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TIME-SPLITTING METHODS TO SOLVE THE STOCHASTIC INCOMPRESSIBLE STOKES EQUATION*

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Abstract. For the stochastic incompressible time-dependent Stokes equation, we study different time-splitting methods that decouple the computation of velocity and pressure iterates in every iteration step. Optimal strong convergence is shown for Chorin's time-splitting scheme in the case of solenoidal noise, while computational counterexamples show poor convergence behavior in the case of general stochastic forcing. This suboptimal performance may be traced back to the nonregular pressure process in the case of general noise. A modified version of the deterministic time-splitting method that distinguishes between the deterministic and stochastic pressure removes this deficiency, leading to optimal convergence behavior.

Key words. discretization of stochastic partial differential equation, splitting scheme, error analysis

AMS subject classifications. 60H15, 65M12, 65M60, 76D07

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1. Introduction. Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space, and let $D \subset \mathbb{R}^d$, $d = 2, 3$, be a bounded polyhedral domain. We consider the d-dimensional stochastic Stokes equation which $\mathbb{P}$-a.s. satisfies

$$\begin{aligned}
\mathbf{u}_t - \Delta \mathbf{u} + \nabla p &= \mathbf{f} + \mathbf{B}(\cdot, \mathbf{u})\dot{\mathbf{W}} && \text{on } D_T := (0,T) \times D,\\
\operatorname{div} \mathbf{u} &= 0 && \text{on } D_T,\\
\mathbf{u}(0) &= \mathbf{u}_0 && \text{on } D,\\
\mathbf{u} &= \mathbf{0} && \text{on } \partial D_T := (0,T) \times \partial D.
\end{aligned} \tag{1.1}$$

Here, the velocity $\mathbf{u} = (u_1, \dots, u_d)$ and the pressure p are unknown random fields on D_T, and $\mathbf{W}$ is an $\mathbb{F}$-adapted cylindrical Wiener process on $\boldsymbol{\mathcal{H}}$, where $\boldsymbol{\mathcal{H}}$ is a separable Hilbert space. Possible choices of $\boldsymbol{\mathcal{H}}$ include $\mathbf{L}^2(D)$ or $\mathbf{H}^1_0(D)$. Finally, let $t \mapsto \mathbf{B}\big(t, \mathbf{u}(t)\big)$ be an appropriate operator-valued map to be specified later; as an example. As an example (see section 6 for details), we may consider a constant operator which maps from $\boldsymbol{\mathcal{H}}$ to a finite dimensional subspace of $\mathbf{H}^1_0(D)$. The study of the stochastic incompressible Stokes system is, e.g., motivated from modeling microfluids, where inertial effects are generally negligible, and microscopic fluctuations are relevant contributions to fluid flow dynamics; cf. [18, 7].

Let $\boldsymbol{\mathcal{V}} := \{\boldsymbol{\psi} \in \mathbf{C}^\infty_0(D) : \operatorname{div} \boldsymbol{\psi} = 0 \text{ in } D\}$ denote the space of solenoidal functions with closures $\mathbf{H} := \overline{\boldsymbol{\mathcal{V}}}^{\mathbf{L}^2(D)}$ and $\mathbf{V} := \overline{\boldsymbol{\mathcal{V}}}^{\mathbf{H}^1_0(D)}$. Then, strong solutions $\mathbf{u} \in L^2(\Omega; C([0,T]; \mathbf{H})) \cap L^2(\Omega_T; \mathbf{V})$, where $\Omega_T := \Omega \times (0,T)$, of (1.1) for proper operators $\mathbf{B}$ are usually obtained by a Galerkin method which employs *divergence-free* approximates from finite dimensional spaces $\mathbf{H}_n \subset \mathbf{H}$ $(n \geq 1)$ to remove the pressure from the problem. This strategy is different from a numerical setting, where the choice of

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the finite dimensional ansatz space for the pressure, as well as regularity properties of the pressure from (1.1), crucially determine both stability and convergence behavior of the resulting scheme; see, e.g., [9].

Properly handling the incompressibility constraint numerically is a nontrivial issue and is usually accomplished in a variational rather than a pointwise sense. As it is well-known for the corresponding deterministic problem, discretization strategies based on implicit methods cause a significant computational effort due to the coupled computation of both velocity and pressure iterates. Moreover, choices of stable finite element pairings are restricted by the LBB constraint. As a consequence, splitting algorithms turn out to be a very promising alternative to reducing the complexity of actual computations by successively updating velocity and pressure iterates; we refer to [10] for a recent survey on this topic. It is evident that such a strategy is desirable to solve the stochastic partial differential equation (1.1), where a significant number of trajectories has to be computed to obtain statistically relevant results for quantities of interest. The goal of this paper is to show that the interplay of time-splitting strategies and the "stochastic nature" of problem (1.1) is subtle, leading to a poor convergence behavior of known time-splitting schemes which perform well in the deterministic case. Computational experiments detail this assertion, which is rooted in the nonregular pressure process in (1.1). In a second step, an optimally convergent stochastic time-splitting scheme is constructed that distinguishes between approximations of the (nonregular) stochastic pressure and the (more regular) deterministic pressure.

To illustrate the problematic issue to construct a proper time-splitting scheme for a stochastic equation, we start with Chorin's projection method [4, 6, 19], which is one of the first splitting schemes to solve the deterministic incompressible (Navier–) Stokes equation. Consider the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. Let $\mathbf{f}^{m+1} := \mathbf{f}(t_{m+1}, \cdot) \in L^2(\Omega, \mathbf{L}^2)$, suppose that $\mathbf{u}_0 \in L^2(\Omega, \mathbf{V})$ is given, and consider independently and identially distributed stochastic increments $\Delta \mathbf{W}_{m+1} := \mathbf{W}(t_{m+1}) - \mathbf{W}(t_m)$, where $k = t_{m+1} - t_m > 0$ denotes the mesh-size of the equidistant grid $I_k := \{t_m\}_{m=0}^M$ covering $[0, T]$.

ALGORITHM 1.1. 1. Let $m \geq 0$. For given $\mathbf{u}^m \in L^2(\Omega, \mathbf{V})$ and $\widetilde{\mathbf{u}}^m \in L^2(\Omega, \mathbf{H}_0^1(D))$, find $\widetilde{\mathbf{u}}^{m+1} \in L^2(\Omega, \mathbf{H}_0^1(D))$ such that $\mathbb{P}$-a.s.

$$(1.2) \qquad \big(\widetilde{\mathbf{u}}^{m+1} - \mathbf{u}^m\big) - k\, \Delta \widetilde{\mathbf{u}}^{m+1} = k\, \mathbf{f}^{m+1} + \mathbf{B}(t_m, \widetilde{\mathbf{u}}^m) \Delta \mathbf{W}_{m+1} \quad \text{on } D.$$

2. Compute $\mathbf{u}^{m+1} \in L^2(\Omega, \mathbf{H})$ and $p^{m+1} \in L^2(\Omega, H^1(D)/\mathbb{R})$ such that $\mathbb{P}$-a.s.

$$(1.3) \qquad \mathbf{u}^{m+1} - \widetilde{\mathbf{u}}^{m+1} + k\, \nabla p^{m+1} = 0, \qquad \operatorname{div} \mathbf{u}^{m+1} = 0 \qquad \text{on } D,$$

$$(1.4) \qquad \langle \mathbf{u}^{m+1}, \mathbf{n} \rangle = 0 \qquad \text{on } \partial D.$$

We start a discussion of the scheme which ignores the stochastic term for a moment: the latter step can be reformulated as a problem for the pressure function only,

$$(1.5) \qquad -\Delta p^{m+1} = -\frac{1}{k} \operatorname{div} \widetilde{\mathbf{u}}^{m+1} \quad \text{on } D, \qquad \partial_{\mathbf{n}} p^{m+1} = 0 \quad \text{on } \partial D.$$

Hence, each step consists of (1.2), (1.5), and the algebraic update (1.3) to obtain $\mathbf{u}^{m+1} \in \mathbf{H}$.

In order to understand error effects inherent to discretization in time, and operator splitting in Chorin's scheme, we shift the index in $(1.3)_1$ back and add the resulting equation to (1.2); together with (1.5), we then arrive at

$$(1.6)\quad \big(\widetilde{\mathbf{u}}^{m+1}-\widetilde{\mathbf{u}}^{m}\big)-k\,\Delta\widetilde{\mathbf{u}}^{m+1}+k\,\nabla p^{m}=k\,\mathbf{f}^{m+1}+\mathbf{B}(t_m,\widetilde{\mathbf{u}}^{m})\Delta\mathbf{W}_{m+1}\qquad \text{on } D,$$

$$(1.7)\quad \operatorname{div}\widetilde{\mathbf{u}}^{m+1}-k\,\Delta p^{m+1}=0\qquad \text{on } D,$$

$$(1.8)\quad \partial_{\mathbf{n}}p^{m+1}=0\qquad \text{on } \partial D,$$

and $\widetilde{\mathbf{u}}^0\equiv\mathbf{u}_0$ on D. We make the following observations: (i) iterates $\{\widetilde{\mathbf{u}}^m\}_{m\geq 0}$ of Algorithm 1.1 are no longer divergence-free but satisfy the "quasi-compressibility equation" (1.7) with a penalization parameter equal to k; (ii) iterates of the pressure satisfy a homogeneous Neumann boundary condition, which is in contrast to pressure $p:D_T\to\mathbb{R}$ from (1.1); and (iii) the pressure iterate in (1.6) is used in an explicit fashion, which rules out an immediate discrete energy law, where test functions $\mathbf{u}^{m+1}$ and p^{m+1} are used.

For the deterministic case, by assuming $D\subset\mathbb{R}^d$ to be a convex polyhedral domain, $\mathbf{u}_0\in\mathbf{V}\cap\mathbf{H}^2(D)$, and $\mathbf{f}\in W^{2,\infty}(0,T;\mathbf{L}^2(D))$, the following optimal estimates are proved in [16, Theorem 6.1]:

$$(1.9)\quad \max_{1\leq m\leq M}\Big\{\sqrt{\tau^m}\|\mathbf{u}(t_m,\cdot)-\widetilde{\mathbf{u}}^m\|_{\mathbf{L}^2}+\sqrt{k}\|\mathbf{u}(t_m,\cdot)-\widetilde{\mathbf{u}}^m\|_{\mathbf{H}^1}\Big\}\leq Ck,$$

where $\tau^m:=\min\{1,t_m\}$. Its proof consists of three steps. First, optimal error estimates for the implicit Euler discretization using solenoidal velocity fields are derived, where its derivation benefits from valid regularity properties of solutions $\mathbf{u}\in C([0,T];\mathbf{V}\cap\mathbf{H}^2)\cap H^2(0,T;\mathbf{V}')$, where $\mathbf{X}'$ denotes the dual of the Banach space $\mathbf{X}$. Then, a modified version of (1.6)–(1.8) is studied with respect to both convergence and stability properties, where the pressure iterate p^m in (1.6) is shifted to p^{m+1}; a key property here is the existing bound $p\in L^\infty(0,T;H^1/\mathbb{R})$ for the deterministic evolutionary incompressible Stokes problem. We remark that this pressure-stabilization method (with parameter $\varepsilon=k$) is of its own interest, since it allows for more choices of finite element pairings [2, 12], which are usually restricted by the discrete LBB condition. Finally, the third step accounts for the explicit treatment of the pressure in (1.6), which strongly benefits from an upper bound for $\int_0^T\tau(s)\|\nabla p_t(s)\|^2\,\mathrm{d}s$ for the pressure from (1.1) in terms of the data $\mathbf{u}_0$, $\mathbf{f}$, and D_T, where $\tau(s)=\min\{1,s\}$.

The goal of the present work is to study convergence properties of $\mathbf{H}^1_0(D)$-valued iterates $\{\widetilde{\mathbf{u}}^m\}_m$ from Algorithm 1.1 to approximate solutions of (1.1). The main difficulties which enter in the stochastic setting are due to *restricted regularity properties* (in time) of solutions $(\mathbf{u},p)$ to (1.1), which are due to the driving stochastic term; for instance, the pressure which is constructed by Helmholtz decomposition after $\mathbf{u}$ is found need not even be absolutely continuous with respect to time [13] (see (2.7)), but its regularity properties are crucial for the convergence analysis of our splitting method as detailed above. Hence, there is the question of whether splitting effects inherent to Algorithm 1.1 will deteriorate convergence rates of computed iterates $\{\widetilde{\mathbf{u}}^m\}_m$— if compared to divergence-free velocity iterates $\{\mathbf{w}^m\}_m\subset L^2(\Omega;\mathbf{V})$, approximating $\{\mathbf{u}(t_m,\cdot)\}_m$, and solving the coupled Euler–Maruyama time discretization of (1.1), which holds $\mathbb{P}$-a.s.,

(1.10)
$$\big(\mathbf{w}^{m+1}-\mathbf{w}^m\big)-k\Delta\mathbf{w}^{m+1}+k\nabla q^{m+1}=k\mathbf{f}^{m+1}+\mathbf{B}(t_m,\mathbf{w}^m)\Delta\mathbf{W}_{m+1}\qquad \text{on } D,$$

$$(1.11)\quad \operatorname{div}\mathbf{w}^{m+1}=0\qquad \text{on } D,$$

where $\mathbb{P}$-a.s. $\mathbf{w}^0 = \mathbf{u}_0$ on D. Note that the pressure $q^{m+1} : \Omega \times D \to \mathbb{R}$, which approximates $p(t_m, \cdot)$, will be eliminated from the convergence analysis when solenoidal test functions are used. As a consequence, the following rates of strong convergence of Euler iterates $\{\mathbf{w}^m\}_m$ are proved in [11]:

$$\max_{1\le m\le M}\Big(\mathbb{E}\Big[\|\mathbf{u}(t_m,\cdot)-\mathbf{w}^m\|^2_{\mathbf{L}^2}\Big]\Big)^{1/2} + \Big(\mathbb{E}\Big[k\sum_{1\le m\le M}\|\nabla\big(\mathbf{u}(t_m,\cdot)-\mathbf{w}^m\big)\|^2_{\mathbf{L}^2}\Big]\Big)^{1/2} \le C_T\sqrt{k}. \tag{1.12}$$

In fact, [11, Theorem 3.1] provides rates of convergence for a finite dimensional Wiener process. However, the proof can be modified in such a way that the same result holds for a Wiener process having a covariance operator with finite trace. Since this will be the case (see Assumption (2.3) below), Theorem 3.1 of [11] is applicable.

The first main result in this paper is Theorem 3.1, which shows property (1.12) for iterates $\{\widetilde{\mathbf{u}}^m\}$ from Algorithm 1.1 in the case of solenoidal noise. A discretization in space using equal-order finite elements is studied in section 5, and overall error estimates for related finite element iterates $\{\widetilde{\mathbf{u}}^m\}$ are given in Theorem 5.1. Then, computational studies are provided in section 6 which compare convergence behavior of iterates from Algorithm 1.1 and (1.10)–(1.11) for solenoidal and general $\mathbf{L}^2$-noise and highlight that solenoidal noise is imperative for optimal convergence behavior of the splitting Algorithm 1.1, which in the case of general noise deteriorates to a poor convergence behavior. Those computational studies motivate the new time-splitting scheme (Algorithm 4.1) in section 4, which distinguishes between approximate deterministic and stochastic pressure iterates. As a consequence, optimal rate of convergence for general noise is shown both theoretically (see Theorem 4.1) and computationally.

The remainder of this work is organized as follows. Necessary background for the stochastic partial differential equation (1.1) and useful stability bounds for Euler iterates $\{\mathbf{w}^m\}_m$ solving (1.10)–(1.11) are provided in section 2. In section 3, we estimate the additional different perturbation effects due to the quasi-compressibility constraint (1.7) and the splitting character of Algorithm 1.1 due to the explicit treatment of the pressure in (1.6), which then leads to Theorem 3.1. In section 5, a finite element discretization of Algorithm 1.1 is proposed, where the study of the coupled error effects due to time discretization, time splitting, and spatial discretization leads to Theorem 5.1. Computational evidence to highlight the failure of Chorin's method in the case of general noise is reported in section 6, as well as the modified Algorithm 4.1 that performs optimally for general noise.

2. Preliminaries.

For a domain $D \subset \mathbb{R}^d$, $d = 2, 3$, let $L^2(D)$ denote the usual Lebesgue space of square integrable functions, endowed with the scalar product $(\cdot,\cdot)$ and norm $\|\cdot\|_{L^2}$. We employ the usual Sobolev spaces $W^{k,p}(D)$ and $W^{1,p}_0(D)$ with norm $\|\cdot\|_{W^{k,p}}$, while for $p = 2$ we use the notation $H^k(D) = W^{k,2}(D)$. Since we always work on D, most of the time the letter D will be dropped. For vector-valued functions, the corresponding spaces are denoted by bold letters, e.g., $(H^1)^d = \mathbf{H}^1$.

2.1. The problem.

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete probability space with continuous filtration $\mathbb{F} = \{\mathcal{F}_t;\, t \ge 0\}$, and let $\{\beta^j(t);\, t \ge 0\}_j$, $j \in \mathbb{N}$, be a sequence of independent identically distributed $\mathbb{R}$-valued Brownian motions on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. Let $\boldsymbol{\mathcal{H}}$ be a Hilbert space and $\{\mathbf{e}_j;\, j = 1, 2, \dots\}$ be an orthonormal basis of $\boldsymbol{\mathcal{H}}$. We denote by $\mathbf{W} = \{\mathbf{W}(t);\, t \ge 0\}$ the cylindrical Wiener process on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, which is defined as

$$[0,\infty) \ni t \mapsto \mathbf{W}(t) = \sum_{j=1}^{\infty} \beta^j(t)\mathbf{e}_j.$$

For $p \geq 1$ and $\boldsymbol{\mathcal{K}}$ being a Hilbert space, we denote by $M_{\mathbb{F}}^p(0,T;\boldsymbol{\mathcal{K}})$ $(:= M_{\mathbb{F}}^p(\Omega \times [0,T];\boldsymbol{\mathcal{K}}))$ the space of all $\mathbb{F}$-adapted processes belonging to $L^p(\Omega \times [0,T];\boldsymbol{\mathcal{K}})$. Moreover, let $\mathcal{L}^2(\boldsymbol{\mathcal{H}},\boldsymbol{\mathcal{K}})$ denote the space of linear operators from $\boldsymbol{\mathcal{H}}$ to $\boldsymbol{\mathcal{K}}$ having a finite Hilbert–Schmidt norm. For any process $\boldsymbol{\varphi} \in M_{\mathbb{F}}^2(0,T;\mathcal{L}^2(\boldsymbol{\mathcal{H}},\boldsymbol{\mathcal{K}}))$ we define the stochastic integral $\int_0^t \boldsymbol{\varphi}(s)\,\mathrm{d}\mathbf{W}(s)$, $0 \leq t \leq T$, as the unique continuous $\boldsymbol{\mathcal{K}}$-valued $\mathbb{F}$-martingale such that for all $\mathbf{h} \in \boldsymbol{\mathcal{K}}$ it holds that

$$\left(\int_0^t \boldsymbol{\varphi}(s)\,\mathrm{d}\mathbf{W}(s),\mathbf{h}\right)_{\boldsymbol{\mathcal{K}}} = \lim_{J\to\infty}\sum_{j=1}^{J}\int_0^t \big(\boldsymbol{\varphi}(s)\mathbf{e}_j,\mathbf{h}\big)_{\boldsymbol{\mathcal{K}}}\,\mathrm{d}\beta^j(s) \qquad \forall\, 0 \leq t \leq T.$$

Moreover, its second moment satisfies Ito's isometry property,

$$\mathbb{E}\left[\left\|\int_0^T \boldsymbol{\varphi}(s)\,\mathrm{d}\mathbf{W}(s)\right\|_{\boldsymbol{\mathcal{K}}}^2\right] = \mathbb{E}\left[\int_0^T \|\boldsymbol{\varphi}(s)\|^2_{\mathcal{L}^2(\boldsymbol{\mathcal{H}},\boldsymbol{\mathcal{K}})}\,\mathrm{d}s\right].$$

Remark 2.1. This construction includes the case of a $\mathbf{Q}$-Wiener process with values in $\boldsymbol{\mathcal{H}}$ and a positive self-adjoint covariance operator $\mathbf{Q}:\boldsymbol{\mathcal{H}}\to\boldsymbol{\mathcal{H}}$ of trace class, i.e., with eigenvalues $\{\lambda_j\}_{j=1}^{\infty}$ satisfying $\sum_{j=1}^{\infty}\lambda_j < \infty$, and eigenfunctions $\{\mathbf{e}_j\}_{j=1}^{\infty} \subset \boldsymbol{\mathcal{H}}$ which build an orthonormal basis. Then we may represent a $\mathbf{Q}$-Wiener process with covariance operator $\mathbf{Q}$ in the form

$$[0,\infty) \ni t \mapsto \mathbf{W}(t) = \sum_{j=1}^{\infty} \sqrt{\lambda_j}\,\beta^j(t)\mathbf{e}_j.$$

Thus, Ito's isometry reads

$$\mathbb{E}\left[\left\|\int_0^T \boldsymbol{\varphi}(s)\,\mathrm{d}\mathbf{W}(s)\right\|_{\boldsymbol{\mathcal{K}}}^2\right] = \mathbb{E}\left[\int_0^T \|\boldsymbol{\varphi}(s)\circ\mathbf{Q}^{1/2}\|^2_{\mathcal{L}^2(\boldsymbol{\mathcal{H}},\boldsymbol{\mathcal{K}})}\,\mathrm{d}s\right].$$

We recover the stochastic integral with the cylindrical Wiener process if we set $\mathbf{Q} = \mathbf{Id}$ (which has no finite trace). For such a $\mathbf{Q}$-Wiener process it is possible to enlarge the class of integrands to the space $M_{\mathbb{F}}^2(0,T;\mathcal{L}^2(\mathbf{Q}^{1/2}(\boldsymbol{\mathcal{H}}),\boldsymbol{\mathcal{K}}))$, which includes $M_{\mathbb{F}}^2(0,T;\mathcal{L}^2(\boldsymbol{\mathcal{H}},\boldsymbol{\mathcal{K}}))$.

Recall the Stokes operator $\mathbf{A} \equiv -\mathbf{P_H}\Delta$ with domain $D(\mathbf{A}) = \mathbf{V}\cap\mathbf{H}^2(D)$, which is endowed with the norm $\|\cdot\|_{D(\mathbf{A})} = \|\mathbf{A}\,\cdot\|_{\mathbf{L}^2}$. Here, $\mathbf{P_H}:\mathbf{L}^2\to\mathbf{H}$ denotes the (Leray) projection operator. For Lipschitz domains $D\subset\mathbb{R}^d$ and on $\mathbf{V}\cap\mathbf{H}^2(D)$, the operator norm $\|\cdot\|_{D(\mathbf{A})}$ is equivalent to the $\mathbf{H}^2(D)$-norm. Throughout this work, we assume that the domain D is such that for the solution of the Stokes equations

$$-\Delta\mathbf{u} + \nabla\pi = \mathbf{f} \quad \text{on } D, \quad \operatorname{div}\mathbf{u} = 0 \quad \text{on } D, \quad \mathbf{u} = 0 \quad \text{on } \partial D,$$

there holds the bound

$$\|\mathbf{u}\|_{\mathbf{H}^2} + \|\pi\|_{H^1} \leq C\|\mathbf{f}\|_{\mathbf{L}^2} \tag{2.1}$$

if $\mathbf{f} \in \mathbf{L}^2$. In two dimensions this is known to be true for convex polygonal domains, while in three dimensions this holds for C^2 boundaries (see [20, Proposition 2.2]), and it is believed to hold for convex polyhedra as well. Throughout the paper, let

$$\mathbf{u}_0 \in L^2\big(\Omega, \mathcal{F}_0, \mathbb{P}; \mathbf{V}\big) \quad \text{and} \quad \mathbf{f} \in L^2\big(\Omega \times (0,T); \mathbf{L}^2(D)\big). \tag{2.2}$$

Suppose that $\mathbf{B} : [0,T] \times \mathbf{H}^1_0 \to \mathcal{L}^2(\boldsymbol{\mathcal{H}}; \boldsymbol{\mathcal{K}})$ is measurable, Lipschitz, and sublinear for $\boldsymbol{\mathcal{K}} = \mathbf{L}^2$ and $\boldsymbol{\mathcal{K}} = \mathbf{H}^1_0$; more precisely, there exists a constant $C_T > 0$ such that for $\boldsymbol{\mathcal{K}} = \mathbf{L}^2$ and $\boldsymbol{\mathcal{K}} = \mathbf{H}^1_0$,

$$\big\|\mathbf{B}(t,\mathbf{v}) - \mathbf{B}(t,\mathbf{w})\big\|_{\mathcal{L}^2(\boldsymbol{\mathcal{H}},\boldsymbol{\mathcal{K}})} \le C_T \|\mathbf{v} - \mathbf{w}\|_{\boldsymbol{\mathcal{K}}} \qquad \forall\, t \in [0,T] \quad \forall\, \mathbf{v}, \mathbf{w} \in \mathbf{H}^1_0, \tag{2.3}$$

and for $\mathbf{v} \in \mathbf{H}^1_0$

$$\mathbf{B}(\cdot, \mathbf{v}) \in L^2\Big(\Omega; L^2\big(0,T; \mathcal{L}^2(\boldsymbol{\mathcal{H}}, \mathbf{H}^1_0)\big)\Big). \tag{2.4}$$

We call an $\mathbb{F}$-adapted stochastic process a strong solution of (1.1) (in the stochastic sense) if $\mathbf{u} \in L^2(\Omega; C([0,T]; \mathbf{H})) \cap M^2_{\mathbb{F}}(0,T; \mathbf{V})$ such that for all $t \in [0,T]$ and all $\boldsymbol{\psi} \in \mathbf{V}$ there holds $\mathbb{P}$-a.s.,

$$\big(\mathbf{u}(t), \boldsymbol{\psi}\big) + \int_0^t \big(\nabla \mathbf{u}(s), \nabla \boldsymbol{\psi}\big)\,\mathrm{d}s = \big(\mathbf{u}_0, \boldsymbol{\psi}\big) + \int_0^t \big(\mathbf{f}(s), \boldsymbol{\psi}\big)\,\mathrm{d}s + \left(\int_0^t \mathbf{B}(s,\mathbf{u})\mathrm{d}\mathbf{W}(s), \boldsymbol{\psi}\right).$$

The existence of a unique strong solution $\mathbf{u} \in L^2(\Omega; C([0,T]; \mathbf{V})) \cap M^2_{\mathbb{F}}(0,T; D(\mathbf{A}))$ which satisfies $\mathbb{P}$-a.s. the energy equation

$$\begin{aligned} &\|\mathbf{u}(t)\|^2_{\mathbf{L}^2} + 2\int_0^t \|\nabla \mathbf{u}(s)\|^2_{\mathbf{L}^2}\,\mathrm{d}s \\ &\quad = \|\mathbf{u}_0\|^2_{\mathbf{L}^2} + 2\int_0^t \big(\mathbf{f}(s), \mathbf{u}(s)\big)\,\mathrm{d}s + 2\int_0^t \Big(\mathbf{B}\big(s, \mathbf{u}(s)\big)\mathrm{d}\mathbf{W}(s), \mathbf{u}(s)\Big) \\ &\qquad + \int_0^t \|\mathbf{B}\big(s,\mathbf{u}(s)\big)\|^2_{\mathcal{L}^2(\boldsymbol{\mathcal{H}};\mathbf{L}^2)}\,\mathrm{d}s \qquad \forall\, t \in [0,T] \end{aligned} \tag{2.5}$$

is well-known; see, for instance, [15, Theorem 6.19]. Moreover, standard arguments yield for $\mathbf{u}_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; \mathbf{V})$ that $\mathbf{u} \in L^2(\Omega; C([0,T]; \mathbf{V})) \cap M^2_{\mathbb{F}}(0,T; D(\mathbf{A}))$, and

$$\begin{aligned} &\mathbb{E}\Big[\sup_{0\le t\le T} \|\mathbf{u}(t)\|^2_{\mathbf{H}^1}\Big] + \mathbb{E}\left[\int_0^T \|\mathbf{A}\mathbf{u}(s)\|^2_{\mathbf{L}^2}\,\mathrm{d}s\right] \\ &\quad \le C_T \left\{ 1 + \mathbb{E}\big[\|\mathbf{u}_0\|^2_{\mathbf{H}^1}\big] + \mathbb{E}\left[\int_0^T \|\mathbf{f}(s)\|^2_{\mathbf{L}^2}\,\mathrm{d}s\right] \right. \\ &\qquad \left. + \mathbb{E}\left[\int_0^T \|\mathbf{B}(s, \mathbf{0})\|^2_{\mathcal{L}^2(\boldsymbol{\mathcal{H}},\mathbf{L}^2)}\,\mathrm{d}s\right] \right\}; \end{aligned} \tag{2.6}$$

see, e.g., [5, Theorem 4.4 and section 5]. In contrast, the limited available analytical results about the pressure in (1.1) indicate very restricted smoothness. For $\mathbf{B} : [0,T] \times \mathbf{L}^2(D) \to \mathcal{L}^2(\boldsymbol{\mathcal{H}}, \mathbf{L}^2(D))$, there exists a unique (distributional) pressure (see [13, Theorem 2.2])

$$p \in L^1\Big(\Omega, \mathbb{F}, \mathbb{P}; W^{-1,\infty}\big(0,T; L^2(D)\big)\Big) \tag{2.7}$$

such that $\mathbb{P}$-a.s.

$$\mathbf{u}_t - \Delta \mathbf{u} + \nabla p = \mathbf{f} + \mathbf{B}(\cdot, \mathbf{u})\dot{\mathbf{W}} \qquad \text{in } \big(\mathcal{D}'(D_T)\big)^d, \tag{2.8}$$

$$\int_D p \, \mathrm{d}\mathbf{x} = 0 \qquad \text{in } \mathcal{D}'(0,T). \tag{2.9}$$

This result gives evidence of a deregularizing effect upon the pressure in (1.1) which is exerted by a general noise. This feedback effect of general noise onto the (lack of) regularity of the pressure may be avoided by analytical constructions using Leray projection but causes severe deterioration with respect to accuracy of well-known numerical schemes where accurate pressures are needed.

As will be shown in Lemma 2.1 below, pressure iterates of the (coupled) Euler–Maruyama scheme (1.10)–(1.11) are more regular for noise that is solenoidal, which is why we assume

$$\mathbf{B} : [0,T] \times \mathbf{H}^1_0(D) \to \mathcal{L}^2(\boldsymbol{\mathcal{H}}, \mathbf{V}) \tag{2.10}$$

in sections 3 and 5. Conversely, computational experiments in section 6.1 show that Chorin's projection method performs optimally only in the case of solenoidal noise. In the case $\mathbf{B} : [0,T] \times \mathbf{H}^1_0 \to \mathcal{L}^2\big(\boldsymbol{\mathcal{H}}; \mathbf{V}\big)$, related a priori estimates in Lemma 2.1 motivate

$$p \in L^1\Big(\Omega, \mathbb{F}, \mathbb{P}; L^2(0,T; H^1(D)/\mathbb{R})\Big), \tag{2.11}$$

which provides enough regularity of the pressure such that the splitting scheme performs optimally. However, we are not aware of a rigorous analytical motivation of this assertion for the limiting equations (1.1).

Remark 2.2. A velocity field $\mathbf{u}$ that solves the stochastic incompressible (Navier–) Stokes equations is usually constructed by an ("inner approximation") Galerkin method that employs solenoidal test functions and thus eliminates the pressure p from the problem in a first step; a pressure p is then later obtained by de Rham's theorem; see, e.g., [1, 8, 5, 13]. A different strategy is to obtain solutions by perturbing the incompressibility constraint ("quasi-compressibility method") to avoid the saddle-point character of the problem, for example, ($\varepsilon > 0$):

(i) $\operatorname{div} \mathbf{u}^\varepsilon + \varepsilon p^\varepsilon = 0$ on D_T,

(ii) $\operatorname{div} \mathbf{u}^\varepsilon - \varepsilon \Delta p^\varepsilon = 0$ on D_T, $\quad \partial_{\mathbf{n}} p^\varepsilon = 0$ on ∂D_T,

(iii) $\operatorname{div} \mathbf{u}^\varepsilon + \varepsilon p^\varepsilon_t = 0$ on D_T, $\quad p^\varepsilon(0) = p(0)$ on D,

(iv) $\operatorname{div} \mathbf{u}^\varepsilon - \varepsilon \Delta p^\varepsilon_t = 0$ on D_T, $\quad \partial_{\mathbf{n}} p^\varepsilon = 0$ on ∂D_T, $\quad p^\varepsilon(0) = p(0)$ on D.

The penalty method (i) is used in [3], and the artificial compressibility method (iii) in [14] to construct solutions of the stochastic incompressible Navier–Stokes equations. The pressure stabilization ansatz (ii) is related to Algorithm 1.1, where $\varepsilon = k$ is chosen in (1.5). The pressure correction method (iv) is used for numerical schemes as well; cf. [16] for further details.

2.2. Euler scheme. Suppose that (2.2)–(2.4) and (2.10) are valid throughout the section. For every $m \geq 0$, there exists a solution $\mathbf{w}^{m+1} \in L^2\big(\Omega; \mathbf{V}\big)$ such that $\mathbf{w}^0 = \mathbf{u}_0$ and $\mathbb{P}$-a.s.

$$\begin{aligned}(\mathbf{w}^{m+1} - \mathbf{w}^m, \boldsymbol{\varphi}) &+ k(\nabla \mathbf{w}^{m+1}, \nabla \boldsymbol{\varphi}) \\ &= k(\mathbf{f}^{m+1}, \boldsymbol{\varphi}) + \Big(\mathbf{B}(t_m, \mathbf{w}^m)\Delta \mathbf{W}_{m+1}, \boldsymbol{\varphi}\Big) \qquad \forall \boldsymbol{\varphi} \in \mathbf{V}.\end{aligned} \tag{2.12}$$

Both $\mathbb{P}$-a.s. existence and uniqueness of iterates from (1.10)–(1.11) follow by $\mathbf{V}$-coercivity of the bilinear form related to the Stokes equations. Moreover, solutions satisfy the error estimate given in (1.12) and shown in [11].

Some bounds for solutions of (1.10)–(1.11) in strong norms will be useful later, where the first one mimics (2.5) on a discrete level.

LEMMA 2.1. *Let* $\{\mathbf{w}^m\}_{m\geq 1} \subset L^2(\Omega;\mathbf{V})$ *be a solution of* (2.12), *and let* (2.2), (2.3), (2.4), *and* (2.10) *be valid. Then*

$$\text{(i)}\quad \max_{1\leq m\leq M} \mathbb{E}\left[\|\mathbf{w}^m\|_{\mathbf{L}^2}^2\right] + \mathbb{E}\left[\sum_{m=1}^M \|\mathbf{w}^m - \mathbf{w}^{m-1}\|_{\mathbf{L}^2}^2\right] + \mathbb{E}\left[k\sum_{m=1}^M \|\nabla\mathbf{w}^m\|_{\mathbf{L}^2}^2\right] \leq C_T\left\{\mathbb{E}\left[\|\mathbf{u}_0\|_{\mathbf{L}^2}^2\right] + \mathbb{E}\left[k\sum_{m=1}^M \|\mathbf{f}^m\|_{\mathbf{L}^2}^2\right]\right\},$$

$$\text{(ii)}\quad \max_{1\leq m\leq M} \mathbb{E}\left[\|\nabla\mathbf{w}^m\|_{\mathbf{L}^2}^2\right] + \mathbb{E}\left[\sum_{m=1}^M \|\nabla[\mathbf{w}^m - \mathbf{w}^{m-1}]\|_{\mathbf{L}^2}^2\right] + \mathbb{E}\left[k\sum_{m=1}^M \|\mathbf{A}\mathbf{w}^m\|_{\mathbf{L}^2}^2\right] \leq C_T\left\{\mathbb{E}\left[\|\nabla\mathbf{u}_0\|_{\mathbf{L}^2}^2\right] + \mathbb{E}\left[k\sum_{m=1}^M \|\mathbf{f}^m\|_{\mathbf{L}^2}^2\right]\right\},$$

$$\text{(iii)}\quad \mathbb{E}\left[k\sum_{m=1}^M \|\nabla q^m\|_{\mathbf{L}^2}^2\right] \leq C_T\left\{\mathbb{E}\left[\|\nabla\mathbf{u}_0\|_{\mathbf{L}^2}^2\right] + \mathbb{E}\left[k\sum_{m=1}^M \|\mathbf{f}^m\|_{\mathbf{L}^2}^2\right]\right\},$$

where $C_T \equiv C(\mathbf{u}_0, \mathbf{B}, \mathbf{f}, D, T) > 0$ *is a generic constant that does not depend on* k.

Proof. Assertion (i). Choose $\boldsymbol{\varphi} = \mathbf{w}^{m+1}$ in (2.12), and use the algebraic identity $2\langle \mathbf{a}-\mathbf{b}, \mathbf{a}\rangle = |\mathbf{a}|^2 - |\mathbf{b}|^2 + |\mathbf{a}-\mathbf{b}|^2$ to obtain

$$\begin{aligned}(2.13)\qquad &\frac{1}{2}\Big(\|\mathbf{w}^{m+1}\|_{\mathbf{L}^2}^2 - \|\mathbf{w}^m\|_{\mathbf{L}^2}^2 + \|\mathbf{w}^{m+1} - \mathbf{w}^m\|_{\mathbf{L}^2}^2\Big) + k\|\nabla\mathbf{w}^{m+1}\|_{\mathbf{L}^2}^2 \\ &= k\big(\mathbf{f}^{m+1}, \mathbf{w}^{m+1}\big) + \Big(\mathbf{B}(t_m, \mathbf{w}^m)\Delta\mathbf{W}_{m+1}, \mathbf{w}^{m+1} - \mathbf{w}^m\Big) \\ &\quad + \Big(\mathbf{B}(t_m, \mathbf{w}^m)\Delta\mathbf{W}_{m+1}, \mathbf{w}^m\Big).\end{aligned}$$

Taking expectations puts the last term in (2.13) to zero. For the remaining stochastic term, we use the Ito isometry and (2.3), (2.4) to conclude that

$$\begin{aligned}&\mathbb{E}\Big[\Big|\Big(\mathbf{B}(t_m, \mathbf{w}^m)\Delta\mathbf{W}_{m+1}, \mathbf{w}^{m+1} - \mathbf{w}^m\Big)\Big|\Big] \\ &\leq k\mathbb{E}\Big[\|\mathbf{B}(t_m, \mathbf{w}^m)\|_{\mathcal{L}^2(\boldsymbol{\mathcal{H}},\mathbf{L}^2)}^2\Big] + \frac{1}{4}\mathbb{E}\Big[\|\mathbf{w}^{m+1} - \mathbf{w}^m\|_{\mathbf{L}^2}^2\Big] \\ &\leq C_T k\Big(1 + \mathbb{E}\big[\|\mathbf{w}^m\|_{\mathbf{L}^2}^2\big]\Big) + \frac{1}{4}\mathbb{E}\Big[\|\mathbf{w}^{m+1} - \mathbf{w}^m\|_{\mathbf{L}^2}^2\Big].\end{aligned}$$

We now use the discrete version of Gronwall's lemma in (2.13) to obtain assertion (i).

Assertion (ii). Formally take $\boldsymbol{\varphi} = \mathbf{A}\mathbf{w}^{m+1}$ in (2.12), and proceed as before. We use (2.10) and integrate by parts in the stochastic term to find

$$\begin{aligned}(2.14)\qquad &\Big(\nabla\big[\mathbf{B}(t_m, \mathbf{w}^m)\Delta\mathbf{W}_{m+1}\big], \nabla\mathbf{w}^m\Big) \\ &\quad + \Big(\nabla\big[\mathbf{B}(t_m, \mathbf{w}^m)\Delta\mathbf{W}_{m+1}\big], \nabla\big[\mathbf{w}^{m+1} - \mathbf{w}^m\big]\Big).\end{aligned}$$

After taking expections, only the second term is nonzero; by Ito's isometry, (2.3), and (2.4), an upper bound for it is

$$\frac{1}{4}\mathbb{E}\Big[\|\nabla[\mathbf{w}^{m+1}-\mathbf{w}^m]\|^2_{\mathbf{L}^2}\Big] + Ck\mathbb{E}\Big[\|\mathbf{B}(t_m,\mathbf{w}^m)\|^2_{\mathcal{L}^2(\boldsymbol{\mathcal{H}},\mathbf{H}^1)}\Big]$$
$$\leq \frac{1}{4}\mathbb{E}\Big[\|\nabla[\mathbf{w}^{m+1}-\mathbf{w}^m]\|^2_{\mathbf{L}^2}\Big] + C_T k\Big(1+\mathbb{E}\big[\|\mathbf{w}^m\|^2_{\mathbf{H}^1}\big]\Big).$$

Putting things together and using the discrete Gronwall inequality then leads to assertion (ii).

Assertion (iii). For every $m \geq 0$, consider (2.12) in strong form on $L^2(\Omega;\mathbf{L}^2)$, which is justified from the previous step. Termwise multiplication with ∇q^{m+1} and integration in space then leads to

$$\frac{k}{2}\|\nabla q^{m+1}\|^2_{\mathbf{L}^2} \leq Ck\Big(\|\Delta\mathbf{w}^{m+1}\|^2_{\mathbf{L}^2} + \|\mathbf{f}^{m+1}\|^2_{\mathbf{L}^2}\Big),$$

where we use (2.10). Assertion (ii) and (2.1) then validate the claim. ∎

3. Perturbation effects in Algorithm 1.1: Quasi-compressibility and operator-splitting. Solutions $\{\widetilde{\mathbf{u}}^m\}_m \subset L^2\big(\Omega;\mathbf{H}^1_0(D)\big)$ of Algorithm 1.1 satisfy (1.6)–(1.8), which illustrates the different error effects due to time discretization, quasi-incompressibility, and splitting character in the scheme. The main result of this section is the following theorem.

THEOREM 3.1. *Let $T > 0$ and $D \subset \mathbb{R}^d$, $d = 2, 3$, be such that* (2.1) *holds, and let* (2.2)–(2.4), (2.10) *be valid. Denote by $\mathbf{u} \in L^2(\Omega;C([0,T];\mathbf{V})) \cap L^2(\Omega;L^2(0,T;\mathbf{H}^2))$ the strong solution of* (1.1), *and $\{\widetilde{\mathbf{u}}^m\}_m \subset L^2(\Omega;\mathbf{H}^1_0(D))$ solves Algorithm* 1.1. *There exists a constant $C \equiv C(\mathbb{E}[\|\mathbf{u}_0\|^2_{\mathbf{H}^1}], D_T) > 0$ such that*

$$\max_{1\leq m\leq M}\Big(\mathbb{E}\Big[\|\mathbf{u}(t_m,\cdot)-\widetilde{\mathbf{u}}^m\|^2_{\mathbf{L}^2}\Big]\Big)^{1/2} + \left(\mathbb{E}\left[k\sum_{m=1}^{M}\|\nabla\big(\mathbf{u}(t_m,\cdot)-\widetilde{\mathbf{u}}^m\big)\|^2_{\mathbf{L}^2}\right]\right)^{1/2} \leq C\sqrt{k}. \tag{3.1}$$

The proof is split into several steps: first, we study solutions $\{(\mathbf{v}^m, r^m)\}_m \subset L^2(\Omega;\mathbf{H}^1_0(D)) \times L^2(\Omega, H^1(D)/\mathbb{R})$ of an auxiliary problem, where $\mathbb{P}$-a.s.

$$\begin{aligned}\big(\mathbf{v}^{m+1}-\mathbf{v}^m\big) - k\Delta\mathbf{v}^{m+1} + k\nabla r^{m+1} &= k\mathbf{f}^{m+1} + \mathbf{B}(t_m,\mathbf{v}^m)\Delta\mathbf{W}_{m+1} && \text{on } D,\\ \operatorname{div}\mathbf{v}^{m+1} - k\Delta r^{m+1} &= 0 && \text{on } D,\\ \partial_{\mathbf{n}} r^{m+1} &= 0 && \text{on } \partial D,\end{aligned} \tag{3.2}$$

and $\mathbf{v}^0 \equiv \mathbf{u}_0$ on D. Note that in contrast to (1.6), where the approximation of the pressure is given from the previous time-step, it is here computed by an implicit procedure. Our goal is to show both convergence of iterates $\{\mathbf{v}^m, r^m\}_m$ toward the solution of (1.10) and stability behavior. Then, we study convergence behavior for solutions of (3.2) to that of (1.6)–(1.8).

Proof. Step 1. *The pressure stabilization problem* (3.2): *Rates of convergence.* We show the following convergence estimate for solutions $\{\mathbf{w}^m\}_m \subset L^2(\Omega;\mathbf{V})$ of (2.12) and $\{\mathbf{v}^m\}_m \subset L^2(\Omega, \mathbf{H}_0^1(D))$ of (3.2):

$$\max_{1\le m\le M}\Big(\mathbb{E}\Big[\|\mathbf{w}^m - \mathbf{v}^m\|_{\mathbf{L}^2}^2\Big]\Big)^{1/2} + \left(\mathbb{E}\left[k\sum_{1\le m\le M}\|\nabla\big(\mathbf{w}^m - \mathbf{v}^m\big)\|_{\mathbf{L}^2}^2\right]\right)^{1/2}$$
$$+\left(\mathbb{E}\left[k^2\sum_{1\le m\le M}\|\nabla\big(q^m - r^m\big)\|_{\mathbf{L}^2}^2\right]\right)^{1/2} \le C\sqrt{k}. \tag{3.3}$$

Let $\mathbf{e}^m := \mathbf{w}^m - \mathbf{v}^m \in L^2(\Omega, \mathbf{H}^1(D))$ and $\chi^m := q^m - r^m \in L^2(\Omega, H^1(D)/\mathbb{R})$. Taking the difference of (1.10) and (3.2) then leads to $\mathbb{P}$-a.s.

$$\begin{aligned}\big(\mathbf{e}^{m+1} - \mathbf{e}^m\big) - k\Delta\mathbf{e}^{m+1} + k\nabla\chi^{m+1} &= \Big(\mathbf{B}(t_m, \mathbf{w}^m) - \mathbf{B}(t_m, \mathbf{v}^m)\Big)\Delta\mathbf{W}_{m+1} && \text{on } D,\\ \operatorname{div}\mathbf{e}^{m+1} - k\Delta\chi^{m+1} &= -k\Delta q^{m+1} && \text{on } D,\\ \partial_{\mathbf{n}}\chi^{m+1} &= \partial_{\mathbf{n}} q^{m+1} && \text{on } \partial D,\end{aligned} \tag{3.4}$$

and $\mathbf{e}^0 \equiv \mathbf{0}$ in D. By testing the first equation with $\mathbf{e}^{m+1}$ and using Lipschitz continuity of $\mathbf{B}$ and testing the second with χ^{m+1} and using $(3.4)_3$ for integration by parts, adding both identities and using Young's inequality then leads to $\mathbb{P}$-a.s.

$$\begin{aligned}&\frac{1}{2}\Big(\|\mathbf{e}^{m+1}\|_{\mathbf{L}^2}^2 - \|\mathbf{e}^m\|_{\mathbf{L}^2}^2 + \|\mathbf{e}^{m+1} - \mathbf{e}^m\|_{\mathbf{L}^2}^2\Big) + k\|\nabla\mathbf{e}^{m+1}\|_{\mathbf{L}^2}^2 + k^2\|\nabla\chi^{m+1}\|_{\mathbf{L}^2}^2\\ &\quad\le \Big([\mathbf{B}(t_m, \mathbf{w}^m) - \mathbf{B}(t_m, \mathbf{v}^m)]\Delta\mathbf{W}_{m+1}, \mathbf{e}^m\Big) + \frac{1}{4}\|\mathbf{e}^m - \mathbf{e}^{m-1}\|_{\mathbf{L}^2}^2\\ &\qquad + \|[\mathbf{B}(t_m, \mathbf{w}^m) - \mathbf{B}(t_m, \mathbf{v}^m)]\Delta\mathbf{W}_{m+1}\|_{\mathbf{L}^2}^2 + \frac{1}{4}k^2\|\nabla\chi^{m+1}\|_{\mathbf{L}^2}^2 + k^2\|\nabla q^{m+1}\|_{\mathbf{L}^2}^2.\end{aligned}$$

The leading term on the right-hand side vanishes when we take its expectation. By Ito's isometry, and (2.3), there holds for the remaining stochastic integral term

$$\mathbb{E}\Big[\|[\mathbf{B}(t_m, \mathbf{w}^m) - \mathbf{B}(t_m, \mathbf{v}^m)]\Delta\mathbf{W}_{m+1}\|_{\mathbf{L}^2}^2\Big] \le Ck\Big(1 + \mathbb{E}\big[\|\mathbf{e}^m\|_{\mathbf{L}^2}^2\big]\Big).$$

We now take expectation termwise and sum over all steps $0 \le m \le m^* \le M-1$; because of $\mathbb{E}[\|\mathbf{e}^0\|_{\mathbf{L}^2}^2] = 0$, Lemma 2.1(iii), and the discrete version of Gronwall's inequality, after summation we arrive at

$$\frac{1}{2}\mathbb{E}\Big[\|\mathbf{e}^{m^*+1}\|_{\mathbf{L}^2}^2\Big] + \frac{1}{4}\mathbb{E}\left[\sum_{m=0}^{m^*}\|\mathbf{e}^{m+1} - \mathbf{e}^m\|_{\mathbf{L}^2}^2\right] + \mathbb{E}\left[k\sum_{m=0}^{m^*}\|\nabla\mathbf{e}^{m+1}\|_{\mathbf{L}^2}^2\right]$$
$$+\frac{3}{4}\mathbb{E}\left[k^2\sum_{m=0}^{m^*}\|\nabla\chi^{m+1}\|_{\mathbf{L}^2}^2\right] \le C_{t_{m^*}}\mathbb{E}\left[k^2\sum_{m=0}^{m^*}\|\nabla q^{m+1}\|_{\mathbf{L}^2}^2\right] \le C_T k. \tag{3.5}$$

Step 2. *The pressure stabilization problem* (3.2)*: Stability.* Proper bounds are needed for the pressure in (3.2) to validate optimal error estimates between solutions of (3.2) and (1.6)–(1.8) below. We show

$$\max_{1\le m\le M}\mathbb{E}\Big[\|\mathbf{v}^m\|_{\mathbf{H}^1}^2\Big] + \mathbb{E}\left[k\sum_{m=1}^{M}\|\mathbf{v}^m\|_{\mathbf{H}^2}^2\right] + \mathbb{E}\left[k\sum_{m=1}^{M}\|\nabla r^m\|_{\mathbf{L}^2}^2\right] \le C_T. \tag{3.6}$$

Hence, for solutions of problem (3.2) there hold the same estimates which are valid for solutions of (2.12) provided in Lemma 2.1.

Property $(3.6)_1$ follows from the term $(3.3)_3$ and Lemma 2.1(ii), and property $(3.6)_3$ is a consequence of $(3.3)_3$ and Lemma 2.1(iii). A formal derivation of $(3.6)_2$ uses $(3.2)_1$, which we multiply by $-\Delta \mathbf{v}^{m+1}$ and then integrate over D. After summing up over all $0 \leq m \leq M-1$, by taking expectations and absorbing terms we arrive at

$$
\begin{aligned}
&\frac{1}{2}\mathbb{E}\Big[\|\nabla \mathbf{v}^M\|_{\mathbf{L}^2}^2\Big] + \frac{1}{2}\mathbb{E}\left[\sum_{m=0}^{M-1}\|\nabla(\mathbf{v}^{m+1}-\mathbf{v}^m)\|_{\mathbf{L}^2}^2\right] + \frac{1}{4}\mathbb{E}\left[k\sum_{m=0}^{M-1}\|\Delta \mathbf{v}^{m+1}\|_{\mathbf{L}^2}^2\right] \\
&\leq \frac{1}{2}\mathbb{E}\big[\|\nabla \mathbf{v}^0\|_{\mathbf{L}^2}^2\big] + C\mathbb{E}\left[k\sum_{m=0}^{M-1}\|\nabla r^{m+1}\|_{\mathbf{L}^2}^2\right] + C\mathbb{E}\left[k\sum_{m=0}^{M-1}\|\mathbf{f}^{m+1}\|_{\mathbf{L}^2}^2\right] \\
&\quad + k\mathbb{E}\left[\sum_{m=0}^{M-1}\|\mathbf{B}(t_m,\mathbf{v}^m)\|_{\mathcal{L}^2(\boldsymbol{\mathcal{H}},\mathbf{H}^1)}^2\right] + \frac{1}{4}\mathbb{E}\left[\sum_{m=0}^{M-1}\|\nabla[\mathbf{v}^{m+1}-\mathbf{v}^m]\|_{\mathbf{L}^2}^2\right],
\end{aligned} \tag{3.7}
$$

where we use the fact that $\mathbb{E}[\sum_{m=1}^{M}(\nabla \mathbf{B}(t_m,\mathbf{v}^m)\Delta \mathbf{W}_{m+1},\nabla \mathbf{v}^m)] = 0$ and Ito's isometry. By (2.3), (2.4), we have $\mathbb{E}[\|\mathbf{B}(t_m,\mathbf{v}^m)\|_{\mathcal{L}^2(\boldsymbol{\mathcal{H}},\mathbf{H}^1)}] \leq C_T(1+\|\mathbf{v}^m\|_{\mathbf{H}^1})$. The bounds $(3.6)_{1,3}$ then allow us to conclude $(3.6)_2$ from (3.7) after using the discrete version of Gronwall's inequality.

Step 3. *The splitting error: Comparison of problems* (3.2) *and* (1.6)–(1.8). We estimate the differences $\boldsymbol{\varepsilon}^m := \mathbf{v}^m - \widetilde{\mathbf{u}}^m \in L^2\big(\Omega, \mathbf{H}_0^1(D)\big)$ and $\eta^m := \chi^m - p^m \in L^2\big(\Omega, H^1(D)/\mathbb{R}\big)$, which are determined by the following system of equations, which hold $\mathbb{P}$-a.s.:

$$
\begin{aligned}
\big(\boldsymbol{\varepsilon}^{m+1}-\boldsymbol{\varepsilon}^m\big) - k\Delta\boldsymbol{\varepsilon}^{m+1} + k\nabla\eta^m &= \boldsymbol{\mathcal{E}}^{m+1} && \text{on } D,\\
\operatorname{div}\boldsymbol{\varepsilon}^{m+1} - k\Delta\eta^{m+1} &= 0 && \text{on } D,\\
\partial_{\mathbf{n}}\eta^{m+1} &= 0 && \text{on } \partial D,
\end{aligned} \tag{3.8}
$$

where $\boldsymbol{\varepsilon}_0 \equiv \mathbf{0}$, and

$$
\boldsymbol{\mathcal{E}}^{m+1} := -k\nabla[r^{m+1}-r^m] + \Big(\mathbf{B}(t_m,\mathbf{v}^m) - \mathbf{B}(t_m,\widetilde{\mathbf{u}}^m)\Big)\Delta\mathbf{W}_{m+1}. \tag{3.9}
$$

Upon testing $(3.8)_1$ by $\boldsymbol{\varepsilon}^{m+1}$ and $(3.8)_2$ by η^{m+1}, adding both identities, using Young's inequality with $\delta_1 > 0$, and absorbing terms then yields

$$
\begin{aligned}
&\frac{1}{2}\Big(\|\boldsymbol{\varepsilon}^{m+1}\|_{\mathbf{L}^2}^2 - \|\boldsymbol{\varepsilon}^m\|_{\mathbf{L}^2}^2 + \|\boldsymbol{\varepsilon}^{m+1}-\boldsymbol{\varepsilon}^m\|_{\mathbf{L}^2}^2\Big) + k\|\nabla\boldsymbol{\varepsilon}^{m+1}\|_{\mathbf{L}^2}^2 + k^2\big(\nabla\eta^{m+1},\nabla\eta^m\big)\\
&\quad\leq \Big([\mathbf{B}(t_m,\mathbf{v}^m) - \mathbf{B}(t_m,\widetilde{\mathbf{u}}^m)]\Delta\mathbf{W}_{m+1},\boldsymbol{\varepsilon}^m\Big) - k\Big(\nabla(r^{m+1}-r^m),\boldsymbol{\varepsilon}^{m+1}\Big)\\
&\qquad + C_{\delta_1}\|[\mathbf{B}(t_m,\mathbf{v}^m) - \mathbf{B}(t_m,\widetilde{\mathbf{u}}^m)]\Delta\mathbf{W}_{m+1}\|_{\mathbf{L}^2}^2 + \delta_1\|\boldsymbol{\varepsilon}^{m+1}-\boldsymbol{\varepsilon}^m\|_{\mathbf{L}^2}^2.
\end{aligned} \tag{3.10}
$$

Again, the expectation of the leading term on the right-hand side vanishes; Ito's isometry and (2.3) then imply

$$
\mathbb{E}\Big[\|[\mathbf{B}(t_m,\mathbf{v}^m) - \mathbf{B}(t_m,\widetilde{\mathbf{u}}^m)]\Delta\mathbf{W}_{m+1}\|_{\mathbf{L}^2}^2\Big] \leq C_T k\Big(1+\mathbb{E}[\|\boldsymbol{\varepsilon}^m\|_{\mathbf{L}^2}^2]\Big).
$$

It remains to deal with terms which contain pressures. We use $(3.8)_2$ and Young's inequality with $\delta_2 > 0$ to conclude that

$$
\begin{aligned}
k^2\big(\nabla\eta^{m+1}, \nabla\eta^m\big) &= k^2\|\nabla\eta^{m+1}\|^2_{\mathbf{L}^2} - k^2\Big(\nabla\eta^{m+1}, \nabla[\eta^{m+1} - \eta^m]\Big) \\
&= k^2\|\nabla\eta^{m+1}\|^2_{\mathbf{L}^2} - k\big(\nabla\eta^{m+1}, \boldsymbol{\varepsilon}^{m+1} - \boldsymbol{\varepsilon}^m\big) \\
&\geq \left(1 - \frac{1}{4\delta_2}\right) k^2\|\nabla\eta^{m+1}\|^2_{\mathbf{L}^2} - \delta_2\|\boldsymbol{\varepsilon}^{m+1} - \boldsymbol{\varepsilon}^m\|^2_{\mathbf{L}^2}.
\end{aligned}
\tag{3.11}
$$

The remaining crucial term in (3.10) is bounded as follows:

$$
k^2\Big(\nabla[r^{m+1} - r^m], \nabla\eta^{m+1}\Big) \leq k^2\delta_3\|\nabla\eta^{m+1}\|^2_{\mathbf{L}^2} + \frac{k^2}{4\delta_3}\|\nabla[r^{m+1} - r^m]\|^2_{\mathbf{L}^2},
$$

where we used Young's inequality with $\delta_3 > 0$. To keep the corresponding terms in (3.12) nonnegative, we choose parameters $\delta_i > 0$, $i = 1, 2, 3$, such that

$$
1 - \frac{1}{4\delta_2} - \delta_3 \geq 0 \qquad \text{and} \qquad \frac{1}{2} - \delta_1 - \delta_2 \geq 0.
$$

Next, we sum over all $0 \leq m \leq m^* \leq M - 1$ in (3.10) and take expectations. Then, by the discrete version of Gronwall's inequality,

$$
\begin{aligned}
&\mathbb{E}\Big[\|\boldsymbol{\varepsilon}^{m^*+1}\|^2_{\mathbf{L}^2}\Big] + \left(\frac{1}{2} - \delta_1 - \delta_2\right)\mathbb{E}\left[\sum_{m=0}^{m^*}\|\boldsymbol{\varepsilon}^{m+1} - \boldsymbol{\varepsilon}^m\|^2_{\mathbf{L}^2}\right] + \mathbb{E}\left[k\sum_{m=0}^{m^*}\|\nabla\boldsymbol{\varepsilon}^{m+1}\|^2_{\mathbf{L}^2}\right] \\
&\quad + k^2\left(1 - \frac{1}{4\delta_2} - \delta_3\right)\mathbb{E}\left[\sum_{m=0}^{m^*}\|\nabla\eta^{m+1}\|^2_{\mathbf{L}^2}\right] \leq C_T 4\frac{k^2}{4\delta_3}\sum_{m=0}^{m^*+1}\|\nabla r^m\|^2_{\mathbf{L}^2} \leq C_{t_T} k,
\end{aligned}
\tag{3.12}
$$

where the last estimate uses $(3.6)_3$ and $r^0 \equiv 0$ is a consistent choice from of $(3.2)_{2,3}$.

Putting together results (1.12), (3.3), and (3.12) yields the error bound

$$
\begin{aligned}
&\max_{1\leq m\leq M}\Big(\mathbb{E}\Big[\|\mathbf{u}(t_m,\cdot) - \widetilde{\mathbf{u}}^m\|^2_{\mathbf{L}^2}\Big]\Big)^{1/2} \\
&\quad + \left(\mathbb{E}\left[k\sum_{1\leq m\leq M}\|\nabla\big(\mathbf{u}(t_m,\cdot) - \widetilde{\mathbf{u}}^m\big)\|^2_{\mathbf{L}^2}\right]\right)^{1/2} \leq C\Big(\sqrt{k} + \sqrt{k} + \sqrt{k}\Big),
\end{aligned}
\tag{3.13}
$$

which proves Theorem 3.1. ∎

The following stability result for solutions of Algorithm 1.1 will be helpful in section 5, where we consider an optimally convergent, practical finite element discretization.

LEMMA 3.1. *Let $T > 0$, $D \subset \mathbb{R}^d$, $d = 2, 3$, be such that* (2.1) *holds, and let* (2.2)–(2.4), (2.10) *be valid. Let $\{\widetilde{\mathbf{u}}^m\}_{m\geq 1} \subset L^2(\Omega, \mathbf{H}^1_0(D))$ be the solution of Algorithm* 1.1. *Then, all estimates given in Lemma* 2.1 *remain valid.*

Proof. First, we observe that since (1.6) requires solving a linear elliptic boundary value problem, we have $\tilde{\mathbf{u}}^m \in L^2(\Omega; \mathbf{H}^2(D) \cap \mathbf{H}^1_0(D))$. Then we use (3.6), together with (3.12), to validate bounds (i), (iii), and $(\text{ii})_{1,2}$ in Lemma 2.1 for $\{\widetilde{\mathbf{u}}^m\}$. In order to (formally) verify $\mathbb{E}[k\sum_{m=1}^M \|\Delta\widetilde{\mathbf{u}}^m\|^2_{\mathbf{L}^2}] \leq C$, we multiply (1.6) by $-\Delta\widetilde{\mathbf{u}}^{m+1}$, integrate over D, and consider expectations. Similar arguments as above lead to

$$
\begin{aligned}
&\frac{1}{2}\mathbb{E}\left[\|\nabla\widetilde{\mathbf{u}}^{m+1}\|^2_{\mathbf{L}^2} - \|\nabla\widetilde{\mathbf{u}}^m\|^2_{\mathbf{L}^2} + \frac{3}{4}\|\nabla[\widetilde{\mathbf{u}}^{m+1} - \widetilde{\mathbf{u}}^m]\|^2_{\mathbf{L}^2}\right] + \frac{3k}{4}\mathbb{E}\big[\|\Delta\widetilde{\mathbf{u}}^{m+1}\|^2_{\mathbf{L}^2}\big] \\
&\quad \leq \mathbb{E}\big[k\|\nabla p^m\|^2_{\mathbf{L}^2}\big] + Ck\Big(1 + \mathbb{E}\big[\|\widetilde{\mathbf{u}}^m\|^2_{\mathbf{H}^1}\big]\Big) + Ck\mathbb{E}\big[\|\mathbf{f}^{m+1}\|^2_{\mathbf{L}^2}\big].
\end{aligned}
$$

We now sum up $0 \le m \le M$ and may use the available bound $\mathbb{E}[k\sum_{m=1}^{M}\|\nabla p^m\|_{\mathbf{L}^2}^2] \le C$ to obtain

$$\mathbb{E}\left[k\sum_{m=1}^{M}\|\Delta\widetilde{\mathbf{u}}^m\|_{\mathbf{L}^2}^2\right] \le C_T. \qquad \square$$

4. Chorin scheme with stochastic pressure correction. As has been shown so far, we can prove optimal convergence behavior for Algorithm 1.1 only in the case of solenoidal noise. Here we try to modify Algorithm 1.1 in order to validate optimal convergence behavior also in the case that the sequence of random variables $\{\mathbf{B}(t_m, \widetilde{\mathbf{u}}^m)\Delta\mathbf{W}_{m+1}\}_{m\ge 0}$ approximates general noise. The scheme that we propose is the following.

ALGORITHM 4.1. Let $m \ge 0$.

1. For given $\widetilde{\mathbf{u}}^m \in L^2(\Omega, \mathbf{H}_0^1(D))$, find $\boldsymbol{\xi}^{m+1} \in L^2(\Omega, \mathbf{H})$ such that $\mathbb{P}$-a.s.

$$\begin{aligned} \boldsymbol{\xi}^{m+1} + \nabla s^{m+1} &= \frac{1}{\sqrt{k}}\mathbf{B}(t_m, \widetilde{\mathbf{u}}^m)\Delta\mathbf{W}_{m+1} \quad &&\text{on } D,\\ \operatorname{div}\boldsymbol{\xi}^{m+1} &= 0 &&\text{on } D,\\ \langle\boldsymbol{\xi}^{m+1}, \mathbf{n}\rangle &= 0 &&\text{on } \partial D. \end{aligned} \tag{4.1}$$

2. For given $\mathbf{u}^m \in L^2(\Omega, \mathbf{H})$, find $\widetilde{\mathbf{u}}^{m+1} \in L^2(\Omega, \mathbf{H}_0^1(D))$ such that $\mathbb{P}$-a.s.

$$\left(\widetilde{\mathbf{u}}^{m+1} - \mathbf{u}^m\right) - k\Delta\widetilde{\mathbf{u}}^{m+1} = k\mathbf{f}^{m+1} + \sqrt{k}\boldsymbol{\xi}^{m+1} \quad \text{on } D. \tag{4.2}$$

3. Compute $\mathbf{u}^{m+1} \in L^2(\Omega, \mathbf{H})$ and $p^{m+1} \in L^2(\Omega, H^1(D)/\mathbb{R})$ from the following equations, which hold $\mathbb{P}$-a.s.:

$$\begin{aligned} \mathbf{u}^{m+1} - \widetilde{\mathbf{u}}^{m+1} + k\nabla p^{m+1} &= 0 \quad \text{on } D,\\ \operatorname{div}\mathbf{u}^{m+1} &= 0 \quad \text{on } D,\\ \langle\mathbf{u}^{m+1}, \mathbf{n}\rangle &= 0 \quad \text{on } \partial D. \end{aligned} \tag{4.3}$$

4. Compute the approximation of the pressure p via

$$r^{m+1} = p^{m+1} + \frac{1}{\sqrt{k}}s^{m+1}.$$

The underlying idea for this algorithm is to distinguish between deterministic and stochastic (forcing) terms on the right-hand side of (1.1), which scale differently in a time-discretization scheme. Corresponding Helmholtz decompositions of both terms involve gradient functions, which are then referred to as *deterministic* and *stochastic pressures*. It is by step 1 that the gradient of the *stochastic pressure* $\{s^m\}_{m\ge1}$ in $(4.1)_1$ (which is the Lagrange multiplier resulting from the Leray projection) has no influence on computing velocity iterates in steps 1 to 3, where only the deterministic pressure $\{p^m\}_{m\ge1}$ is involved. This argument is further detailed by the following formal computation for Euler iterates from (1.10)–(1.11):

$$\begin{aligned} \mathbf{w}^{m+1} - k\Delta\mathbf{w}^{m+1} + k\nabla p^{m+1} &= \mathbf{w}^m + k\mathbf{f}^{m+1} + \mathbf{B}(t_m, \mathbf{w}^m)\Delta\mathbf{W}_{m+1}\\ &= \mathbf{w}^m + k\mathbf{f}^{m+1} + \sqrt{k}\boldsymbol{\xi}^{m+1} + \sqrt{k}\nabla s^{m+1}, \end{aligned}$$

where $\boldsymbol{\xi}^{m+1} = \frac{1}{\sqrt{k}}\mathbf{P}_{\mathbf{H}}[\mathbf{B}(t_m, \mathbf{w}^m)\Delta\mathbf{W}_{m+1}]$. As a consequence, we get

$$\mathbf{w}^{m+1} - k\Delta\mathbf{w}^{m+1} + k\nabla\pi^{m+1} = \mathbf{w}^m + k\mathbf{f}^{m+1} + \mathbf{P}_{\mathbf{H}}\mathbf{B}(t_m, \mathbf{w}^m)\Delta\mathbf{W}_{m+1},$$

where

$$\pi^{m+1} = p^{m+1} - \frac{1}{\sqrt{k}} s^{m+1}.$$

In fact, Algorithm 4.1 is Algorithm 1.1, which is applied to the same equation with projected noise. So, the proof of the convergence rate follows directly from Theorem 3.1. Unfortunately, for $\mathbf{v} \in \mathbf{H}^1_0(D)$ the projection $\mathbf{P_H v} \in \mathbf{H}^1(D)$ is not an element of $\mathbf{H}^1_0(D)$. As a consequence, in formula (2.14) we obtain an additional boundary integral which is difficult to bound, and properties (ii) and (iii) of Lemma 2.1 are not evident to remain valid in this setting anymore. To avoid this problematic issue, we consider Problem 1.1 with space periodic boundary conditions on a set $Q = [0, L]^d$, $L > 0$. Let $\mathbf{H}_{per}$, $\mathbf{V}_{per}$, and $\mathbf{H}^n_{per}$ denote the space periodic analogues of the spaces $\mathbf{H}$, $\mathbf{V}$, and $\mathbf{H}^n$. In this case optimal convergence rates of the splitting Algorithm 4.1 also hold for general noise. We have the following theorem.

THEOREM 4.1. *Let $T > 0$, $D \subset \mathbb{R}^d$, $d = 2, 3$, be such that* (2.1) *holds, and let* (2.2)–(2.4) *be valid. Denote by $\mathbf{u} \in L^2(\Omega; C([0,T]; \mathbf{V}_{per})) \cap L^2(\Omega; L^2(0,T; \mathbf{H}^2_{per}))$ the strong solution of* (1.1), *and $\{\widetilde{\mathbf{u}}^m\}_m \subset L^2(\Omega, \mathbf{H}^1_{per}(D))$ solves Algorithm* 4.1. *There exists a constant $C \equiv C(\mathbb{E}[\|\mathbf{u}_0\|_{\mathbf{H}^1}], D_T) > 0$ such that*

(4.4)
$$\max_{1\le m\le M}\Big(\mathbb{E}\Big[\|\mathbf{u}(t_m,\cdot) - \widetilde{\mathbf{u}}^m\|^2_{\mathbf{L}^2}\Big]\Big)^{1/2} + \left(\mathbb{E}\left[k\sum_{m=1}^{M}\|\nabla\big(\mathbf{u}(t_m,\cdot) - \widetilde{\mathbf{u}}^m\big)\|^2_{\mathbf{L}^2}\right]\right)^{1/2} \le C\sqrt{k}.$$

Again, note that condition (2.10) is not needed in this case to validate (4.4).

5. Finite element discretization of Algorithm 1.1. Let $\mathcal{T}_h$ be a quasi-uniform triangulation of the polygonal or polyhedral bounded Lipschitz domain $D \subset \mathbb{R}^d$ into triangles or tetrahedra for $d = 2$ or $d = 3$, respectively. We define the lowest order finite element space

$$H_h = \big\{\Phi \in C(\overline{D}) : \Phi\big|_K \in \mathcal{P}_1(K) \quad \forall\, K \in \mathcal{T}_h\big\},$$

where $\mathcal{P}_1(K)$ denotes the set of polynomials of degree less than or equal to one if restricted to the element $K \in \mathcal{T}_h$. We introduce equal-order finite element function spaces

$$\mathbf{H}_h := \big[H_h\big]^d \qquad \text{and} \qquad L_h := H_h \cap L^2(D)/\mathbb{R}$$

and $\mathbf{H}^0_h := \mathbf{H}_h \cap \mathbf{H}^1_0(D)$. We recall the $\mathbf{L}^2$-orthogonal projection $\mathbf{P}^0_h : \mathbf{L}^2 \to \mathbf{H}^0_h$, where

$$\big(\boldsymbol{\phi} - \mathbf{P}^0_h\boldsymbol{\phi}, \boldsymbol{\xi}\big) = 0 \qquad \forall \boldsymbol{\xi} \in \mathbf{H}^0_h,$$

for which holds

$$\|\boldsymbol{\phi} - \mathbf{P}^0_h\boldsymbol{\phi}\|_{\mathbf{L}^2} + h\|\nabla(\boldsymbol{\phi} - \mathbf{P}^0_h\boldsymbol{\phi})\|_{\mathbf{L}^2} \le Ch^2\|\boldsymbol{\phi}\|_{\mathbf{H}^2} \qquad \forall \boldsymbol{\phi} \in \mathbf{H}^2.$$

Accordingly, there holds for $P^1_h : H^1(D)/\mathbb{R} \to L_h$, where

$$\big(\nabla[\chi - P^1_h\chi], \nabla\eta\big) = 0 \qquad \forall\, \eta \in L_h,$$

that

$$\|\chi - P^1_h\chi\|_{L^2} + h\|\nabla[\chi - P^1_h\chi]\|_{L^2} \le Ch^2\|\chi\|_{H^2} \qquad \forall\, \chi \in H^1/\mathbb{R} \cap H^2.$$

Below we use finite elements for a fully discrete version of Algorithm 1.1. Moreover, for simplicity we assume that $\mathbf{B}$ is independent of time.

ALGORITHM 5.1. Let $m \geq 0$. *Set* $\widetilde{\mathbf{U}}^0 := \mathbf{U}^0$ *for* $\mathbf{U}^0 \in L^2(\Omega, \mathbf{H}_h^0)$ *such that* $(\mathbf{U}^0, \nabla\chi) = 0$ *for all* $\chi \in L_h$.

1. For given $\mathbf{U}^m \in L^2(\Omega; \mathbf{H}_h^0)$, find $\widetilde{\mathbf{U}}^{m+1} \in L^2(\Omega; \mathbf{H}_h^0)$ such that $\mathbb{P}$-a.s.

$$\begin{aligned}&\big(\widetilde{\mathbf{U}}^{m+1} - \mathbf{U}^m, \boldsymbol{\Psi}\big) + k\big(\nabla\widetilde{\mathbf{U}}^{m+1}, \nabla\boldsymbol{\Psi}\big)\\ &\qquad = k\big(\mathbf{f}^{m+1}, \boldsymbol{\Psi}\big) + \Big(\mathbf{B}(\widetilde{\mathbf{U}}^m)\Delta\mathbf{W}_{m+1}, \boldsymbol{\psi}\Big) \qquad \forall\, \boldsymbol{\Psi} \in \mathbf{H}_h^0.\end{aligned} \tag{5.1}$$

2. For given $\widetilde{\mathbf{U}}^{m+1} \in L^2(\Omega; \mathbf{H}_h^0(D))$, compute $P^{m+1} \in L^2(\Omega; L_h)$ such that $\mathbb{P}$-a.s.

$$\big(\nabla P^{m+1}, \nabla\chi\big) = \frac{1}{k}\big(\widetilde{\mathbf{U}}^{m+1}, \nabla\chi\big) \qquad \forall\, \chi \in L_h. \tag{5.2}$$

3. Update $\mathbb{P}$-a.s.

$$\big(\mathbf{U}^{m+1}, \boldsymbol{\varphi}\big) = \big(\widetilde{\mathbf{U}}^{m+1}, \boldsymbol{\varphi}\big) - k\big(\nabla P^{m+1}, \boldsymbol{\varphi}\big) \qquad \forall\, \boldsymbol{\varphi} \in \mathbf{H}_h^0\,.$$

The following result provides error estimates for the fully discrete scheme.

THEOREM 5.1. *Suppose that the assumptions in Lemma* 3.1 *hold. Let* $\{\widetilde{\mathbf{U}}^m\}_{m\geq 1} \subset L^2(\Omega; \mathbf{H}_h^0)$ *be computed from Algorithm* 5.1. *Then*

$$\begin{aligned}&\max_{1\leq m\leq M}\Big(\mathbb{E}\Big[\|\mathbf{u}(t_m,\cdot) - \widetilde{\mathbf{U}}^m\|_{\mathbf{L}^2}^2\Big]\Big)^{1/2}\\ &\qquad + \left(\mathbb{E}\left[k\sum_{m=1}^{M}\|\nabla[\mathbf{u}(t_m,\cdot) - \widetilde{\mathbf{U}}^m]\|_{\mathbf{L}^2}^2\right]\right)^{1/2} \leq C\left(\sqrt{k} + h + \frac{h^2}{\sqrt{k}}\right).\end{aligned}$$

Because of Theorem 3.1, it is sufficient to control the error between the solutions of Algorithms 1.1 and 5.1, for which Lemma 3.1 is relevant. Balancing the coupling error $\mathcal{O}(\frac{h^2}{\sqrt{k}})$ with the other two errors due to time discretization, splitting, and spatial discretization motivates a (noncritical) balancing $h \leq C\sqrt{k}$. We remark that this coupling is well-known in the deterministic setting, where stability of equal-order finite element pairings using the pressure stabilization ansatz

$$\operatorname{div}\mathbf{u}^\varepsilon - \varepsilon\Delta p^\varepsilon = 0 \quad \text{in } D, \qquad \partial_{\mathbf{n}} p^\varepsilon = 0 \quad \text{on } \partial D$$

requires choices $\varepsilon \geq Ch^2$; cf. [12, 16]: since $\varepsilon = k$ in (5.2), the restriction $k \geq Ch^2$ then leads to a stable discretization in space by equal-order finite element pairings.

Proof. For every $m \geq 1$, let

$$\big(\mathbf{e}^m, \eta^m\big) := \Big(\widetilde{\mathbf{u}}^m - \widetilde{\mathbf{U}}^m, p^m - P^m\Big) \in L^2\big(\Omega; \mathbf{H}_0^1 \times H^1/\mathbb{R}\big)$$

be the solution of the following set of error equations, which hold $\mathbb{P}$-a.s.:

$$\begin{aligned}&\big(\mathbf{e}^{m+1} - \mathbf{e}^m, \boldsymbol{\Psi}\big) + k\big(\nabla\mathbf{e}^{m+1}, \nabla\boldsymbol{\Psi}\big) + k\big(\nabla\eta^m, \boldsymbol{\Psi}\big)\\ &\qquad = \Big([\mathbf{B}(\widetilde{\mathbf{u}}^m) - \mathbf{B}(\widetilde{\mathbf{U}}^m)]\Delta\mathbf{W}_{m+1}, \boldsymbol{\Psi}\Big) \qquad \forall\, \boldsymbol{\Psi} \in \mathbf{H}_h^0,\end{aligned} \tag{5.3}$$

$$\big(\operatorname{div}\mathbf{e}^{m+1}, \chi\big) + k\big(\nabla\eta^{m+1}, \nabla\chi\big) = 0 \qquad \forall \chi \in L_h, \tag{5.4}$$

and $\|\mathbf{e}^0\|_{\mathbf{L}^2} \leq Ch^2$ in D. Observe that $\eta^0 = 0$ is a consistent choice, taking into account (1.7), (1.8), (5.2), together with the fact that $(\mathbf{U}^0, \nabla\chi) = 0$ for all $\chi \in L_h$.

The equations follow from the reformulation of Algorithm 1.1 in the form (1.6)–(1.8) and corresponding equations for (5.1), (5.2). We may choose $\mathbf{\Psi} = \mathbf{P}_h^0 \mathbf{e}^{m+1}$ as a test function in (5.3). We use Young's inequality and $\mathbf{L}^2$-stability of $\mathbf{P}_h^0$ to conclude for any $\delta_1 > 0$,

$$\begin{aligned}
&\frac{1}{2}\Big(\|\mathbf{e}^{m+1}\|^2_{\mathbf{L}^2} - \|\mathbf{e}^{m}\|^2_{\mathbf{L}^2} + \|\mathbf{e}^{m+1}-\mathbf{e}^{m}\|^2_{\mathbf{L}^2}\Big) + \frac{3k}{4}\|\nabla \mathbf{e}^{m+1}\|^2_{\mathbf{L}^2} + k\big(\nabla \eta^m, \mathbf{P}_h^0 \mathbf{e}^{m+1}\big) \\
&\quad \leq \Big([\mathbf{B}(\widetilde{\mathbf{u}}^m) - \mathbf{B}(\widetilde{\mathbf{U}}^m)]\Delta \mathbf{W}_{m+1}, \mathbf{P}_h^0 \mathbf{e}^m \Big) + k\|\nabla[\widetilde{\mathbf{u}}^{m+1} - \mathbf{P}_h^0 \widetilde{\mathbf{u}}^{m+1}]\|^2_{\mathbf{L}^2} \\
&\qquad + C_{\delta_1}\big\|[\mathbf{B}(\widetilde{\mathbf{u}}^m) - \mathbf{B}(\widetilde{\mathbf{U}}^m)]\Delta \mathbf{W}_{m+1}\big\|^2_{\mathbf{L}^2} + \delta_1\|\mathbf{e}^{m+1}-\mathbf{e}^{m}\|^2_{\mathbf{L}^2}.
\end{aligned} \tag{5.5}$$

A lower bound for the last term on the left-hand side is as follows ($\delta_2 > 0$):

$$\begin{aligned}
k\big(\nabla \eta^m, \mathbf{P}_h^0 \mathbf{e}^{m+1}\big) &\geq k\big(\nabla P_h^1 \eta^m, \mathbf{e}^{m+1}\big) - \delta_2 k^2 \|\nabla \eta^m\|^2_{\mathbf{L}^2} - \frac{1}{4\delta_2}\|\widetilde{\mathbf{u}}^{m+1} - \mathbf{P}_h^0 \widetilde{\mathbf{u}}^{m+1}\|^2_{\mathbf{L}^2} \\
&\quad - k\|p^m - P_h^1 p^m\|^2_{L^2} - \frac{k}{4}\|\nabla \mathbf{e}^{m+1}\|^2_{\mathbf{L}^2}.
\end{aligned} \tag{5.6}$$

We use $\chi = P_h^1 \eta^m$ in (5.4) to conclude $k(\nabla P_h^1 \eta^m, \mathbf{e}^{m+1}) = k^2(\nabla P_h^1 \eta^m, \nabla \eta^{m+1})$. We use properties of P_h^1 to conclude

$$\begin{aligned}
k^2\big(\nabla P_h^1 \eta^m, \nabla \eta^{m+1}\big) &= k^2\big(\nabla P_h^1 \eta^{m+1}, \nabla \eta^{m+1}\big) - k^2\big(\nabla P_h^1[\eta^{m+1} - \eta^m], \nabla \eta^{m+1}\big) \\
&= k^2\|\nabla \eta^{m+1}\|^2 + k^2\big(\nabla[p^{m+1} - P_h^1 p^{m+1}], \nabla \eta^{m+1}\big) \\
&\quad - k^2\big(\nabla[\eta^{m+1} - \eta^m], \nabla P_h^1 \eta^{m+1}\big).
\end{aligned}$$

Because of (5.4) we may now conclude ($\delta_3, \delta_4 > 0$)

$$\begin{aligned}
k^2\big(\nabla P_h^1 \eta^m, \nabla \eta^{m+1}\big) &= k^2\big(1-\delta_3\big)\|\nabla \eta^{m+1}\|^2_{\mathbf{L}^2} - C_{\delta_3} k^2 \|\nabla[p^{m+1} - P_h^1 p^{m+1}]\|^2_{\mathbf{L}^2} \\
&\quad - k\big(\mathbf{e}^{m+1} - \mathbf{e}^m, \nabla P_h^1 \eta^{m+1}\big) \\
&\geq k^2\big(1-\delta_3-\delta_4\big)\|\nabla \eta^{m+1}\|^2_{\mathbf{L}^2} - C_{\delta_3} k^2 \|\nabla p^{m+1}\|^2_{\mathbf{L}^2} \\
&\quad - \frac{1}{4\delta_4}\|\mathbf{e}^{m+1} - \mathbf{e}^m\|^2_{\mathbf{L}^2}.
\end{aligned}$$

Because of standard approximation results and Lemma 3.1, arising interpolation error terms in (5.5)–(5.6) may be controlled as follows:

$$\begin{aligned}
\mathbb{E}\left[k\sum_{m=1}^{M+1}\|\nabla(\widetilde{\mathbf{u}}^m - \mathbf{P}_h^0\widetilde{\mathbf{u}}^m)\|^2_{\mathbf{L}^2}\right] &\leq Ch^2\mathbb{E}\left[k\sum_{m=1}^{M+1}\|\Delta\widetilde{\mathbf{u}}^m\|^2_{\mathbf{L}^2}\right] \leq Ch^2, \\
\mathbb{E}\left[\sum_{m=1}^{M+1}\|\widetilde{\mathbf{u}}^m - \mathbf{P}_h^0\widetilde{\mathbf{u}}^m\|^2_{\mathbf{L}^2}\right] &\leq Ch^4\mathbb{E}\left[\sum_{m=1}^{M+1}\|\Delta\widetilde{\mathbf{u}}^m\|^2_{\mathbf{L}^2}\right] \leq C\frac{h^4}{k}, \\
\mathbb{E}\left[k\sum_{m=1}^{M+1}\|p^m - P_h^1 p^m\|^2_{L^2}\right] &\leq Ch^2\mathbb{E}\left[k\sum_{m=1}^{M+1}\|\nabla p^m\|^2_{\mathbf{L}^2}\right] \leq Ch^2,
\end{aligned} \tag{5.7}$$

where $(5.7)_2$ comes from (5.6), which involves a coupling of discretization scales in space and time.

To keep the corresponding terms in (5.8) nonnegative, it is possible to choose $\delta_i > 0$ such that

$$1 - \delta_2 - \delta_3 - \delta_4 > 0 \qquad \text{and} \qquad \frac{1}{2} - \delta_1 - \frac{1}{4\delta_4} \geq 0.$$

Next, we sum over all $0 \leq m \leq m^* \leq M-1$ in (3.10), and take expectations. Then, by the discrete Gronwall inequality and (5.7),

$$\begin{aligned} &\mathbb{E}\Big[\|\mathbf{e}^{m^*+1}\|_{\mathbf{L}^2}^2\Big] + \left(\frac{1}{2} - \delta_1 - \frac{1}{4\delta_4}\right)\mathbb{E}\left[\sum_{m=0}^{m^*}\|\mathbf{e}^{m+1} - \mathbf{e}^m\|_{\mathbf{L}^2}^2\right] + \mathbb{E}\left[k\sum_{m=0}^{m^*}\|\nabla\mathbf{e}^{m+1}\|_{\mathbf{L}^2}^2\right] \\ &+ k^2\Big(1 - \delta_2 - \delta_3 - \delta_4\Big)\mathbb{E}\left[\sum_{m=0}^{m^*}\|\nabla\eta^{m+1}\|_{\mathbf{L}^2}^2\right] \leq C_{t_{m^*}}\left(h^2 + \frac{h^4}{k}\right). \end{aligned} \tag{5.8}$$

This proves the theorem. ◻

Remark 5.1. The same techniques may be used to find a corresponding error bound for the finite element discretization of Algorithm 4.1.

6. Computational experiments. In this section, we report on comparative computational studies for both the Euler method (1.10)–(1.11) and the splitting Algorithm 5.1. For a stable discretization in space, we use the LBB-stable MINI element; cf. [2, 12] for details. In this section we assume that (1.1) is driven by a finite dimensional noise. For the underlying domain $D = (0,1)^2 \subset \mathbb{R}^2$ and a deterministic applied forcing term $\mathbf{f}$, we consider the constant finite dimensional forcing term $\mathbf{B}\big(t, \mathbf{u}(t)\big) \equiv \mathbf{B} \in \mathcal{L}^2(\boldsymbol{\mathcal{H}}, \boldsymbol{\mathcal{K}})$, where $\boldsymbol{\mathcal{H}}$, $\boldsymbol{\mathcal{K}} \subset \mathbf{H}^1$ are finite dimensional subsets; see below. Then

$$\int_0^t \mathbf{B}\,\mathrm{d}\mathbf{W}(s) = \sum_{j,k=1}^N \int_0^t \lambda_{j,k}\,\mathrm{d}\beta_{j,k}(s)\mathbf{e}_{j,k} = \sum_{j,k=1}^N \lambda_{j,k}\beta_{j,k}(t)\mathbf{e}_{j,k} \qquad (1 \leq N < \infty),$$

where $\{\beta_{j,k}\}_{j,k=1}^N$ are independent $\mathbb{R}$-valued Wiener processes and $\{\mathbf{e}_{j,k}\}_{j,k=1}^N$ are orthonormal functions. Since the above sum is finite, the operator $\mathbf{B}$ is Hilbert–Schmidt. The orthonormal functions $\mathbf{e}_{j,k}$ are defined by $\mathbf{e}_{j,k} = \mathbf{g}_{j,k}\|\mathbf{g}_{j,k}\|_{\mathbf{L}^2}^{-1}$, where

(i) nonsolenoidal functions

$$\mathbf{g}_{j,k}(x,y) := \Big(\sin(j\pi x)\sin(k\pi y), \sin(j\pi x)\sin(k\pi y)\Big)^\top \quad \text{and}$$

(ii) solenoidal functions

$$\mathbf{g}_j(x,y) := \left(-\cos\left(j\pi x - \frac{\pi}{2}\right)\sin\left(j\pi y - \frac{\pi}{2}\right), \sin\left(j\pi x - \frac{\pi}{2}\right)\cos\left(j\pi y - \frac{\pi}{2}\right)\right)^\top$$

are used. Then $\lambda_{j,k} = \frac{1}{(j+k)^2}\|\mathbf{g}_{j,k}\|_{\mathbf{L}^2}$. Note that the index in the solenoidal functions depends only on j, in order to have orthogonality. Hence $\boldsymbol{\mathcal{H}} = \boldsymbol{\mathcal{K}} = \mathrm{span}\{\mathbf{e}_{1,1}, \ldots, \mathbf{e}_{N,N}\} \subset \mathbf{H}_0^1$ for the basis from (i), and $\boldsymbol{\mathcal{H}} = \boldsymbol{\mathcal{K}} = \mathrm{span}\{\mathbf{e}_1, \ldots, \mathbf{e}_N\} \subset \mathbf{H} \cap \mathbf{H}^2$ in the case (ii). Thus, the expansion of the noise in the case (ii) is given by

$$\int_0^t \mathbf{B}\,\mathrm{d}\mathbf{W}(s) = \sum_{j=1}^N \lambda_j\beta_j(t)\mathbf{e}_j \qquad (1 \leq N < \infty)$$

for $\lambda_j = \frac{1}{j^2}\|\mathbf{g}_j\|_{\mathbf{L}^2}$.

In the experiments below we take $N = 4$ and address the following topics in sections 6.1 and 6.2:

(A) How does nonsolenoidal (resp., solenoidal) noise affect strong approximation properties of Algorithm 5.1? Is Theorem 3.1 sharp with respect to the restriction to solenoidal noise? Is improved convergence behavior of iterates of Algorithm 4.1 for general noise observed computationally?

(B) Chorin's projection scheme in the deterministic setting is known to exhibit anisotropic error structures for the pressure, such as boundary layers of magnitude $\mathcal{O}(\sqrt{k}\,|\log k|)$; cf., e.g., [17]. What may be concluded accordingly in the stochastic setting for both trajectories and expectations of pressure iterates of Algorithms 5.1 and 4.1?

It is evident that if compared to Euler's method the splitting schemes discussed here cause reduced computational effort, which in particular pays off in the present stochastic setting, where a significant number of realizations have to be computed to obtain expectations.

For the experiments below we use $T = 1$ and compute on cartesian meshes of size $h = \frac{1}{50}$, for a number of realizations $N_p = 3000$, a minimum time discretization parameter $k_0 = \frac{1}{4096}$ and a constant operator $\mathbf{B}$ (see beginning of the present section). To approximate strong errors $(1 \leq M^* \leq M)$

(6.1)
$$\Big(\mathbf{E}\big[\|\mathbf{U}_{k_0}^{M^*} - \mathbf{U}_{k_i}^{M^*}\|_{\mathbf{L}^2}^2\big]\Big)^{1/2} \approx \left(\frac{1}{N_p}\sum_{\ell=1}^{N_p} \|\mathbf{U}_{k_0}^{M^*}(\omega_\ell) - \mathbf{U}_{k_i}^{M^*}(\omega_\ell)\|_{\mathbf{L}^2}^2\right)^{1/2} \qquad (i \geq 1),$$

we use $\mathbf{U}_{k_0}^{M^*} \approx \mathbf{u}(t_{M^*}, \cdot)$ as the (approximate) solution to (1.1) which is computed for the smallest $k_0 \ll 1$, whereas $\{\mathbf{U}_{k_i}^{M^*}\}_{i\geq 1}$ are obtained from Algorithm 5.1 for $k_i = 2^i k_0$ with $i = 1, 2, 3, \dots$.

6.1. Strong errors for different noise. We compare computed velocity iterates of both the Euler scheme (1.10)–(1.11) and Algorithm 5.1 for both solenoidal and nonsolenoidal noise. The theoretical study in the previous sections needed the uniform bound

$$\mathbb{E}\left[k\sum_{m=1}^{M} \|\nabla q^m\|_{\mathbf{L}^2}^2\right] \leq C \tag{6.2}$$

for pressure iterates of (1.10)–(1.11); this property is shown in Lemma 2.1(iii) in the case of solenoidal noise and in Lemma 3.1 for pressure iterates of Algorithm 1.1 in this case as well. The computational results in Figure 1 evidence $\frac{1}{2}$ as convergence rate at time $T = 1$ for the $\mathbf{L}^2$-error of velocity iterates from Algorithm 5.1, which is in accordance with Theorem 5.1. Figure 2 reports corresponding results for applied nonsolenoidal noise; we observe a reduction of the convergence rate for velocity iterates of Algorithm 5.1 by approximately 50%, while Euler iterates still converge optimally in the $\mathbf{L}^2$-norm. To further evidence this loss of accuracy for iterates of the splitting Algorithm 5.1 in the presence of nonsolenoidal noise, our computations in Figure 6 (left) suggest

$$\left(\mathbb{E}\left[k\sum_{m=1}^{M} \|\nabla P^m\|_{\mathbf{L}^2}^2\right]\right)^{1/2} \approx \frac{C}{\sqrt{k}},$$

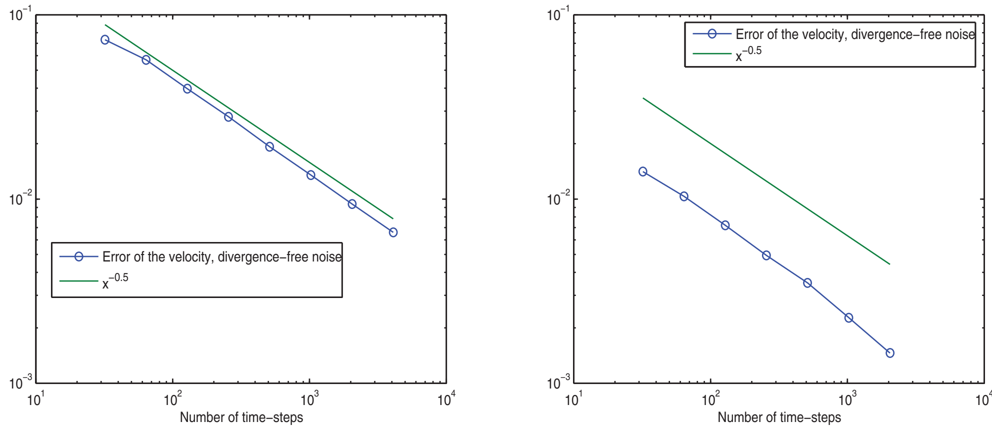

Fig. 1. *Solenoidal noise: Rates of convergence for velocity iterates of Algorithm* 5.1 *(left) and corresponding Euler iterates from the space discretization of* (1.10)–(1.11) *(right), both with respect to the norm given in* (6.1).

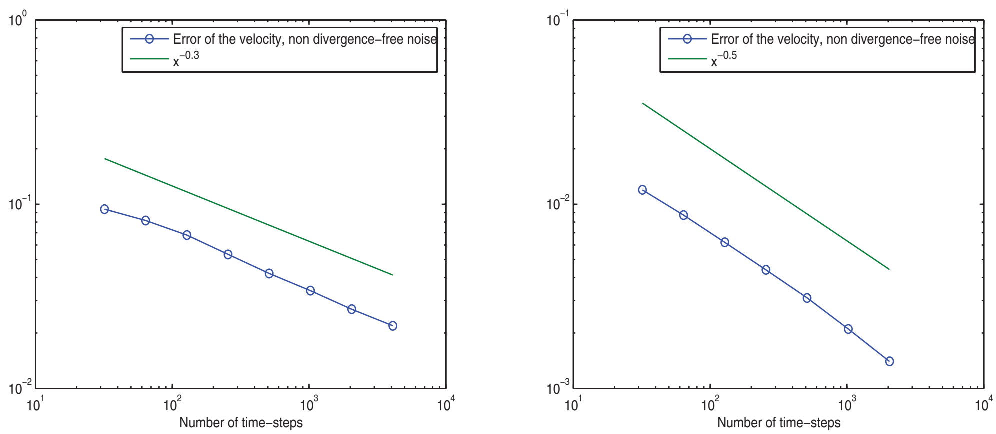

Fig. 2. *Nonsolenoidal noise: Rates of convergence for velocity iterates of Algorithm* 5.1 *(left) and corresponding Euler iterates from the space discretization of* (1.10)–(1.11) *(right), both with respect to the norm given in* (6.1).

which is a bound that we obtain for $\{q^m\}_{m=1}^M$ instead of Lemma 2.1(iii) for applied nonsolenoidal noise.

6.2. Approximation of pressures. The reformulation (1.6)–(1.8) of Algorithm 1.1 evidences error effects due to homogeneous boundary conditions, which are well-known in the deterministic setting to cause artificial boundary layers of thickness $\mathcal{O}(\sqrt{k}\,|\log k|)$; see, e.g., [10, 16, 17] and the literature cited in these works. Hence, it is reasonable to ask if corresponding anisotropic errors for pressure iterates from Algorithm 1.1 occur in the stochastic setting as well. We remark that no results regarding (rates of) convergence of iterates $\{P^m\}_{m\geq 1}$ from Algorithm 1.1 were obtained in the previous sections. The following results show error profiles for the pressure computed by Algorithm 5.1 both pathwise and expectationwise, computed for $h = 1/30$ and $k_0 = 1/512$. Again, we distinguish between computations for applied solenoidal and nonsolenoidal noise.

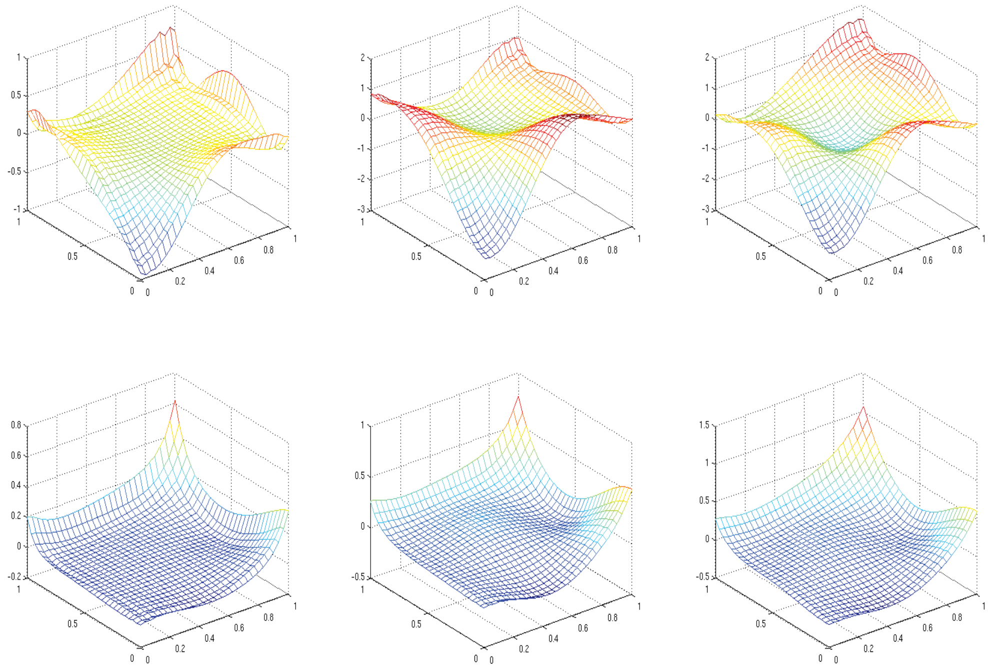

FIG. 3. *Solenoidal noise: Error of pressure from Algorithm* 5.1 *at* $T = 1$ *for* $k_i = \frac{1}{512}, \frac{1}{256}, \frac{1}{128}$ *for one realization (first line) and its expectation (second line).*

Pressure error functions in the case of solenoidal noise for different time-step sizes are depicted in Figure 3 for both a single path (first line) and expectations (second line); in both cases, we observe an anisotropic structure of error profiles, which for expectations are similar to the corresponding deterministic scenario and are more pronounced in the case of single paths, which grow for increasing time-steps $k_i > 0$.

The influence of applied nonsolenoidal noise on the accuracy of pressure iterates can be deduced from the plots in Figure 4: no local error structures are visible for a single realization; this is different from corresponding plots for expectations which still show boundary layers that dominate error profiles and increase for growing values $k_i > 0$.

6.3. Stochastic pressure correction. Here we give some numerical motivations for the new Algorithm 4.1 by considering the same setting as at the beginning of this section. Figure 5 shows error plots for different types of noise. We observe an improvement in the case of general noise to almost optimal order, which is rooted in the improved regularity of the deterministic pressure, which is exclusively needed to make sure optimal convergence behavior of this time-splitting scheme,

$$\mathbb{E}\left[k \sum_{m=1}^{M} \|\nabla p^m\|_{\mathbf{L}^2}^2\right] \leq C.$$

This is shown in Figure 6 for our example with nonsolenoidal noise. There the function

$$k_i \mapsto \mathbb{E}\left[k_i \sum_{m=1}^{M} \|\nabla P_{k_i}^m\|_{\mathbf{L}^2}^2\right]^{1/2}$$

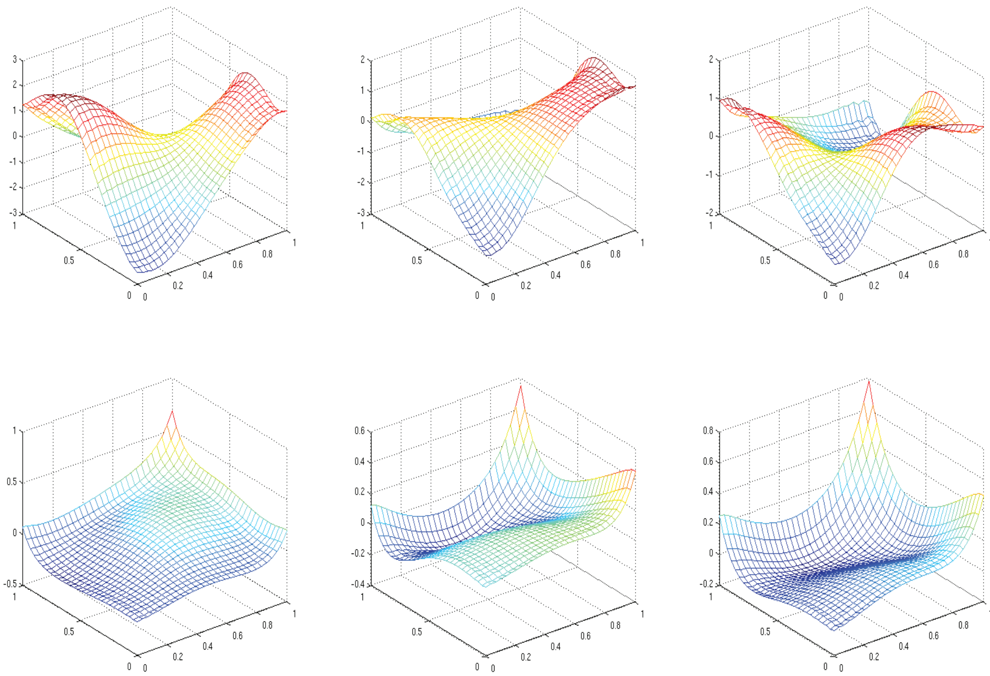

FIG. 4. *Nonsolenoidal noise: Error of pressure from Algorithm* 5.1 *at* $T = 1$ *for* $k_i = \frac{1}{512}, \frac{1}{256}, \frac{1}{128}$ *for one realization (first line) and its expectation (second line).*

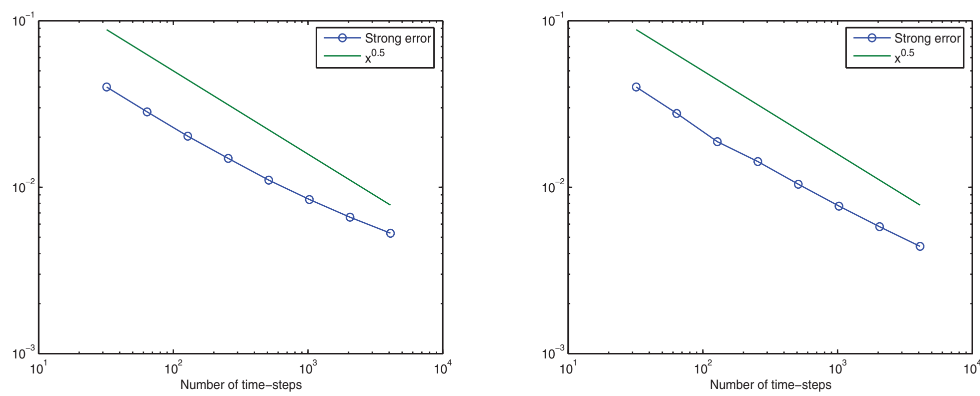

FIG. 5. *Rates of convergence for velocity iterates of Algorithm* 4.1 *with nonsolenoidal noise (left) and solenoidal noise (right), plotted with respect to the number of time-steps, both with respect to the norm given in* (6.1).

is plotted for the Chorin scheme (left) and for the scheme with the stochastic pressure correction, showing the norm of the pressure for small time-steps. Our result suggests that the *deterministic pressure* from Algorithm 4.1 has significantly better regularity properties than the pressure from Algorithm 5.1. We conjecture that the observed reduced growth with respect to the time-step $k > 0$ for the deterministic pressure is due to interacting boundary layer effects and space discretization effects of the nonsolenoidal noise.

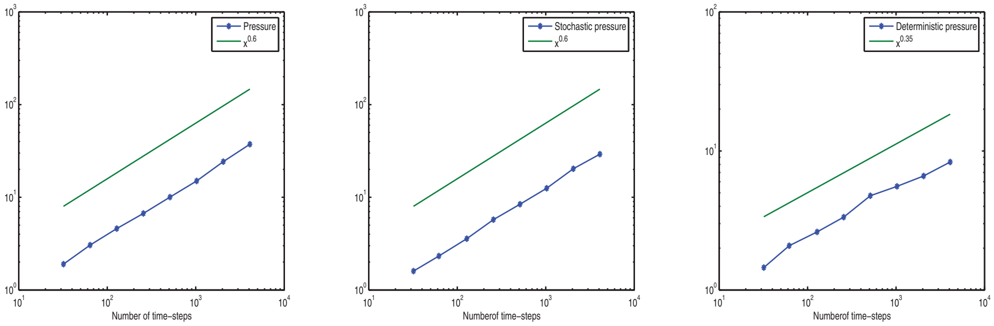

FIG. 6. *Nonsolenoidal noise: Evolution of the $H^1/\mathbb{R}$-norm of pressure iterates for Algorithm* 5.1 *(left), stochastic pressure (middle), and deterministic pressure for Algorithm* 4.1.

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