sequence $\{S_i\}$ converges weakly to S , then we will write $S_i \rightharpoonup S$.

Let S be a hypersurface in a Riemannian manifold \bar{M} . Given $p \in S$ and $r > 0$ we denote by $B_p(r)$ the tangent ball in $T_p S \subset T_p \bar{M}$ centered at $0 \in T_p S$ with radius r .

$$B_r(p) = \{v \in T_p S : |v| < r\}$$

Let N be a unit normal vector to S at p . Fixed a sufficiently small $\varepsilon > 0$, let $W_{r,\varepsilon}(p)$ be the solid cylinder around p , that is,

$$W_{r,\varepsilon}(p) := \{\exp_p(q + tN) : q \in B_r(p) \text{ and } |t| < \varepsilon\},$$

where \exp is the exponential map of the manifold \bar{M} . Given a smooth function $f : B_r(p) \rightarrow (-\varepsilon, \varepsilon)$, the set

$$\text{Graph}[f] := \{\exp_p(q + u(q)N) : q \in B_r(p)\},$$

is called the graph of f over $B_r(p)$. Finally, the convergence in C^∞ -topology can be defined as follows.

Definition 2.2. Let $\{S_i\}$ be a sequence of smooth codimension one submanifolds of a smooth manifold \bar{M} . We say that $\{S_i\}$ converges in C^∞ -topology with finite multiplicity to a smooth embedded submanifold S if

- a. S consists of accumulation points of $\{S_i\}$, that is, for each $p \in S$ there exists a sequence $\{p_i\}$ such that $p_i \in S_i$, for each $i \in \mathbb{N}$, and $p = \lim_i p_i$;
- b. For every $p \in S$ there exist $r, \varepsilon > 0$ such that $S \cap W_{r,\varepsilon}(p)$ can be represented as a graph of a function f over $B_r(p)$;
- c. For i large enough, the set $S_i \cap W_{r,\varepsilon}(p)$ consists of a finite number, say k , independent of i , of graphs of functions f_i^1, \dots, f_i^k over $B_r(p)$ that converge smoothly to f .

The multiplicity of a given point $p \in S$ is defined by k . If $\{S_i\}$ converges smoothly to S , then we will write $S_i \rightarrow S$.

In this setting, the strong compactness theorem for integral varifolds can be stated as follows.

Theorem 2.3 (Strong compactness theorem). Let $\{S_i\}$ be a sequence of stable minimal hypersurfaces in Riemannian manifold \bar{M}^{n+1} , without boundary, such that

$$\limsup \mu_{S_i}(\bar{M}) < \infty,$$

where μ_{S_i} is the (Riemannian) measure associated to S_i . Suppose that there exists a sequence $\{p_i\}$ so that $p_i \in S_i$ and $p_i \rightarrow p_\infty \in \bar{M}$. Then there exist a closed set, $\text{sing } S_\infty$, and a stationary integral varifold, S_∞ , such that a subsequence of $\{S_i\}$ converges smoothly to S_∞ away from $\text{sing } S_\infty$ and $S_i \rightarrow S_\infty$ with $p_\infty \in \text{spt } S_\infty$. Moreover, the set $\text{sing } S_\infty$ has Hausdorff dimension at most $n - 7$. Hence, it is empty for $n < 7$.

Remark 2.4. *This version is a consequence of regularity and compactness results due to R. Schoen and L. Simon in [23] and Corollary 17.8 in [25]. See also [27].*

In very recent works about regularity and compactness of stable CMC integral varifolds C. Bellettini and N. Wickramasekera, in [2], [3] and [4], have extended the previous theorem to the class of all stable CMC integral varifolds in the following way:

Theorem 2.5 (General compactness theorem). *Let $\{S_i\}$ be a sequence of stable CMC hypersurfaces in Riemannian manifold \bar{M}^{n+1} , without boundary, satisfying*

$$\limsup\{\mu_{S_i}(\bar{M}) + |H_i|\} < \infty,$$

where μ_{S_i} is the (Riemannian) measure associated to S_i and H_i is the mean curvature of S_i . Suppose that there exists a sequence $\{p_i\}$ so that $p_i \in S_i$ and $p_i \rightarrow p_\infty \in \bar{M}$. Then there exist a real number H_∞ , a closed set $\text{sing } S_\infty$, an integral varifold S_∞ and subsequence $\{S_{i_j}\}$ and subsequence of $\{H_{i_j}\}$ such that $S_i \rightarrow S_\infty$, $H_i \rightarrow H_\infty$ and $p_\infty \in \text{spt } S_\infty$. Moreover $\text{sing } S_\infty \setminus \text{sing}_T S_\infty$ has Hausdorff dimension at most $n-7$, and $\text{sing}_T S_\infty \subset \text{gen-reg } S_\infty$, $\text{sing } S_\infty$ is locally contained in a smooth submanifold of dimension $n-1$, and $\text{gen-reg } S_\infty$ is a classical CMC immersion with mean curvature H_∞ .

Remark 2.6. *Both Theorem 2.3 and Theorem 2.5, as stated above, are weak versions of the compactness theorems by Bellettini and Wickramasekera.*

In the statement of Theorem 2.5 the sets $\text{sing}_C S_\infty$, $\text{reg}_1 S$, $\text{sing}_T S_\infty$ and $\text{gen-reg } S_\infty$ are defined as follows. Here $B_\varepsilon(x)$ denotes an open ball of radius ε and center x in \bar{M} .

Definition 2.7 (Set of classical singularities). $\text{sing}_C S_\infty$ denotes the set of all $x \in \text{spt } S$ such that, for some $\alpha \in (0, 1]$, there exists $\varepsilon > 0$ so that $\text{spt } S \cap B_\varepsilon(x)$ is the union of three or more embedded $C^{1,\alpha}$ hypersurface with boundary meeting pairwise only along their common $C^{1,\alpha}$ boundary γ containing x and such that at least one pair of the hypersurfaces with boundary meet transversely everywhere along γ .

Definition 2.8 (C^1 regular set). $\text{reg}_1 S$ denotes the set of all $x \in \text{spt } S$ such that there exists $\varepsilon > 0$ so that $\text{spt } S \cap B_\varepsilon(x)$ is an embedded hypersurface of $B_\varepsilon(x)$ of class C^1 .

Definition 2.9 (Set of touching singularities). $\text{sing}_T S_\infty$ denotes the set of all $x \in \text{sing } S_\infty \setminus \text{reg}_1 S_\infty$ so that $x \notin \text{sing}_C S_\infty$ and there exists $\varepsilon > 0$ such that $\text{spt } S_\infty \cap B_\varepsilon(x)$ is the union of two embedded $C^{1,\alpha}$ hypersurfaces of $B_\varepsilon(x)$ which touch at x .

Definition 2.10 (Generalized regular set). $\text{gen-reg } S_\infty$ denotes the set of all points $x \in \text{spt } S_\infty$ so that either $x \in \text{reg } S_\infty$ or $x \in \text{sing}_T S_\infty$ and there exists $\varepsilon > 0$ such that $\text{spt } S_\infty \cap B_\varepsilon(x)$ is the union of two embedded C^∞ hypersurfaces of $B_\varepsilon(x)$.

3. TRANSLATING GRAPHS

In this section we present some preliminaries on translating and minimal graphs. We will first study the behaviour of the translating graphs in the product space $M \times \mathbb{R}$ and in the end we prove our main theorem for translating and minimal graphs. Here M is a complete Riemannian manifold whose metric is denoted by σ . Following [15] we say that a hypersurface Σ in $M \times \mathbb{R}$ is a translating soliton with respect to the parallel vector field $X = \partial_t$ (with translation speed $c \in \mathbb{R}$) if

$$\mathbf{H} = c X^\perp,$$

where \mathbf{H} is the mean curvature vector field of Σ and $^\perp$ indicates the projection onto the normal bundle of Σ . In particular, if N is a normal vector field along Σ , then we have

$$H = c \langle X, N \rangle, \quad (3.1)$$

where $\langle \cdot, \cdot \rangle$ denotes the Riemannian product metric in $M \times \mathbb{R}$.

Although translating solitons seem to define a completely new class of hypersurfaces, the next result due to T. Ilmanen [13] tells that they are actually minimal hypersurfaces with respect to the so-called Ilmanen's metric

$$g_c = e^{\frac{2c}{m}t}(\sigma + dt^2) \quad (3.2)$$

which is a metric conformal to the Riemannian product metric $g_0 = \sigma + dt^2$ in $M \times \mathbb{R}$, see also [15].

Hence (3.1) can be interpreted as the fact that the mean curvature of Σ with respect to g_c vanishes, i.e.

$$e^{\frac{c}{m}t} \tilde{H} = H - c \langle X, N \rangle = 0. \quad (3.3)$$

This is the Euler-Lagrange equation (for normal variations) of the volume functional

$$\mathcal{A}_c[\Sigma] = \int_{\Sigma} e^{c\eta} d\mu_{\Sigma}$$

where $\eta = t|_{\Sigma}$ and $e^{c\eta} d\mu_{\Sigma}$ is the volume element in Σ induced by g_c .

Remark 3.1. *In geometric measure theory, $\mathcal{A}_c[\Sigma]$ is usually called the mass of the varifold associate to Σ . Indeed, the standard notation to $\mathcal{A}_c[\Sigma]$ is $M_c(\Sigma)$, where we put the index c to indicate the dependence of the metric g_c . However, we will use the notation $\mathcal{A}_c[\Sigma]$ to denote the integral above.*

Lemma 3.2 (T. Ilmanen). *Translating solitons with translation speed $c \in \mathbb{R}$ are minimal hypersurfaces in the product $M \times \mathbb{R}$ with respect to the Ilmanen's metric $g_c = e^{\frac{2c}{m}t}(\sigma + dt^2)$.*

With this interpretation, it is natural to consider the second variation of the volume and the corresponding Jacobi operator

$$L_c[v] = \Delta v + c\langle X, \nabla v \rangle + (|A|^2 + \overline{\text{Ric}}(N, N))v, \quad v \in C^2(\Sigma),$$

where $|A|$ is the norm of the second fundamental form and $\overline{\text{Ric}}$ is the Ricci curvature of $M \times \mathbb{R}$, both calculated with respect to the Riemannian product metric. We refer to [1], [15] and [29] for further details. Taking derivatives on both sides of (3.1) we obtain

$$\nabla H = -c AT, \quad (3.4)$$

where $T = X^\top$ is the tangential component of X . Since X is parallel in $M \times \mathbb{R}$ with respect to the product metric g_0 , we also have

$$\nabla_V T = \frac{H}{c} AV \quad (3.5)$$

for any $V \in \Gamma(T\Sigma)$. Hence, using Codazzi equation for any $U, V \in \Gamma(T\Sigma)$ we have

$$\begin{aligned} \langle \nabla_U \nabla H, V \rangle &= -c\langle (\nabla_U A)T, V \rangle - H\langle AU, AV \rangle \\ &= -c\langle (\nabla_T A)U, V \rangle + c\langle \bar{R}(U, T)N, V \rangle - H\langle AU, AV \rangle. \end{aligned}$$

Therefore

$$\Delta H = -c\langle \nabla H, T \rangle + c \text{Ric}_{\bar{M}}(T, N) - |A|^2 H.$$

Since

$$0 = \overline{\text{Ric}}(X, N) = \overline{\text{Ric}}(T, N) + \langle X, N \rangle \overline{\text{Ric}}(N, N) = \overline{\text{Ric}}(T, N) + \frac{1}{c} H \overline{\text{Ric}}(N, N)$$

we conclude that

$$L_c[H] = \Delta H + c\langle X, \nabla H \rangle + (|A|^2 + \text{Ric}_{\bar{M}}(N, N))H = 0. \quad (3.6)$$

Therefore, the function $h := \langle X, N \rangle$ satisfies $L_c[h] = 0$.

Outside the points where a translating soliton is vertical, it can be described locally in non-parametric terms as a graph

$$\Sigma = \{(x, u(x)) : x \in \Omega\}$$

of a smooth function u defined in a domain $\Omega \subset M$ with regular boundary (possibly empty.) In this case, we denote $\Sigma = \text{Graph}[u]$ and we refer to those solitons as *translating graphs*. From (3.1) we can check that u satisfies the following partial differential equation

$$\text{div} \left(\frac{\nabla u}{W} \right) = \frac{1}{W}, \quad (3.7)$$

where $W := \sqrt{1 + |\nabla u|^2}$, and the gradient and divergence operators are taken with respect to the Riemannian metric σ of M . In this case, Σ can be oriented by the normal vector field

$$N = \frac{1}{W}(X - \nabla u)$$

with ∇u translated from $x \in \Omega$ to the point $(x, u(x)) \in \Sigma$. Proceeding as in [29] we prove the following result.

Lemma 3.3 (Shahriyari-Zhou). *All translating graphs are stable in $M \times \mathbb{R}$ endowed with Ilmanen's metric g_c .*

Remark 3.4. *If Σ is a graphical translator and N is the (upward-pointing) unit-normal vector field to Σ , then $\langle \partial_t, N \rangle$ is a positive Jacobi field. Consequently, Σ is a stable g_c -minimal surface. Therefore a sequence of translating graphs will converge, subsequentially, to a translator. Moreover the vertical translates of Σ are also g_c -minimal and foliate a cylinder $\Omega \times \mathbb{R}$, where Ω is the region over which Σ is a graph. As a consequence, Σ is a g_c -area minimizing surface in $\Omega \times \mathbb{R}$.*

The proof of the last assertion in the previous remark is quite simple, as we will see in the next lemma. Recall that a complete hypersurface is called area-minimizing if any compact piece is area-minimizing among all the hypersurfaces with the same boundary.

Lemma 3.5. *Let $\Omega \subset M$ be a bounded domain and Σ a translating graph over $\bar{\Omega}$. For any hypersurface Σ' in $\bar{\Omega} \times \mathbb{R}$ with $\partial\Sigma = \partial\Sigma'$, we have*

$$\mathcal{A}_c[\Sigma] = \int_{\Sigma} e^{ct} d\mu_{\Sigma} \leq \int_{\Sigma'} e^{ct} d\mu_{\Sigma'} = \mathcal{A}_c[\Sigma'],$$

and the equality holds if, and only if, $\Sigma = \Sigma'$.

Proof. Let $\Psi_{\tau}(x, t) = \Psi((x, t), \tau) = (x, t + \tau)$ the flow generated by $X = \partial_t$ in $M \times \mathbb{R}$. Consider the one-parameter family of translated copies of Σ given by

$$\Sigma_{\tau} = \Psi_{\tau}(\Sigma)$$

and let N_{τ} be the vector field in $\bar{\Omega} \times \mathbb{R}$ given by

$$N_{\tau}(\Psi_{\tau}(x, t)) = \Psi_{\tau*}(x, t) \cdot N(x, t),$$

where N is a unit normal vector field along Σ . It is obvious that N_{τ} is a normal vector field along Σ_{τ} . Consider the vector field in $\bar{\Omega} \times \mathbb{R}$ defined by

$$Y = e^{ct} N_{\tau}. \tag{3.8}$$

Let $\{\mathbf{e}_i\}_{i=1}^m$ be a local orthonormal frame tangent to Σ_τ . Therefore, denoting the Riemannian connection and the divergence in $(M \times \mathbb{R}, g_0)$ by $\bar{\nabla}$ and div , we obtain

$$\begin{aligned} \text{div } Y|_{\Sigma_\tau} &= ce^{ct} \langle \bar{\nabla} t, N_\tau \rangle + e^{ct} \text{div } N_\tau \\ &= ce^{ct} \langle X, N_\tau \rangle + e^{ct} \sum_{i=1}^m \langle \bar{\nabla}_{\mathbf{e}_i} N_\tau, \mathbf{e}_i \rangle + e^{ct} \langle \bar{\nabla}_{N_\tau} N_\tau, N_\tau \rangle \\ &= ce^{ct} \langle X, N \rangle - e^{ct} H_\tau, \end{aligned}$$

where we used the fact that $|N_\tau|^2 = 1$. Here $H_\tau(\Psi_\tau(x, t)) = H(x, t)$ is the mean curvature of Σ . Using (3.1), we conclude that

$$\text{div } Y = e^{ct} (H - c \langle X, N \rangle) = 0 \quad (3.9)$$

in $\bar{\Omega} \times \mathbb{R}$. First, suppose that $\Sigma' \subset \bar{\Omega} \times \mathbb{R}$ lies in one side of Σ with $\partial \Sigma = \partial \Sigma'$ and denote by U the domain bounded by $\Sigma \cup \Sigma'$. We have

$$\begin{aligned} 0 &= \int_U \text{div } Y = \int_\Sigma \langle Y, N \rangle d\mu_\Sigma - \int_{\Sigma'} \langle Y, N' \rangle d\mu_{\Sigma'} \\ &= \int_\Sigma e^{ct} d\mu_\Sigma - \int_{\Sigma'} e^{ct} \langle N, N' \rangle d\mu_{\Sigma'} \\ &\geq \int_\Sigma e^{ct} d\mu_\Sigma - \int_{\Sigma'} e^{ct} d\mu_{\Sigma'}, \end{aligned}$$

where N' and $d\mu_{\Sigma'}$ define the orientation and the volume element in Σ' , respectively. This completes the proof in this particular case. The general case follows by breaking up the hypersurface Σ' into regions that lie on one side of Σ and applying the previous argument to get the inequality in each one of these regions. \square

Remark 3.6. *Similar result was proved by Y. L. Xin [28] for translating graphs in \mathbb{R}^{n+1} .*

Remark 3.7. *Notice that by Lemma 3.5 translating graphs are, in fact, area-minimizing with respect to g_c .*

Next we provide a proof for the first equality in (3.3), that is, the relation between the mean curvatures with respect to g_c and g_0 . The Riemannian connections $\tilde{\nabla}$ and ∇ for the metrics g_c and g_0 , respectively, are related by

$$\tilde{\nabla}_V W = \nabla_V W + \frac{c}{m} (\langle V, \partial_t \rangle W + \langle W, \partial_t \rangle V - \langle V, W \rangle \partial_t).$$

If N is a unit normal vector field along Σ with respect to g_0 , here it is not necessary to suppose that Σ is a graph. Then the normal vector field with respect to g_c is given by $\tilde{N} = e^{-\frac{c}{m}t} N$. Therefore, the second fundamental forms \tilde{II} and II of Σ

with respect to g_c and g_0 are related by

$$\widetilde{II} = e^{\frac{c}{m}t} (II - \frac{c}{m} \langle \partial_t, N \rangle g_0|_{\Sigma})$$

Taking traces with respect to $g_c|_{\Sigma}$ one gets

$$\widetilde{H} = e^{-\frac{c}{m}t} (H - c \langle \partial_t, N \rangle) \quad (3.10)$$

as we have stated above.

As an application of (3.10), we have the next lemma.

Lemma 3.8. *Suppose that Λ is a hypersurface in M . Then the mean curvature $\widetilde{H}_{\Lambda \times \mathbb{R}}$ of $\Lambda \times \mathbb{R}$ in $(M \times \mathbb{R}, g_c)$ is given by*

$$\widetilde{H}_{\Lambda \times \mathbb{R}}(x, t) = e^{-\frac{c}{m}t} H_{\Lambda}(x). \quad (3.11)$$

for all $(x, t) \in \Lambda \times \mathbb{R}$. Here H_{Λ} is the mean curvature of Λ in (M, σ) .

Proof. First, observe that

$$H_{\Lambda \times \mathbb{R}}(x, t) = H_{\Lambda}(x)$$

for all $(x, t) \in \Lambda \times \mathbb{R}$, where both mean curvatures are calculated with respect to g_0 . Moreover, $\langle \partial_t, N \rangle = 0$ along $\Lambda \times \mathbb{R}$ since N is merely the horizontal lift of the unit normal vector field along Λ in M . Hence, (3.10) yields (3.11). \square

Before proving the main theorem of this section, we recall what we mean by a complete graph.

Definition 3.9. *Let $\Omega \subset M$ be a domain, not necessarily regular, and let $\Lambda \subset \partial\Omega$ be a smooth open set. We say that a smooth function $u: \Omega \rightarrow \mathbb{R}$ is complete as we approach Λ , if*

$$\lim_{x \rightarrow x_0} u(x) = \pm\infty, \text{ for any } x_0 \in \Lambda.$$

Theorem 3.10. *Let M be a complete Riemannian manifold and $\Omega \subset M$ be a domain (not necessarily regular). Let $\Lambda \subset \partial\Omega$ be a smooth open set and Σ a translating or minimal graph of a smooth function $u: \Omega \rightarrow \mathbb{R}$ that is complete as we approach Λ . Then $H_{\Lambda} = 0$.*

Proof. Fix $x_0 \in \Lambda$ and take a sequence $\{x_i\}$ in Ω with $x_i \rightarrow x_0$. Assume that $u(x_i) \rightarrow \infty$ and define the sequence of hypersurfaces $\{\Sigma_i := \text{Graph}[u - u(x_i)]\}$ in $M \times \mathbb{R}$ endowed with the Ilmanen's metric g_c .

Fix a closed ball B around $(x_0, 0)$ in $M \times \mathbb{R}$ so that B does not intersect $\partial\Lambda \times \mathbb{R}$. Notice that each Σ_i intersects B for i sufficiently large. So, up to subsequence, we can suppose that each Σ_i intersects B . For each i let S_i be the connected component of $\Sigma_i \cap B$ so that $(x_i, 0) \in S_i$. By Lemma 3.5

$$\mathcal{A}_c[S_i] \leq \mathcal{A}_c[\partial B],$$

for all i . Hence, by Theorem 2.3, up to a subsequence, we may assume $S_i \rightarrow S_\infty$ in $\text{int } B \setminus \text{Sing}(S_\infty)$ and $S_i \rightarrow S_\infty$ in $\text{int } B$. Note also that $(x_0, 0) \in (\Lambda \times \mathbb{R}) \cap \text{spt } S_\infty$. Then we have two possibilities for $(x_0, 0)$: either it is a regular point or not.

Suppose first that $(x_0, 0)$ is a regular point of S_∞ . We claim that a small neighbourhood of $(x_0, 0)$ in S_∞ lies on the cylinder $\Lambda \times \mathbb{R}$. In fact, if this does not hold for any small neighbourhood of $(x_0, 0)$, then we could choose a small domain S in S_∞ near $(x_0, 0)$ and away from the singular set, such that S lies in $\Omega \times \mathbb{R}$. Therefore, we could take a small compact cylinder C through S in $M \times \mathbb{R}$ such that it does not touch $\partial\Omega \times \mathbb{R}$. Since $S_i \rightarrow S_\infty$ in $\text{int } B \setminus \text{Sing}(S_\infty)$, it follows that, for i sufficiently large, S_i must intersect C . However the assumption $u(x_i) \rightarrow \infty$ implies that, for sufficiently large i , $S_i \cap C = \emptyset$, which is a contradiction. Therefore a neighbourhood of $(x_0, 0)$ in the minimal surface S_∞ lies on $\Lambda \times \mathbb{R}$ and, in particular, we conclude that $\tilde{H}_{\Lambda \times \mathbb{R}}(x_0, 0) = 0$, that is, the mean curvature of $\Lambda \times \mathbb{R}$ with respect to g_c vanishes at $(x_0, 0) = 0$.

If $(x_0, 0)$ is not a regular point of S_∞ , take a neighborhood W of $(x_0, 0)$ in S_∞ . As the Hausdorff dimension of $\text{Sing}(\Sigma_\infty)$ is less than $n - 7$, we know that $W \setminus \text{Sing}(S_\infty)$ is an open dense subset of W . Furthermore, we can apply the previous argument to prove that any connected component of $W \setminus \text{Sing}(S_\infty)$, which is regular, must lie on $\Lambda \times \mathbb{R}$. Hence, we can take a sequence $\{y_i\} \subset \Lambda \times \mathbb{R}$ such that $y_i \rightarrow (x_0, 0)$ and $\tilde{H}_{\Lambda \times \mathbb{R}}(y_i) = 0$. By continuity $\tilde{H}_{\Lambda \times \mathbb{R}}(x_0, 0) = 0$. Therefore in both cases, Lemma 3.8 implies that we must have $H_\Lambda = 0$ on Λ . \square

Finally, using Theorem 2.5 and the idea of the proof of Theorem 3.10, we can obtain a proof of a result that was obtained by M. Eichmair, J. Metzger for $2 \leq n \leq 7$ [7, Appendix B].

Remark 3.11. *Let M be a complete Riemannian manifold and $\Omega \subset M$ be a domain whose boundary is not necessarily regular. Let Σ be a graph of a smooth function $u: \Omega \rightarrow \mathbb{R}$ with constant mean curvature $H_0 > 0$. Let $\Lambda \subset \partial\Omega$ be a smooth open set and suppose that u is complete as we approach Λ .*

- a. *If $u \rightarrow \infty$ on Λ , then $H_\Lambda = H_0$ with respect to the inward normal to $\partial\Omega$.*
- b. *If $u \rightarrow -\infty$ on Λ , then $H_\Lambda = -H_0$ with respect to the inward normal to $\partial\Omega$.*

The idea of the proof (which was already present in [7]) is the following. We fix $x_0 \in \Lambda$ and take a sequence $\{x_i\}$ in Ω with $x_i \rightarrow x_0$. Assume first that $u(x_i) \rightarrow \infty$ and define the sequence of hypersurfaces $\{\Sigma_i := \text{Graph}[u - u(x_i)]\}$ in $M \times \mathbb{R}$. Fix a closed ball B around $(x_0, 0)$ in $M \times \mathbb{R}$ so that B does not intersect $\partial\Lambda \times \mathbb{R}$, and suppose that each Σ_i intersects B . For each i let S_i be the connected component of $\Sigma_i \cap B$ so that $(x_i, 0) \in S_i$. Reasoning as in Lemma 3.5, we deduce that

$$\mathcal{A}_0[S_i] \leq \mathcal{A}_0[\partial B] + H_0 \text{Vol}(B),$$

for all i . Moreover, it is easy to check that S_i is stable for all i . Hence Theorem 2.5 implies that, up to a subsequence, we may assume $S_i \rightarrow S_\infty$ in $\text{int } B$ and $(x_0, 0) \in (\Lambda \times \mathbb{R}) \cap \text{spt } S_\infty$.

We claim that $\text{spt } S_\infty \subset \Lambda \times \mathbb{R}$. Indeed, suppose that $y_0 \notin \Lambda \times \mathbb{R}$ and take any small ball B' around y_0 so that it does not intersect $\Lambda \times \mathbb{R}$. If φ is any smooth function with support in B' , our definition of S_i give us that

$$\int_B \varphi d\mu_{S_i} = 0$$

for all sufficiently large i . Hence,

$$\int_B \varphi d\mu_{S_\infty} = \lim_i \int_B \varphi d\mu_{S_i} = 0.$$

This proves that $y_0 \notin \text{spt } S_\infty$. Consequently we must have $\text{spt } S_\infty \subset \Lambda \times \mathbb{R}$, and by regularity of S_∞ according to Theorem 2.5, we can argue as in Theorem 3.10 and conclude that $H_\Lambda = H_0$. On the other hand, if $u(x_i) \rightarrow -\infty$, we can argue as above and get $-H_\Lambda = H_0$, where $-H_\Lambda$ is the mean curvature with respect to the outward normal to Ω . So $H_\Lambda = -H_0$ with respect to the inward normal to Ω .

3.1. The Euclidean case. In the Euclidean case $M = \mathbb{R}^n$ endowed with the Euclidean metric $\langle \cdot, \cdot \rangle$, we can obtain a better result. In this particular case, it is very natural to impose on a translators $\Sigma \subset \mathbb{R}^{n+1}$ the condition that

$$\sup_{x \in \mathbb{R}^{n+1}, r > 0} \left(\frac{\text{Area}(\Sigma \cap B(x, r))}{r^n} \right) < \infty. \quad (3.12)$$

In particular, if Σ arise as a blow up of some mean curvature flow, then it has to satisfy (3.12) by [5, Corollary 2.13].

The theorem is then:

Theorem 3.12. *Suppose $u: \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}$ is a smooth function whose graph is a complete translator satisfying (3.12). Then the following holds.*

- (i) *If $n < 8$, then $\partial\Omega$ is a smooth minimal hypersurface.*
- (ii) *For general n , $\partial\Omega$ is a smooth minimal hypersurface except for a closed singular set of Hausdorff dimension at most $n - 8$.*

Proof. Fix $x_0 \in \partial\Omega$ and take a sequence $\{x_i\}$ in Ω with $x_i \rightarrow x_0$. As before, we define the sequence of hypersurfaces $\{\Sigma_i := \text{Graph}[u - u(x_i)]\}$ in \mathbb{R}^{n+1} endowed with the Ilmanen's metric g_c . The idea of the proof is that the result of Schoen and Simon [23] shows that a weak limit V of smooth n -dimensional stable minimal hypersurfaces, is smooth except for an $(n - 7)$ -dimensional singular set, assuming we have local area bounds. These local area bounds are provided by the assumption (3.12).

Furthermore, any point of V where the tangent cone is a plane (possibly with multiplicity) is a regular point. Now we can apply the Schoen-Simon theorem to $V = (\partial\Omega) \times \mathbb{R}$, which is the limit of our sequence Σ_i . At first glance, it seems that we get a singular set of dimension at most $n - 7$, not $n - 8$. But since V is translation-invariant in the vertical direction, in fact one can get one dimension better. One way to see that is the following.

Since $(\partial\Omega) \times \mathbb{R}$ is stable (and has a small singular set) for the Ilmanen metric, it follows that for any sufficiently small ball B in \mathbb{R}^n , $(\partial\Omega) \cap B$ is stable for the Euclidean metric. Hence the Schoen-Simon theorem implies that the singular set of $(\partial\Omega) \cap B$ has dimension at most $n - 8$. \square

Remark 3.13. *Under additional hypotheses on the geometry of the translator, it is possible to prove that $\partial\Omega$ is not only minimal, but totally geodesic (see [12].) In the case of \mathbb{R}^n , this means that $\partial\Omega$ consists of (disjoint) affine hyperplanes.*

4. JENKINS-SERRIN TRANSLATING SOLITONS

In this section we study the existence of Jenkins-Serrin solutions for the translating soliton equation on Riemannian surfaces M . To be more precise, we will prove an existence theorem for type 1 Jenkins-Serrin solutions for the translating soliton equation. Let us recall that u is a solution of the translating soliton equation if

$$\operatorname{div} \left(\frac{\nabla u}{W} \right) = \frac{1}{W},$$

where $W = \sqrt{1 + |\nabla u|^2}$, and the gradient and divergence are taken with respect to the Riemannian metric σ of M . The principal concepts we are going to need are the following:

Definition 4.1 (Nitsche curve). *Let Ω be a domain in M and $\Gamma \subset M \times \mathbb{R}$ be a Jordan curve. We say that Γ is a Nitsche curve, if it admits a parametrization $\Gamma(t) = \{(\alpha(t), \beta(t)) : t \in \mathbb{S}^1\}$ such that $\alpha(t)$ is a monotone parametrization of $\partial\Omega$. This means that $\alpha : \mathbb{S}^1 \rightarrow \partial\Omega$ is continuous and monotone, and there exist closed disjoint intervals J_1, \dots, J_v such that $\alpha|_{J_i}$ is constant for all i and $\alpha|_{\mathbb{S}^1 \setminus \cup J_i}$ is one-to-one and smooth.*

Definition 4.2 (Admissible domain). *Let Ω be a connected domain in M . We say that Ω is an admissible domain if it is geodesically convex and $\partial\Omega$ is a union of geodesic arcs $A_1, \dots, A_s, B_1, \dots, B_r$, convex arcs C_1, \dots, C_t , the end points of these arcs and that no two arcs A_i and no two arcs B_i have a common endpoint.*

Definition 4.3 (Admissible polygon). *Let Ω be an admissible domain. We say that \mathcal{P} is an admissible polygon if $\mathcal{P} \subset \Omega$ and the vertices of \mathcal{P} are chosen among the vertices of Ω .*

Let Γ be a Nitsche curve over the boundary $\partial\Omega$ of an admissible domain Ω . By a translating soliton with boundary Γ we mean a translating soliton in $\Omega \times \mathbb{R}$ that is a graph over Ω . Using classical results about the solvability of the Plateau problem, we can prove that any Nitsche curve over an admissible domain admits a unique translating soliton with it as the boundary.

Theorem 4.4 (Local existence). *Let Ω be an admissible domain in M and Γ a Nitsche curve over $\partial\Omega$. Then there exists a unique translating soliton with boundary Γ .*

Proof. The proof is similar to that of [22] and therefore we skip some details. First we note that $\Omega \times \mathbb{R}$ is homogeneous in the sense of Meeks and Yau [18, 19], and Morrey [20]. Since the boundary $\partial\Omega$ consists of geodesic and convex arcs, the boundary $\partial(\Omega \times \mathbb{R})$ is mean convex. By Lemma 3.8 it remains mean convex also when we change the metric to Ilmanen's metric g_c . Therefore there exists an embedded minimal (w.r.t. g_c) disk $\Sigma \subset \Omega \times \mathbb{R}$ with boundary Γ . It remains to prove that $\text{int}(\Sigma)$ is a graph over Ω .

First we show that, for all $p \in \text{int}(\Sigma)$, $T_p\Sigma$ is not a vertical plane. On the contrary, suppose that there exists a point $p \in \text{int}(\Sigma)$ such that $p \in M \times \{c\}$ for some $c \in \mathbb{R}$ and that the tangent plane Π to Σ is vertical in $\Omega \times \mathbb{R}$. Take a basis $\{\partial_t, v\}$ tangent to Π at p , where ∂_t is tangent to Σ and v is tangent to $M \times \{c\}$ with $\|v\| = 1$. Let γ be the unique geodesic in $M \times \{c\}$ such that $\gamma(0) = p$ and $\gamma'(0) = v$. Note that γ intersects $\partial(\Omega \times \mathbb{R})$ exactly in two points.

Now $\gamma \times \mathbb{R}$ is a totally geodesic surface, in particular minimal, in $\Omega \times \mathbb{R}$ and by Lemma 3.8 it is minimal also with respect to the metric g_c . Moreover, we have $T_p(\gamma \times \mathbb{R}) = \Pi$ and therefore, near p , $I := \Sigma \cap (\gamma \times \mathbb{R})$ contains at least two curves that intersect transversally at p . If there exists a closed curve α in $I \setminus \partial\Sigma$, then α is the boundary of a minimal disk D in Σ . Thus we could choose a geodesic curve β in D so that the totally geodesic surface $\beta \times \mathbb{R}$ touches D at an interior point. But this is impossible by the maximum principle.

Since I does not contain a closed curve, each of the branches leaving p must go to $\partial\Sigma$. Moreover, γ intersects $\partial\Omega$ at two points so at least two of these branches must go to the same point or vertical segment of $\partial\Sigma$. However, this yields again closed curve that bounds a minimal surfaces and we get a contradiction with the maximum principle. Therefore $T_p\Sigma$ is not a vertical plane,

Finally, the same argument as in [22] shows that each vertical line in $\Omega \times \mathbb{R}$ intersects Σ exactly at one point and therefore Σ is a graph over the interior of Ω . \square

To prove our main theorem, we will need the following maximum principle.

Proposition 4.5 (Maximum principle). *Let $\Omega \subset M$ be an admissible domain. Suppose that u_1 and u_2 satisfy*

$$\operatorname{div} \left(\frac{\nabla u_1}{\sqrt{1 + |\nabla u_1|^2}} \right) \geq \operatorname{div} \left(\frac{\nabla u_2}{\sqrt{1 + |\nabla u_2|^2}} \right),$$

and $\liminf(u_2 - u_1) \geq 0$ for any approach of $\partial\Omega$, with possible exception of finite numbers of points $\{q_1, \dots, q_r\} =: E \subset \partial\Omega$. Then $u_2 \geq u_1$ on $\partial\Omega \setminus E$ with strict inequality unless $u_2 = u_1$.

Proof. The proof follows similar arguments as in [26]. Let K and ε be positive constants, with K large enough and ε small enough. Define a function

$$\varphi := \begin{cases} K - \varepsilon, & \text{if } u_1 - u_2 \geq K; \\ u_1 - u_2 - \varepsilon, & \text{if } \varepsilon < u_1 - u_2 \leq K; \\ 0, & \text{if } u_1 - u_2 \leq \varepsilon. \end{cases}$$

Notice that φ is Lipschitz with $0 \leq \varphi \leq K$, and $\nabla\varphi = \nabla u_1 - \nabla u_2$ in the set $\{\varepsilon < u_1 - u_2 < K\}$ and $\nabla\varphi = 0$ almost everywhere in the complement of $\{\varepsilon < u_1 - u_2 < K\}$. Around any point $q_i \in E$, consider an open geodesic disk $B_\varepsilon(q_i)$ of radius ε and center q_i . Let $\Omega_\varepsilon := \Omega \setminus \cup B_\varepsilon(q_i)$, and suppose that $\partial\Omega_\varepsilon = \tau_\varepsilon \cup \rho_\varepsilon$, where $\rho_\varepsilon = \cup(\partial B_\varepsilon(q_i) \cap \Omega)$ and $\tau_\varepsilon = \partial\Omega_\varepsilon \cap \partial\Omega$. Since $\liminf(u_2 - u_1) \geq 0$ in $\partial\Omega \setminus E$, we have $\varphi \equiv 0$ in a neighbourhood of τ_ε .

Define

$$J := \int_{\rho_\varepsilon} \varphi \left\{ \left\langle \frac{\nabla u_1}{W_1}, \nu \right\rangle - \left\langle \frac{\nabla u_2}{W_2}, \nu \right\rangle \right\}, \quad (4.1)$$

where ν is the unit outer conormal to Ω_ε and $W_i = \sqrt{1 + |\nabla u_i|^2}$. From (4.1) and $0 \leq \varphi \leq K$ we obtain

$$J \leq 2K \sum_{i=1}^r \|\partial B_\varepsilon(q_i)\|, \quad (4.2)$$

where $\|\partial B_\varepsilon(q_i)\|$ denotes the length of $\partial B_\varepsilon(q_i)$. On the other hand, since φ is Lipschitz, we have

$$\operatorname{div} \left(\varphi \left\{ \frac{\nabla u_1}{W_1} - \frac{\nabla u_2}{W_2} \right\} \right) = \nabla\varphi \left\{ \frac{\nabla u_1}{W_1} - \frac{\nabla u_2}{W_2} \right\} + \varphi \left\{ \operatorname{div} \left(\frac{\nabla u_1}{W_1} \right) - \operatorname{div} \left(\frac{\nabla u_2}{W_2} \right) \right\},$$

almost everywhere in Ω . By Stokes theorem we get

$$\begin{aligned}
J &= \int_{\Omega_\varepsilon} \left\{ \left\langle \nabla \varphi, \left(\frac{\nabla u_1}{W_1} - \frac{\nabla u_2}{W_2} \right) \right\rangle + \varphi \left(\operatorname{div} \left(\frac{\nabla u_1}{W_1} \right) - \operatorname{div} \left(\frac{\nabla u_2}{W_2} \right) \right) \right\} \\
&\geq \int_{\Omega_\varepsilon} \left\langle \nabla \varphi, \left(\frac{\nabla u_1}{W_1} - \frac{\nabla u_2}{W_2} \right) \right\rangle.
\end{aligned} \tag{4.3}$$

Now if $N_i := \frac{\partial_t}{W_i} - \frac{\nabla u_i}{W_i}$, then

$$\begin{aligned}
\left\langle \nabla u_1 - \nabla u_2, \left(\frac{\nabla u_1}{W_1} - \frac{\nabla u_2}{W_2} \right) \right\rangle &= \langle N_1 - N_2, W_1 N_1 - W_2 N_2 \rangle \\
&= W_1 - (W_1 + W_2) \langle N_1, N_2 \rangle + W_2 \\
&= \frac{1}{2} (W_1 + W_2) \|N_1 - N_2\|^2.
\end{aligned} \tag{4.4}$$

From (4.2), (4.3) and (4.4) we get

$$2K \sum_{i=1}^r \|\partial B_\varepsilon(q_i)\| \geq \frac{1}{2} \int_{\Omega_\varepsilon \cap \{0 < u_1 - u_2 < K\}} (W_1 + W_2) \|N_1 - N_2\|^2 \geq 0.$$

In particular, when $\varepsilon \rightarrow 0$ we obtain

$$\int_{\{0 < u_1 - u_2 < K\}} (W_1 + W_2) \|N_1 - N_2\|^2 = 0.$$

Therefore $N_1 = N_2$ in $\{0 < u_1 - u_2 < K\}$, so $\nabla u_1 = \nabla u_2$ in $\{0 < u_1 - u_2 < K\}$. As K was arbitrary we may conclude $\nabla u_1 = \nabla u_2$ in the set $\{0 < u_1 - u_2\}$. Suppose now that $\{0 < u_1 - u_2\}$ contains a connected component with non-empty interior. By the previous argument $u_1 = u_2 + c$, where c is a positive constant, so by the maximum principle $u_1 = u_2 + c$ in Ω . On the other hand, as $\liminf(u_2 - u_1) \geq 0$ for any approach of $\partial\Omega \setminus E$, then c is a non-positive constant, which is impossible. This finishes the proof. \square

Finally, we can prove the main theorem of this section.

Theorem 4.6 (Existence of Jenkins-Serrin Solution). *Let $\Omega \subset M$ be an admissible domain with $\{B_i\} = \emptyset$. Given any continuous data $f_i: C_i \rightarrow \mathbb{R}$, there exists a Jenkins-Serrin solution $u: \Omega \rightarrow \mathbb{R}$ for the translating soliton equation with continuous data $u|_{C_i} = f_i$, if for any admissible polygon \mathcal{P} we have*

$$2\alpha(\mathcal{P}) < \ell(\mathcal{P}). \tag{4.5}$$

Proof. Define a Nitsche curve $\Gamma_n = (\alpha_n, \beta_n)$ by setting $\beta_n = n$ on $\{A_i\}$ and $\beta_n = \min\{f_i, n\}$ on C_i for all i . By Theorem 4.4, for all $n \in \mathbb{N}$, there exists $u_n: \Omega \rightarrow \mathbb{R}$ so that $\operatorname{Graph}[u_n]$ is a translating soliton in $\Omega \times \mathbb{R}$ with boundary Γ_n . Notice that

if $n > m$ we have $u_n \geq u_m$ on $\partial\Omega$, so $u_n > u_m$ in Ω by comparison principle. Hence $\{u_n\}$ is a monotone sequence. Taking into account Pinheiro's results [22], (4.5) guarantees that there exists a Jenkins-Serrin solution $v: \Omega \rightarrow \mathbb{R}$ for the minimal graph equation with continuous data f_i . Since

$$\operatorname{div} \left(\frac{v}{\sqrt{1 + |v|^2}} \right) = 0 < \frac{1}{\sqrt{1 + |u_n|^2}} = \operatorname{div} \left(\frac{u_n}{\sqrt{1 + |u_n|^2}} \right)$$

and $\liminf(v - u_n) \geq 0$ on $\partial\Omega \setminus E$, where E is the set of vertices of Ω , Proposition 4.5 implies $v > u_n$ for all n . Therefore $\lim u_n = u$ exists and u satisfies

$$\operatorname{div} \left(\frac{u}{\sqrt{1 + |\nabla u|^2}} \right) = \frac{1}{\sqrt{1 + |u|^2}}$$

in Ω . Clearly $u|_{C_i} = f_i$, by construction, and $u \rightarrow \infty$ as we approach A_i for all i . \square

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