Partial synchronization and clustering in a system of diffusively coupled chaotic oscillators

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Abstract

We examine the problem of partial synchronization (or clustering) in diffusively coupled arrays of identical chaotic oscillators with periodic boundary conditions. The term partial synchronization denotes a dynamic state in which groups of oscillators synchronize with one another, but there is no synchronization among the groups. By combining numerical and analytical methods we prove the existence of partially synchronized states for systems of three and four oscillators. We determine the stable clustering structures and describe the dynamics within the clusters. Illustrative examples are presented for coupled Rössler systems. At the end of the paper, synchronization in larger arrays of chaotic oscillators is discussed. © 2001 IMACS. Published by Elsevier Science B.V. All rights reserved.

Keywords: Partial synchronization; Diffusively coupled chaotic oscillators; Clustering; Rössler system

1. Introduction

The collective motion of systems of interacting nonlinear oscillators is of significant interest in many areas of science and technology. Particularly interesting is the case where the uncoupled oscillators each behave chaotically. This situation may arise in physiology [1–5], electronics [6,7], physics [8], chemistry [9], and in a variety of other fields. The insulin producing β-cells of the pancreas, for instance, are known to display complicated patterns of bursts and spikes in their membrane potentials [3,4], and these dynamics may also become chaotic. The β-cells interact with one another through the diffusive exchange of ions and small molecules via gap junctions, and observations indicate that this interaction leads to synchronization between the cells as well as to waves of calcium that travel across the ensemble of cells.

As described, e.g. in the surveys by Pecora et al. [10] and by Rulkov [6], the problem of two coupled chaotic oscillators has been extensively studied. This problem is particularly interesting in connection with

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the development of new types of secure communication in which one exploits the possibility of masking a
signal by mixing it with a chaotic oscillation. An important question that arises in this connection concerns
the form of the basin of attraction for the synchronized chaotic state, i.e. for which initial conditions the
oscillators will be able to synchronize if they are started out of synchrony. Other important questions
relate to the stability of the synchronized state to a small parameter mismatch between the interacting
oscillators and to the behavior of the coupled system, once the synchronization breaks down. Studies of
these and related problems have lead to the discovery of a variety of new phenomena, including riddled
basins of attraction [11], soft and hard blowout bifurcations, and on–off intermittency [12]. The largest
transverse Lyapunov exponent $\lambda_\perp$ provides a measure of the average stability of the synchronized chaotic
state with respect to perturbations perpendicular to this state. As long as $\lambda_\perp$ is negative, the synchronized
state is at least weakly stable.

The transition in which $\lambda_\perp$ becomes positive is referred to as the blowout bifurcation. This transition
may lead to an abrupt loss of stability for the synchronized chaotic state. Alternatively, one may observe
an interesting form of bursting behavior known as on–off intermittency. Immediately above the blowout
bifurcation, as long as $\lambda_\perp$ is still relatively small, a trajectory started near the synchronized chaotic state
may spend a long time in the neighborhood of this state. However, sooner or later the repulsive character
of the synchronized state manifests itself, and the trajectory exhibits a burst in which it moves far out
in phase space. Provided that the trajectory does not find another limiting state to approach, and that it
is restrained by nonlinear mechanisms from diverging to infinity, after some time it will return to the
neighborhood of the synchronized state, and the process will continue to repeat itself in an apparently
random manner. On–off intermittency distinguishes itself from more conventional forms of intermittency
by the fact that the laminar phase is chaotic.

It is clear from the above discussion that the presence of other stable states in the vicinity of the
synchronized chaotic state plays an important role for the behavior of the coupled system. It is also
important if there is an absorbing region (or trapping zone) around the synchronized chaotic state that
can prevent trajectories from diverging to infinity.

Riddled basins of attraction may be observed on the other side of the blowout bifurcation where $\lambda_\perp < 0$. Even though the synchronized chaotic state is now attracting on the average, particular orbits
embedded in the chaotic set may be transversely unstable. As a result one can observe a situation where the
synchronized chaotic state attracts a positive Lebesgue measure set of points from its vicinity. Arbitrary
close to any such point, however, there will be a positive Lebesgue measure set of points that are repelled
from the synchronized state. The transition in which the first orbit on the synchronized chaotic set loses its
transverse stability is referred to as the riddling bifurcation. For a system of two symmetrically coupled,
identical oscillators this transition typically takes place via a transverse period-doubling or a transverse
pitchfork bifurcation [1,2].

If the coupled system involves more than two coupled chaotic oscillators, a whole new range of addi-
tional phenomena can occur, including partial synchronization (or clustering [13–16]) as well as various
forms of wave-like dynamics. A state of partial synchronization is said to occur when the interacting
oscillators synchronize with one another in different groups, but there is no synchronization among the
groups. Interesting questions in this relation concern the types of partial synchronization that can occur
with different coupling schemes.

The present study is mainly concerned with partial synchronization phenomena in an array of chaotic
oscillators with nearest-neighbor interaction. An alternative problem of significant interest concerns the
behavior of an ensemble of globally (i.e. all-to-all) coupled chaotic oscillators. This problem relates,
for instance, to the dynamics of a group economic sectors, that all respond to the same macroeconomic variations in purchasing power and labor availability, generated at least partly through variations in their own aggregated activity. Examples of locally coupled chaotic oscillators are also found in the living world where many cells or functional units, which individually exhibit complicated nonlinear dynamics, interact to produce a coherent behavior at a higher functional level. The basic model for our investigations is a chain of Rössler systems which are coupled in a diffusive way

\[ \dot{u}_j = f(u_j) + C(u_{j+1} + u_{j-1} - 2u_j), \quad j = 1, \ldots, N \]

with the boundary condition \( u_{N+1} = u_1 \). Here \( u_j \in \mathbb{R}^n \) denotes the phase space coordinates of the individual oscillator and \( C \) is the coupling matrix. Each of the uncoupled oscillators

\[ \begin{pmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \dot{X}_3 \end{pmatrix} = \dot{X} = f(X) = \begin{pmatrix} -X_2 - X_3 \\ X_1 + aX_2 \\ b + X_3(X_1 - c) \end{pmatrix} \]

is known to have an invariant attracting chaotic set \( A \) for \( a = 0.42, b = 2.0, \) and \( c = 4.0 \) [16]. It is also evident that the synchronization manifold \( u_1 = u_2 = \cdots = u_N \) is invariant and contains the invariant chaotic set

\[ A_s = \{ u_1 = \cdots = u_N, \ u_1 \in A \} \]

Let us recall that complete (or full) synchronization takes place when this “synchronous” set \( A_s \) is asymptotically stable. This implies that small deviations from the state (3) tend to zero, i.e. \( \| u_i - u_j \| \to 0, \ i \neq j \) with \( t \to \infty \) for initial conditions \( \varnothing(0) = (u_1(0), \ldots, u_N(0)) \) from some neighborhood \( U \) of \( A_s \).

A system of the form (1) with coupling only via the first component (i.e. \( C = \text{diag}(0, \alpha, 0) \)) was recently considered by Heagy et al. [17]. They discussed an associated size instability that occurs in systems that exhibit a short wavelength bifurcation (e.g. a variant of the Rössler system). This instability limits the number of oscillators capable of sustaining stable synchronous chaos even for large coupling. They also developed a general approach, involving the so-called “master stability function” which makes it possible to investigate different linear coupling schemes [18,19]. Phase synchronization effects in a nonidentical array of diffusively coupled Rössler oscillators with a coupling matrix \( C = \text{diag}(0, \alpha, 0) \) were investigated by Osipov et al. [20]. Systems that are coupled by a common internal field (global coupling) were considered numerically by Zanette and Mikhailov [21]. Nakagawa and Komatsu [22] studied coupled tent maps and introduced the Lyapunov exponent which characterizes the dynamical properties of the collective motion. Networks of coupled cells were considered, e.g. by Golubitsky et al. [23] using symmetry arguments.

By contrast to complete synchronization as defined above, in the case of partial synchronization the coupled system splits into clusters of identically oscillating elements. Problems of partial synchronization were studied by Pyragas [13] in connection with the phenomena of generalized synchronization. Vieira and Lichtenberg [14] showed that partial (in their notation “weak”) synchronization does not necessarily precede complete synchronization. Taborov et al. [15] reported on partial synchronization phenomena in a system of three coupled logistic maps.

The organization of this paper is as follows. In Section 2 we state some useful results for two diffusively coupled systems. Besides the numerical computations we emphasize some analytical results which for some values of coupling parameters prove the existence of a trapping region around the synchronous set.
The results of Section 3 allow us to obtain the conditions for complete synchronization and riddling for a system of three coupled Rössler systems. We also prove that this system admits partial synchronization for some narrow parameter range. The system of four coupled oscillators can be partially synchronized for a “massive” set of parameters. This case is considered in Section 4. Conditions to determine when complete or partial synchronization take place are also given. Finally, Section 5 discusses the case of a large number of diffusively coupled systems.

2. Preliminary results for two coupled systems

The system of two diffusively coupled oscillators has the following form:

\[
\begin{align*}
\dot{u}_1 &= f(u_1) + C(u_2 - u_1), \\
\dot{u}_2 &= f(u_2) + C(u_1 - u_2).
\end{align*}
\] (4)

Synchronization effects for this system are well studied (see, for example the surveys in [6,10]). For simplicity, we shall consider coupling with only one parameter in the form \(C = \alpha I\) where \(I\) is the unit matrix. Denote the transverse coordinates by \(\xi = u_1 - u_2\). The synchronization manifold \(u_1 = u_2\) is invariant for system (4).

As it follows from previous studies [2,7,10,17–20], one can identify the following qualitatively different values of the coupling parameter \(\alpha\).

**Case 1.** Those values of \(\alpha\) for which system (4) admits complete synchronization. Let us denote this set as \(S^c\).

**Case 2.** Values for \(\alpha\) where (4) has a symmetric chaotic attractor \(u_1 = u_2 \in \mathcal{A}\) such that \(\mathcal{A}\) is transversely stable on the average but embedded in \(\mathcal{A}\) there is one (or more) transversely unstable orbit. In this case the largest transverse Lyapunov exponent along any typical trajectory is negative, but \(\mathcal{A}\) is not asymptotically stable. Denote this set as \(S^r\).

**Case 3.** Remaining values of \(\alpha\). Let this set be \(S^u\). It corresponds to the case when the synchronous chaotic set is unstable.

Note that the Case 2 may admit two different dynamical behaviors depending on the global dynamics of the system. First, a globally riddled basin of attraction may occur, where the basin of the synchronized attractor is densely riddled by initial conditions from which the trajectory goes to infinity or approaches some other attractor [11]. Second, due to the existence of nonlinear restricting forces, attractor bubbling or local riddling phenomena may occur where intervals of nearly synchronous motion are intermittent with occasional bursts [12].

In order to be able to arrive at conclusions that are independent on the choice of the specific system let us make the following rather general assumption.

(A) Suppose, that there exist such constants \(\alpha_1\) and \(\alpha_2\) so that \(S^c = \{\alpha > \alpha_1\}\), \(S^r = \{\alpha_2 < \alpha < \alpha_1\}\), and \(S^u = \{\alpha < \alpha_2\}\). In other words, the attractor loses its asymptotic stability via the transverse destabilization of some nontypical orbit embedded in the attractor at \(\alpha = \alpha_1\) (riddling bifurcation) and then, with decreasing coupling parameter, becomes transversely unstable in average at \(\alpha = \alpha_2\) (blowout bifurcation).
Fig. 1. Largest transverse Lyapunov exponent vs. the coupling parameter $\alpha$, calculated for the chaotic attractor (bold curve) and for some low-periodic orbits.

To show that condition (A) is fulfilled for two coupled Rössler systems, we have calculated the largest transverse Lyapunov exponent of the synchronized chaotic attractor versus $\alpha$. We depict this variation as the bold curve in Fig. 1. The intersection of this graph with the horizontal axis (the point $\alpha_2$) determines the moment when the attractor becomes unstable in average. The thin lines in Fig. 1 show the same quantity for individual periodic orbits embedded in the attractor. The rightmost point of intersection of these lines with the axis (the point $\alpha_1$) gives us some approximation of the riddling bifurcation point. This corresponds to the transverse destabilization of the period-1 orbit. As it follows from the numerical calculations, $\alpha_1 \approx 0.060$ and $\alpha_2 \approx 0.042$.

Fig. 2 shows the plot of the transverse distance $||\xi||$ versus time for a parameter value $\alpha = 0.05 \in S'$. It is evident from this plot how small deviations from the synchronization manifold are amplified in the neighborhood of the transversely unstable period-1 orbit. Then global restraining forces return the orbit back to the weakly stable attractor.

Let us now state some analytical results which give conditions for the existence of a trapping region around the synchronous chaotic state $\{u_1 = u_2 \in \mathcal{A}\}$ in system (4). Here some analogy can be observed to the absorbing area in the case of two coupled one-dimensional maps [26]. The only requirements for the system (4) are the following.

1. The form of the coupling is $C = \alpha \text{diag}(1, 1, 1)$.
2. For an individual oscillator $\dot{X} = f(X), X \in R^n$ there exists an invariant attracting set $\mathcal{A} \subset R^n$.
3. There exists a convex trapping neighborhood $U_\mathcal{A}$ of $\mathcal{A}$ in the phase space of the individual attractor such that orbits never escape from it. This neighborhood should have a “good” local structure (for example, it can be diffeomorphic to a manifold with or without boundary).

Then for any $\alpha > 0$ system (4) will have a trapping region (in fact, this is $U_\mathcal{A} \times U_\mathcal{A}$) around the synchronized state $\{u_1 = u_2 \in \mathcal{A}\}$ in the phase space $R^{2n}$. 
The rigorous mathematical proof of this assertion is beyond the scope of the present paper and will be published elsewhere [24]. Nevertheless, we give an outline of the proof containing all the main ideas in Appendix A.

The above result implies that the global riddling phenomena with the existence of orbits diverging to infinity can occur only when the coupling matrix $C$ in Eq. (4) does not coincide with $\alpha I$. Probably, taking into account the proof of the theorem, some components of the matrix $C$ must be negative. Such a case was reported by Yanchuk et al. [2,25] where riddling was observed, e.g. for $C = \text{diag}\{0.1, 1.0, -1.73\}$. 

Fig. 2. Behavior of the transverse perturbation $\|\xi\|$ for some initial value from the neighborhood of the transversely unstable period-1 orbit embedded in the attractor ($\alpha = 0.05$).

Fig. 3. Asynchronous stable limit cycle. Projection onto the $(u_1, u_2)$ plane.
Returning now to the coupled Rössler systems, the result is that after the symmetric chaotic set loses its transverse stability the orbits will still be confined to some bounded region around the synchronization manifold. Another attractor which appears to exists for these parameter values is a stable asynchronous cycle (Fig. 3). This cycle arises in a pitchfork bifurcation at $D_0 = 0.0423$ and disappears at $D_0 = 0.028$. Hence, in our case, the regime of weak synchronization is replaced by asynchronous periodic motion (Fig. 4).

3. Partial synchronization in a system of three coupled oscillators

The symmetry properties of the system (1) with three oscillators

$$
\begin{align*}
\dot{u}_1 &= f(u_1) + \alpha I(-2u_1 + u_2 + u_3), \\
\dot{u}_2 &= f(u_2) + \alpha I(-2u_2 + u_1 + u_3), \\
\dot{u}_3 &= f(u_3) + \alpha I(-2u_3 + u_1 + u_2)
\end{align*}
$$

imply that the synchronous set loses its transverse stability in all transverse directions at the same time. In order to show this, following [18] let us rewrite system (5) into the form

$$
\dot{u} = F(u) + \alpha (G \otimes I) u.
$$

Here, $F(u) = (f(u_1), f(u_2), f(u_3))^T$, $u = (u_1, u_2, u_3)^T$, and the matrix

$$
G = \begin{pmatrix}
-2 & 1 & 1 \\
1 & -2 & 1 \\
1 & 1 & -2
\end{pmatrix}.
$$

The variational equation for the synchronized solutions of (6) can be reduced to three systems of the form

$$
\dot{\xi} = (Df(s) + \mu_i \alpha I) \xi,
$$

where $\mu_1, i = 1, 2, 3$ are the eigenvalues of $G$, and $s(t)$ is a trajectory on $\mathcal{A}$. In our case $\mu_1 = 0$ and $\mu_2 = \mu_3 = -3$. A detailed explanation of the reduction procedure can be found in [18]. Each of the above equations corresponds to some “transverse mode” (i.e. it determines the behavior of transverse perturbations restricted to some direction) except for that for $\mu_1 = 0$. Hence, the equations for both transverse modes are the same.
Therefore all three oscillators will be either synchronized together or desynchronized, provided initial values are chosen in the vicinity of the synchronization manifold. Now we can observe that the transverse variational equation for two coupled systems (4) also has the form (8) with $\mu = -2$. Hence, taking into account that $\alpha \mu \gamma = -2\alpha (\mu_i - 2) I$, we can use the results for the local stability of the synchronous motion for two oscillators and transfer them to a consideration of the stability of the transverse modes, applying the scaling factor $-2/\mu_i$.

Assuming now that the assumption (A) from the previous section is satisfied we arrive at the following conclusions.

1. For coupling strength $\alpha > (2/3)\alpha_1$ system (5) is completely synchronized.
2. For coupling $(2/3)\alpha_2 < \alpha < (2/3)\alpha_1$ either global or local riddling occurs.
3. For $\alpha < (2/3)\alpha_2$ system (5) cannot be fully synchronized.

Using the values of $\alpha_1$ and $\alpha_2$ from the previous section we obtain the thresholds for the coupled triplet: $(2/3)\alpha_1 \simeq 0.040$ and $(2/3)\alpha_2 \simeq 0.028$. For the interval $0.028 < \alpha < 0.040$ it is difficult to observe the bursts away from the synchronization state numerically. But choosing initial conditions in the neighborhood of the symmetric, unstable low-periodic orbit (i.e. the period-1 cycle) we can clearly observe how the desynchronization bursts increase, showing behaviors like those in Fig. 2 for the two modes.

In the above considerations we have performed a local analysis for the stability of the synchronization manifold. After the loss of stability for this set, some stable sets may arise outside this manifold. In particular, if such a set is located in one of the hyperplanes

$$\{u_1 = u_2, u_3\}, \{u_1 = u_3, u_2\}, \{u_2 = u_3, u_1\},$$

then partial synchronization is observed. In order to investigate the existence of the limit sets for the motion confined to these hyperplanes, we shall consider the following nonsymmetric coupling scheme

$$X' = f(X) + \alpha (Y - X), \quad Y' = f(Y) + 2\alpha (X - Y)$$

which was obtained by the factorization $u_1 = u_2 = X$ and $u_3 = Y$.

Let us define the Poincaré return map for system (10) at point $(0.44, 0, 0, 0, 0, 0)$ with normal vector directed along the $X_1$-axis. Calculations shows that this map is defined for all parameter $\alpha$-values in the considered region as well as for the considered initial values. The bifurcation diagram in Fig. 5 shows the evolution of $X_1 - Y_1$ for this map after skipping 300 iterations. We may assume that this procedure reveals the limit sets of our map in the $X_1 - Y_1$-projection and explains the dynamics inside the hyperplanes (9). We clearly observe the loss of synchronization at $\alpha \approx 0.028$ as predicted by the above linear theory. Some periodic windows are also distinguished.

Our next goal is to determine the properties of transverse stability for the limit sets which are located inside the corresponding hyperplane. In order to estimate this stability numerically, we obtain the variational equation for transverse perturbations. Due to the symmetry it is enough to consider the case $u_1 = u_2 = X, u_3 = Y$. Denote the transverse coordinates $\xi = u_1 - u_2$. Then the variational equation for $\xi$ takes the form

$$\delta \xi' = [Df(X(t)) - 3\alpha] \delta \xi,$$

where $X(t)$ is a solution of (10).
The maximal Lyapunov exponent $\lambda_c$ for system (11) is shown in Fig. 6. At the point $\alpha \approx 0.028$ we observe a loss of transverse stability in agreement with linear theory.

It is interesting to note two narrow intervals at the points $\alpha \approx 0.0038$ and $\alpha \approx 0.0198$, where $\lambda_c$ becomes negative. These parameter values correspond to the case when limit sets which are located inside the hyperplane $u_1 = u_2$ become stable in the directions transverse to the hyperplane. This implies clustering. We emphasize that this happens when the in-cluster dynamics, i.e. the dynamics within the hyperplane $u_1 = u_2$ becomes periodic and “asymmetric” with respect to $X_1 = Y_1$, cf. periodic windows in Fig. 5 while the chaotic set is transversely unstable. Fig. 7 shows the evolution of $\|u_1 - u_2\|$ and $\|u_1 - u_3\|$. The calculations confirm that in this case the partially synchronized state is periodic.
Finally note that the symmetry of (5) implies that the similar stable periodic orbit exists in other hyperplanes: $u_2 = u_3$ and $u_1 = u_3$ for the given parameter values. Different types of partial synchronization is realized by varying initial values.

4. Observing clustering phenomena in a system of four coupled oscillators

In this section we shall show that partial synchronization is observed for a more “massive” set of parameters in the case of four coupled oscillators. This is related to the fact that the synchronous set (3) first loses its transverse stability with respect to some special directions while the other transverse directions remain stable. The system of four coupled oscillators can be written in the form (6) with $u = (u_1, u_2, u_3, u_4)^T$, $F(u) = (f(u_1), \ldots, f(u_4))^T$, and the matrix

$$G = \begin{pmatrix} -2 & 1 & 0 & 1 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 1 & 0 & 1 & -2 \end{pmatrix}.$$  \hspace{1cm} (12)

The matrix (12) has the eigenvalues $\mu_{1,2} = -2$ and $\mu_3 = -4$ associated with different transverse modes with different stability properties. To find these modes we calculate the eigenvectors $v_i$ of the matrix $G$.
which correspond to $\mu_i$: $v_1 = (0, 1, 0, -1)$, $v_2 = (1, 0, -1, 0)$, and $v_3 = (-1, 1, -1, 1)$. Therefore, the first two modes involve the coincidence of $u_2 = u_4$ and $u_1 = u_3$, respectively. Their stability properties are described by Eq. (8) with $\mu = -2$. The third mode correspond to $u_1 + u_3 - u_2 - u_4 = 0$ and its stability is determined by Eq. (8) with $\mu = -4$.

Supposing that assumption (A) is satisfied we can use the same arguments as in the previous section to obtain the following conclusions.

1. If the coupling satisfies $\alpha > \alpha_1$ then the system of four coupled oscillators of the form (1) is completely synchronized.
2. For coupling parameters $\alpha_2 < \alpha < \alpha_1$ the two least stable transverse modes $u_2 = u_4$ and $u_1 = u_3$ may be unstable for some initial values at least locally.
3. For $(\alpha_1/2) < \alpha < \alpha_2$ the synchronous motion is stable only with respect to the transverse mode with $\mu_3 = -4$. Taking into account that in this case neither $u_2$ and $u_4$ nor $u_1$ and $u_3$ can coincide we may generally expect the existence of partial synchronization with the following possible clusterings: $M_1 = \{u_1 = u_2, u_3 = u_4\}$ and $M_2 = \{u_1 = u_4, u_2 = u_3\}$. Below we shall show that this holds true for Rössler systems.
4. For $\alpha < (\alpha_2/2)$ the system cannot be fully synchronized.

Referring to coupled Rössler systems, the corresponding quantities will be the following (cf. Section 2): $\alpha_1 = 0.060$, $\alpha_2 = 0.042$, $\alpha_1/2 = 0.030$, and $\alpha_2/2 = 0.021$. Fig. 8 shows the largest transverse Lyapunov exponents for a typical orbit in the synchronous set. This figure supports our conclusions. It is clear, that $\lambda_1$ corresponds to the longitudinal behavior confined to the synchronization manifold. This exponent does not depend on $\alpha$. The next two exponents $\lambda_2$ and $\lambda_3$ have equal values and correspond to the transverse modes determined by the eigenvalues $\mu_{1,2}$ of matrix $G$. The most stable mode corresponds to the Lyapunov exponent $\lambda_4$.

Fig. 8. Four largest Lyapunov exponents with different value of the coupling $\alpha$ for the system of four coupled chaotic Rössler oscillators.
With the stability loss of the first transverse mode, some asynchronous attractors arise away from the synchronization manifold. As in the case of three coupled systems, consider the stability of this asynchronous attractor with respect to perturbations which drive our system out of the partially synchronous state. We shall look for the following clustering structure

\[ u_1 = u_2 = X, \quad u_3 = u_4 = Y, \]  

(13)

(and the symmetric configuration \( u_1 = u_4 \) and \( u_2 = u_3 \)) which comes from the stability analysis of the synchronous set (this correspond to the least stable mode).

In a standard way we obtain the equations for the “perturbations” for this clustering structure. For this, denote the transverse coordinates \( \xi_1 = u_1 - u_2 \) and \( \xi_2 = u_3 - u_4 \). They measure the deviations of the trajectory from the clustered motion (13). The linearized equations admit the form

\[ \delta \xi_1' = [Df(X(t)) - 3\alpha] \delta \xi_1 - \alpha \delta \xi_2, \quad \delta \xi_2' = [Df(Y(t)) - 3\alpha] \delta \xi_2 - \alpha \delta \xi_1, \]  

(14)

where \( X(t) \) and \( Y(t) \) are solutions of two coupled systems. The values of the largest Lyapunov exponents for (14) may serve as criteria for the stability of the linearized system (14), and therefore, the stability of the partially synchronous motion (13). Fig. 9 shows these exponents versus \( \alpha \). It can be seen that after the loss of complete synchronization at \( \alpha_2 \) we have a wide range \( 0.02 \leq \alpha \leq 0.04 \) where \( \lambda_c \) is negative. This corresponds to the existence of a stable partially synchronous structure (13). Due to the symmetry, both clustering structures \( u_1 = u_2, u_3 = u_4 \) and \( u_1 = u_4, u_2 = u_3 \) are realized depending on the initial values.

To obtain the equation of the motion of the clusters in this case, we substitute \( u_1 = u_2 = X \) and \( u_3 = u_4 = Y \) (or \( u_1 = u_4 = X \) and \( u_2 = u_3 = Y \)) into system (1) of four oscillators. The resulting equation is exactly (4). Therefore, the interesting conclusion can be made: the motion of clusters coincides

![Fig. 9. Maximal Lyapunov exponent that determines stability of the partially synchronous motion \( u_1 = u_2, u_3 = u_4 \) in the system of four coupled Rössler oscillators.](image)
Fig. 10. (a) and (b) represent the possible asymptotic modes of behavior for a system of four diffusively coupled Rössler systems with $\alpha = 0.035$. Observe the clustering $u_1 = u_2$ and $u_3 = u_4$ in (a) for one set of initial values and the clustering $u_1 = u_4$ and $u_2 = u_3$ for the other.

with the motion of two coupled system (4). The considered parameter values for the Rössler systems are $\alpha_1/2 < \alpha < \alpha_2$, i.e. they correspond to the moment when our system of two coupled Rössler oscillators lost its transverse stability (cf. Section 2). As follows from Section 2 the pair of coupled Rössler oscillators for these parameters have an asynchronous periodic stable cycle, cf. Fig. 4. Hence, we will obtain the periodic asynchronous motion with $u_1 = u_2$ and $u_3 = u_4$ or with $u_1 = u_4$ and $u_2 = u_3$, depending on the initial values. Fig. 10 illustrates both possibilities for $\alpha = 0.035$.

To clarify the situation we note that there are actually two stable symmetric cycles in the phase space of the coupled system. One of them is contained in the manifold $\{u_1 = u_2, u_3 = u_4\}$ and the other in $\{u_1 = u_4, u_2 = u_3\}$. It is known that each of them has an open basin of attraction. These basins in our case appear to be strongly mixed. In order to show this we have calculated the basins in a two-dimensional cross-section of phase space. This cross-section was defined as $u_{1i} = 1.0$, $u_{4i} = 1.0$, $u_{2j} = 1.0$, $u_{3j} = 1.0$, where $i = 1, 2, 3, 4$, and $j = 2, 3, 4$. The grid was introduced to be $70 \times 70$. The behavior of the orbit starting from a center point in a given square was calculated. The result is shown in
Fig. 11. Cross-section of the basin of attraction of two stable limit cycles representing different clustering structures for $\alpha = 0.035$.

Fig. 11. Black squares correspond to initial conditions from which the system converges to one cycle, leading to the first variant of clustering (cf. Fig. 10) and white points correspond to the second variant of clustering.

5. Coupled arrays

In the previous sections we have considered the mechanisms of clustering and showed when it can be observed. The same analysis can be applied to a large system of diffusively coupled oscillators with possible chaotic behavior. Generalizing the results of the previous section, we can prove the existence of a two-cluster symmetric structure in an array of $2N$ coupled Rössler oscillators (1) with periodic relative motion. This structure is realized when $u_{2k} = X(t), k = 1, \ldots, N$ and $u_{2k+1} = Y(t), k = 0, \ldots, N - 1$. The equation for the relative motion of these clusters becomes of the form (4).

The coupling matrix $G$ in the case of $n$ diffusively coupled oscillators will have the eigenvalues [19] $\mu_k = -4 \sin^2\left(\frac{(k-1)/N}{2}\pi\right), \ k = 1, \ldots, N$. Increasing the coupling parameter $\alpha$ some transverse modes become stable and, as a result, the motion of the system becomes confined to a some manifold of lower dimension for some initial values, i.e. a clustering structure arises. Using the above scaling relation, we obtain the condition for the stability of $k$th transverse mode

$$\alpha > \frac{\alpha_1}{2 \sin^2\left(\frac{(k-1)/N}{2}\pi\right)}.$$

Therefore the condensation process begins when the first transverse mode with $\mu = -4$ becomes stable, i.e. for $\alpha > \alpha_1/2$. With further increase of the coupling strength all the transverse modes become stable, and complete synchronization occurs. This happens when $\alpha > \alpha_1/2 \sin^2\left(\frac{\pi}{N}\right) \simeq N^2\alpha_1/4\pi^2$.

We performed numerical simulations for different array sizes of coupled Rössler oscillators ($N = 5, 6, 7, 8, 9, 10, 12, 15, 20, 30, 50$) and for each size we managed to find parameter values for which the system exhibits periodic relative motion. The corresponding $\alpha$-values belong to those in-
Fig. 12. Periodic solution in an array of 30 coupled Rössler oscillators. $x_1(t_i)$ vs. $N$; $(t_1 = t_5)$.

tervals where only a few transverse modes are unstable and, hence, the system has a relatively small number of transversely unstable modes. Fig. 12 illustrates the periodic motion of an array of 30 coupled Rössler systems with $\alpha = 1$. The first components of each oscillator is plotted for some fixed time moments $t_i$. This parameter value corresponds to the case when only two transverse modes are unstable.

In the case of a more general coupling scheme, when $C = \text{diag}\{d_1, d_2, d_3\}$ with positive $d_i$, it is possible to determine the sufficient conditions for synchronization of the coupled array using the Lyapunov function approach.

Denote $C = d_2 C'$, with $C' = \text{diag}\{d'_1, 1, d'_3\}$, $d'_1 = d_1/d_2$, and $d'_3 = d_3/d_2$. For fixed $d'_1$ and $d'_3$ the equation

$$\frac{d\xi}{dr} = (J - d_2 |\mu_k| C')\xi$$

(15)

determines the stability of $k$th transverse mode. It is stable for sufficiently large values of $\alpha$ because of positiveness of all elements $d_i$, $i = 1, 2, 3$. This fact reveals the existence of a stability threshold function $\alpha'(d'_1, d'_3)$ such that for any $d_2 |\mu_k| > \alpha'(d'_1, d'_3)$ the solution $\xi = 0$ of (15) is asymptotically stable. We construct this function in Appendix B using the Lyapunov function method. Given the function $\alpha'(d'_1, d'_3)$ we may write the condition for the vanishing the $k$th transversal mode

$$d_2 |\mu_k| > \alpha'(d'_1, d'_3).$$

(16)

Hence, the sufficient conditions for synchronization of $N$ oscillators become

$$d_2 \min_{k \neq 0} |\gamma_k| = 4d_2 \sin^2 \frac{\pi}{N} > \alpha \left(\frac{d_1}{d_2}, \frac{d_3}{d_2}\right).$$

(17)
Fig. 13. The maximal number $N_{\text{max}}$ of oscillators that can be fully synchronized for the given coupling parameters. Figures (a) and (b) show the dependence on $d_2$ (with fixed $d_1 = 1.0$ and $d_3 = 3.0$) and $d_3$ (with fixed $d_1 = 1.0$ and $d_2 = 1.0$), respectively.

It follows from (17) that for given positive constants $d_1$, $d_2$, and $d_3$, the maximal number $N = N_{\text{max}}$ of oscillators that can be synchronized may be estimated as the integer part of some function

$$N_{\text{max}} = \text{int} \left[ \frac{\pi}{\arcsin \left( \frac{1}{2} \frac{\sqrt{\alpha(d_1^2 + d_3^2)}}{d_2} \right)} \right].$$

(A.1)

Fig. 13a and b illustrate the variation of $N_{\text{max}}$ with the coupling parameter $d_2$ for fixed $d_1 = 1.0$ and $d_3 = 3.0$, and with $d_3$ for fixed $d_1 = 1.0$ and $d_2 = 1.0$, respectively.

Appendix A

Here we outline the proof of an assertion about the existence of a trapping region around the synchronous state (see Section 2). Assume that conditions 1–3 of Section 2 are satisfied. Evidently, it is enough to prove the existence of such region $M$ in the phase space $(u_1, u_2)$ of the system that vector field (4) is directed inwards to the region $M$ at all points of the boundary $\partial M$ ($M$ must, of course contain the attractor).

Using the new coordinates $\xi = u_1 - u_2$ and $\eta = u_1 + u_2$, we arrive at the following form of system (4)

$$\dot{\xi} = f \left( \frac{(\xi + \eta)}{2} \right) - f \left( \frac{(\eta - \xi)}{2} \right) - 2C\xi, \quad \dot{\eta} = f \left( \frac{(\xi + \eta)}{2} \right) + f \left( \frac{(\eta - \xi)}{2} \right).$$
Denote \( M = (U_A, U_A) := \{ u_1, u_2 : u_1 \in U_A, u_2 \in U_A \} \). As \( \alpha = 0 \) this pair of oscillators becomes uncoupled. Then the specific structure of \( M \) and the property of \( U_A \) imply that orbits of system (4) do not escape \( M \) for \( \alpha = 0 \). Therefore, at any point of the boundary \( \partial M \) the vector field with \( \alpha = 0 \) is directed inwards to the region \( M \) (or tangentially to \( \partial M \)).

Now let us show that \( M \) is convex. Really, for any \((u_1, u_2), (u_1^\ast, u_2^\ast) \in M \) and constant \( 0 < \kappa < 1 \) we have \( \kappa (u_1, u_2) + (1 - \kappa)(u_1^\ast, u_2^\ast) = ((\kappa u_1 + (1 - \kappa)u_1^\ast), (\kappa u_2 + (1 - \kappa)u_2^\ast)) \) which belongs to \( M \) by the convexity of \( U_A \).

Next, exploring the symmetry of (A.1) with respect to an interchange of the oscillators \( \xi \rightarrow -\xi \) and \( \eta \rightarrow \eta \) we conclude that any point \( B_1 = (\xi, \eta) \) on the boundary \( \partial M \) corresponds to the symmetric point \( B_2 = (-\xi, \eta) \in \partial M \). From the convexity of \( M \) follows that all points of the interval \((B_1, B_2)\) will then also belong to \( M \).

Note, that the vector field (A.1) can be obtained from that for \( \alpha = 0 \) by adding at any point the constant vector \((-\alpha \xi, 0)\). But this vector is also directed along the line \((B_1, B_2)\). Hence, the vector \((-\alpha \xi, 0)\) is directed inwards to the region \( M \). Summing the two vectors directed inwards to \( M \) we again obtain a vector field which is directed to the interior of \( M \), provided \( M \) is convex and has some “good” structural properties. In order to complete the proof we note that the above arguments hold true for all points of \( \partial M \).

### Appendix B

In this appendix we derive the stability conditions for some transverse mode using of Lyapunov function
\[ V(\delta \xi) = \| \delta \xi \|^2. \]

For two coupled Rössler systems the equation for transversal perturbation has the form (15). The derivative of \( V \) with respect to time
\[ \frac{dV}{dt} = -4d_1 \left( \delta \xi_1 - \frac{s_3 - 1}{4d_1} \delta \xi_3 \right)^2 + 2\delta \xi_2^2(a - 2d_2) + 2\delta \xi_3^2 \left( -2d_3 - c + s_1 + \frac{(s_3 - 1)^2}{8d_1} \right) \]
will be negative for all \((\delta \xi_1, \delta \xi_2, \delta \xi_3) \neq (0, 0, 0)\) if the following conditions are satisfied
\[ d_1 > 0, \quad d_2 > \frac{a}{2}, \quad d_3 > \frac{1}{2} \left[ -c + s_1 + \frac{1}{8d_1}(s_3 - 1)^2 \right]. \]  

By substituting \( \alpha d_1', \alpha, \) and \( \alpha d_1' \) instead of \( 2d_1, 2d_2, \) and \( 2d_3, \) respectively, we obtain a stability condition for the trivial solution
\[ \alpha > a, \quad s_1 - c - \alpha d_1' + \frac{(s_3 - 1)^2}{4\alpha d_1'} < 0. \]  

Therefore, the threshold of synchronization with respect to \( \alpha' \) (cf. Section 5) is defined by the function
\[ \alpha'(d_1', d_1') = \max \left\{ \alpha, \frac{(s_{1\max} - c)}{2d_3'} + \sqrt{\left( \frac{s_{1\max} - c}{2d_3'} \right)^2 + \frac{(s_{3\max} - 1)^2}{4d_1'd_3'}} \right\}, \]  
which is obtained as the solution of the inequalities (B.2) with respect to \( \alpha \).
References