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SUBJECT MATTER KNOWLEDGE AND THE QUALITY OF MATHEMATICS MADE AVAILABLE TO LEARN: SOME HYPOTHESES

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We offer here some hypotheses about how teachers' subject-matter knowledge is implicated in instruction through the lens of mathematical discourse in instruction (MDI) framework (Adler & Ronda, 2015).

INTRODUCTION

It is a common view that teachers' knowledge of mathematics relates to the mathematics made available to learn in instruction. However, there are only a few studies that provide empirical evidence and explanation of how this knowledge is implicated in instruction. Hill, Blunk, Charalambous et al. (2008) investigated the relationship between mathematical knowledge for teaching (MKT) and the *mathematical quality of instruction* (MQI) seen through the various tasks of teaching. Their in-depth and extensive study of five telling cases, while confirming an overall positive correlation between the teachers' MKT and their MQI, illuminates the complexities of this relationship, and so the challenges in all-encompassing claims. More recently, and with a larger study, Munter & Correnti (2017) examined not only the relation of teachers' MKT to the *quality of their mathematics instruction* but also to teachers' vision of high quality instruction and doing this over time using a tool for examining the presence of ambitious mathematics teaching. Their results illuminate that "teachers' MKT was related to their "current" instructional practice but not to its growth; rather, growth in instructional quality was linked to teachers' vision of quality instruction.

The study we report here aims to contribute towards this emerging field. However, it differs in at least four aspects from these earlier works. First, our 'measure' of teachers' mathematical knowledge does not include pedagogical content knowledge (PCK) which is part of MKT. Our focus is teachers' subject-matter knowledge (SMK). Second, our study is located in a context where traditional or direct teaching is predominant. Third, our study involves a particular and narrowly focused domain of mathematics. We focus on teachers' algebra knowledge and even more specifically to knowledge that can be classified as transformational activities of algebra (Kieran, 2004) as this is the dominant type of activities in our algebra classes. Fourth, possibly a result of our mathematical focus, our study differs in terms of our descriptions of mathematical quality and what we consider as mathematical in instruction. Our description of mathematical quality of instruction is a function of our view of mathematics as a form of scientific knowledge (Vygotsky, 1978) and we contend that mediating this is the critical task of the teacher. Our tool for determining the

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mathematical quality in instruction is the mathematical discourse in instruction (MDI) analytic framework (Adler & Ronda, 2015) which we developed by identifying culturally established instructional tools in the contexts in which we work through which mathematics is mediated, together with those features of mathematical discourse that each tool makes available for learners' engagement. This framework which we present below enables description and evaluation of observable aspects mathematics in and through various mediational tools of instruction.

The level of specificity of the study in terms of its content focus and the dominance of direct teaching is a double edged sword: on the one hand it limits the study's generality particularly in content terms. On the other, it affords a sharpened illustration of a case of how teachers' SMK plays out during instruction. We thus set out to answer the question: *How, if at all, is teachers' subject matter knowledge in algebra related to the algebra they make available to learn in their lessons?*

Transformational algebraic activities

School algebra consists of three interrelated activities: *generational*, *transformational* and *global/meta-level* activities (Kieran, 2004). While these are interrelated activities, each mediates particular aspects of algebra. Each may thus require different mediational means and processes.

Much of the initial-meaning making for school algebra lies in generational activities (Kieran, 2004). The key algebraic object here is the variable and the various relationships between quantities variables represent. School algebra also involves processes where algebra is used as a tool. Global-meta level activities includes problem solving, modelling, studying change, noticing structures, noticing and generalizing relationships, justifying, proving, etc. The multiplicity of processes and ideas that need to be coordinated in global-meta level activities, together with the generational or transformational activities embedded in them, may be cognitively demanding both for the teachers and her learners, but one that is considered to afford more powerful mathematical experiences for learners.

Transformational activities involve various types of algebraic actions or procedures on algebraic objects. These are often referred to as 'rule-based' activities. We assert here that this does not imply mindless learning of procedures and techniques. Transformational activity has its own meaning-making power and this lies in the development of the notion of *equivalence* (Kieran, 2004). Moreover, it provides the general context for applying arithmetical structures e.g., the distributive property or for generalizing operations e.g. subtraction as addition. However, without deliberate foregrounding of the fundamental concept of equivalence beyond numbers to operations and expression; and without deliberate attention to noticing structures and to making explicit the mathematical principles that legitimize the transformational processes, transformational activity can become an exercise for memorizing unrelated techniques. In contexts where transformational activities are dominant, it is thus important to examine whether there is such deliberate attention that leads instead to the development of these notions and an appreciation of general and coherent nature of mathematics, and whether and how this relates to teachers' SMK.

METHOD

Data was sourced from teacher participants in a 20-day subject-matter focussed professional development course spread over one year for Grade 8-10 teachers in disadvantaged schools in the Johannesburg area in South Africa. Out of thirteen teachers who completed the course, seven were teaching transformational activities of algebra when we collected video data at the end of the course. These teachers, the video records and the end of course test results formed our data archive.

Test items for measuring SMK. The end of course test assessed what the teachers learned from the SMK-focused PD the main goal of which was to strengthen teachers' relationship with mathematics, particularly algebra. There were 24 algebra items in all, half of which were purely on transformational tasks, four generational and global meta level tasks where transformational tasks were implicit and eight were on interpreting the properties of function in symbolic form and graphical form. All these items assessed fluency with algebraic expressions in various ways. The seven teachers' results on these test items made up our SMK data.

The video lessons. Our second set of data was the teachers' lessons collected sometime after they took the test. These were video recorded and transcribed. Each transcript was then carefully checked by one of us closely examining and refining the transcript as we watched the video again, and again. Each author individually analysed the lessons at first using the MDI analytic tool and then we compared and discussed our codes. Towards the final round of coding we invited another colleague to do an analysis of the lessons for further confirmation. It is this process of successive video watching, transcript development and analysis that improves the rigour of the analysis.

Data production – constructing mathematical episodes as unit of analysis. A critical task for any lesson analysis is how to divide the lesson into analytic units. Our unit of analysis is a *mathematical episode* so named as our goal is to describe the mathematical 'story' of the lesson as it unfolds. We began by watching the video-recording and simultaneously (re)reading the transcript to identify the intended object of learning that we know is not synonymous with what is enacted (Marton & Tsui, 2004). We then divided the lesson transcript into *mathematical episodes* each identified by a shift in focus of attention with respect to content, typically marked by a new task.

Our Analytic tool. The tool we used to analyse and evaluate the quality of mathematics made available to learn in the lessons is the mathematics discourse in instruction (MDI) framework (Adler & Ronda, 2015). The starting point of the framework is that "teaching and so learning is always about 'something' and bringing this into focus – its mediation - is the teacher's work" (Adler & Ronda, 2015, p. 238). This "something", is the *object of learning* (Marton and Tsui, 2004). For the object of learning to be visible and be in focus during instruction, it needs to be exemplified and experienced by leaners. The most common way this is done is through *tasks*. Embedded in tasks is the mathematical object which is made visible through its *examples*. However, what is being mediated also needs to be *named*, elaborated and explained necessitating use of words and notations to communicate and convey meaning. In addition, what counts as mathematics is also communicated implicitly or explicitly during the lesson reflecting the authority from which mathematical procedures and claims are *legitimated*. In the MDI framework, we claim that these cultural tools of instruction - tasks, examples, naming/word use and legitimations – are means for mediating various aspects of the object of learning and it is in and through these tools that we analyse what is made available to learn. The framework has both descriptive/analytic and evaluative features for each of these mediational means which we present below.

Tasks. The most common and visible mediational means teachers use to engage learners in mathematics is the *task*. A mathematical task has at least two basic components: the object and the action on the object. The MDI framework classified tasks in terms of its cognitive demand. In one of the lessons we analysed here (Teacher 6) the tasks in the main part of lesson were presented and solved one-by-one:

(1) $5x+5y-3x+2y$; (2) Add $(8x^2+4x-6)$ and $(-3x^2+6x+4)$; (3) Subtract $(7x^2+4)$ from $(15x^2-2)$; (4) Subtract $(-3x^2+6x+4)$ from $(8x^2+4x-6)$.

The objects in focus are algebraic expressions and the action required is to add/subtract these. Task (1) involves known facts and procedures because combining similar terms comes before the topic on adding and subtracting algebraic expressions. Task (2) is an applying task because learners are expected to use their knowledge on combining similar terms to add the two multi-term expressions. The same can be said of Task (3) and Task (4) which are also applying tasks. In the MDI analytic framework, Task (1) is coded *R* for recall of facts and procedures and Task (2) is coded *A* for application. Of course, Task (2) may be experienced differently, for example, when the task is stated as *Write 2 algebraic expressions one of which has a negative leading coefficient that give a sum of* $5x^2+10x-2$. This version of the second task embeds the transformational activity within a generational activity and for some learners this could be a problem to solve. This latter version is an example of a task that we code as *C/PS* (connections/problem solving). In this task type, learners can generate many correct answers and potentially encourage them to consider a more systematic and faster way of getting the answers. In this case they would need to see the underlying structure of each term and of the expression itself.

Examples. Examples for us are instantiations of the mathematical object(s) or the object of learning. MDI is interested in two dimensions of example use. First is how the set of examples (and with it the choice of representation) can make visible features that are key to the object of learning. Second is how the set of examples build towards generality. In our analysis of examples, we draw on different patterns of variation proposed in variation theory (Marton & Tsui, 2004). We claim that each pattern of variation and combinations of them within a lesson offers different degrees for experiencing generality of the object that is exemplified. A pattern called similarity

(S) enables local generalization. Similarity occurs when one dimension of the object (its examples) is varied and the rest invariant. If a set of examples brings attention to difference or *contrast* (C), opportunities are made available to recognise boundaries between classes of examples, and so further opportunities to generalise. When examples are *fused* (F), with simultaneous variance/invariance across an example set or through several example sets, opportunities for generalisation is further enhanced. Fusion does not refer to simply presenting examples at random but selecting and sequencing them to make visible the different dimensions of the object that are being varied. In Teacher 6's lesson for instance, what is being exemplified are the various forms in which learners encounter adding and subtracting expressions and the forms of expressions itself. That is, the set of adding and subtracting tasks is the exemplification itself of the object of learning and so we analyse these 'objects' accordingly. Examples 1 and 2 (that is tasks (1) and (2)) differ in one feature: the context for adding expressions. In example 1, adding is in the context of simplifying one expression and so involves looking at individual terms. In example 2, adding is in the context of two algebraic expressions. Note that the expressions in example 2 were bracketed to enable learners to see these as two expressions. Examples 3 and 4 are subtraction tasks with the same structure but the sign of the first terms in the expressions to be subtracted differs. Examples 2 and 4 involve similar expressions but differ in operation. Taken together, the juxtaposition of addition and subtraction operations highlights the contrast between operations. The examples are also sequenced so that there is only one dimension varying at a time. The example set in this lesson is thus coded F for fusion.

Naming. Words or names are signifiers. Each carries a potential meaning and hence its importance in learning and knowledge development. As we stated earlier, what is being mediated needs to be the subject of the talk, and elaborated, examined and explained, necessitating the use of words and notations to communicate and convey meaning. Thus, *what* is named (or not) and *how*, the choice of name to refer to other words, symbols, images, procedures or relationships in the tasks and during elaboration, matters. Naming may involve a range of word use from colloquial nonmathematical words to use of mathematical words as names. We use the following codes to analyse the word use in the lesson. We also exemplify it from some texts in T6 lesson:

1) *Colloquial* (C*) e.g.* everyday language and/or ambiguous pronouns such as this, that, this, to refer to objects in focus; 2) Ms for *math words used as name or labels only* e.g. in reading string of symbols or in referring to the surface feature of the object such as "*constant* is the number beside x"*;* and, 3) Ma for *mathematical words* used in talks about objects, procedures, properties (Ma) e.g *constant is the numerical coefficient of x* or *constant is the number factor in an algebraic expression*.

Legitimations. Part of teaching mathematics is about communicating criteria about what counts as mathematics. Thus, the authority from which the mathematical procedures and claims are *legitimated*, matters. Legitimations ranges from nonmathematical criteria and reliance to teacher authority to use of specific examples and use of mathematical principles. Legitimating criteria may include 1) *Non* *mathematical* e.g. cues are iconic or mnemonic; *a* statement or assertion, typically by the teacher, as if 'fact' or 2) *Mathematical criteria: which may be Local* e.g. a specific or single case (real-life or math) or established shortcut or convention or *General* such as definition, previously established generalization; principles, structures, properties. We identify them when teacher ask why questions or prompt learners with 'because?' or similar prompts.

Having elaborated the elements of the framework and their descriptive coding, we now present the evaluative feature of the framework in Table 1 below. We used this for the summative judgment of the lesson – our 'measure' of mathematical quality in instruction. In the table, the subscripts from 1-4 indicate progression in quality towards the privileged scientific knowledge.

Object of Learning (OoL)			
Tasks	***Example set***	***Naming***	***Legitimations***
T1–only include recall (R) type tasks **T2** – at least includes an application (A) tasks in the main lesson **T3** – A and includes at least a problem solving or task requiring multiple connections (PS/C)	**E1** - example space provides contrast (C) or similarity (S) **E2** - example spaces provide both C *and* S or two example spaces involving S **E3** – example space shows all fusion	**N1** – talk consist mostly of colloquial (C) words; if there are mathematical words (Ms), it is used as label. **N2** - movement between C and Ms but mostly Ms; some MA **N3** – movement between C, Ms and Ma but mostly Ma words	**L0** – no justification provided or if there is it is nonmathematical **L1** – most claims made are legitimated using a single example (Local) **L2** - criteria extend beyond NM and local and attempt to include generality, but this is partial **L3** - General full math legitimation of a concept or procedure, is principled and/or derived/proved

Table 1: MDI indicators of the quality of mathematics made available to learn.

RESULTS

Having illustrated our coding, and how these accumulate across episodes into levels, we can now present the overall results of our study for all seven teachers (Table 2).

Teachers	**Object of Learning**	**SMK (%)**	**MDI** = (Examples, Tasks, Naming, Legitimating)
T1	Evaluating algebraic expressions	23	(E1, T1, N1, L0)
T2	Factorizing Monomials and binomials	39	(E1*, T2->T1, N1, L0)
T3	Multiplying two binomials	57	(E1, T2, N2, L0)

T4	Solving linear and quadratic equations	66	(E3, T2->T1, N2, L1)
T5	Solving quadratic equation	78	(E3, T2->T1, N3, L3)
T6	Adding and subtracting algebraic expressions	78	(E3, T2->T1, N2, L3)
T7	Simplifying/Dividing Algebraic expressions	83	(E3, T2->T1, N2, L2)

E1* - Correct with the example set that can only be coded E1.

Table 2: Teachers' SMK and what is made available to learn in instruction (MDI).

The results in Table 2 indicate that the teachers' SMK is related to quality of mathematics they make available to learn – to some extent – that is only through use of examples and legitimating criteria. The same cannot be said as strongly for the kind of tasks set forth for learners to engage nor in teachers' naming. The teachers that scored below 60% in the test used examples that varied in one dimension only while those who scored above 60% use examples that varied on two or more dimensions. In addition, teachers with scores below 60% did not provide legitimations that were mathematically authorised. In contrast, those with scores above 60% provided mathematical legitimations ranging from giving a specific example to drawing on mathematical principles that were either complete or partially complete.

DISCUSSION

The first hypothesis that we can offer from our focused study is that teachers' subject matter knowledge does matter in how they choose the features of examples critical to the object of learning and that could then offer opportunities for generalizing. Selecting and sequencing of examples is not trivial. The teacher needs to be able to identify those critical aspects of the object of learning that need to be foregrounded. Our second hypothesis is that the domain of knowledge that teachers draw from to substantiate mathematical claims made during the lesson (legitimations) also has strong links to teachers' knowledge of mathematics. It appears appreciating the need to mathematically substantiate the claims made in classroom talk is linked to stronger SMK. Our third hypothesis relates to word use/naming. Teachers' SMK seems related to the quality of mathematical talk in the lesson although we cannot state this hypothesis as strongly as the other first two above. The lessons with predominance of colloquial names were by the two teachers with the weakest SMK. The other teachers used mathematical names in the lesson but this mostly entailed reading of strings of symbols. We can think of two explanations for this. First is a weakness in our coding – that it is not sufficiently differentiated to make visible more nuanced differences across T3-T7. Also, our N2 level covers a wide spectrum of *values* between colloquial and more formal talk and would benefit from further categorization or disaggregation. We have pursued this in our analysis of textbook lessons (Ronda & Adler, 2017) where we differentiated between how objects and procedures are named. A second explanation may be because the lessons were on transformational activities of algebra where predominance of the reading of symbols in the talk is typical. It thus would be interesting to study if the same result will be obtained if the object of learning of the lesson were more on generational or meta-level activities. Our last hypothesis relates to task demand. Tasks that involve problem solving and multiple connections which are features of reform-oriented teaching did not figure in any of the lessons we investigated. Is it possible that this has to do with the rule-based nature of transformational activities or to the predominant traditional teaching of mathematics from which we draw our lessons? Has it to do with teachers having limited exposure to *rich* mathematical tasks or has it to do with teachers vision of quality mathematics teaching (Munter & Correnti, 2017). Is it that teachers position learners as unable to engage with complex processes typical of meta-level algebraic activities? These questions merit further study, particularly whether the hypotheses we presented will hold for algebra lessons that involve generational and/or meta-level activities.

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