High-order regularized regression in Electrical Impedance Tomography

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Abstract

We present a novel approach for the inverse problem in electrical impedance tomography based on regularized quadratic regression. Our contribution introduces a new formulation for the forward model in the form of a nonlinear integral transform, that maps changes in the electrical properties of a domain to their respective variations in boundary data. Using perturbation theory results the kernel of the transform is approximated to yield a high-order misfit function which is then used to derive the regularized inverse problem. In particular, we consider the nonlinear problem to second-order accuracy, hence our approximation method improves upon the local linearization of the forward mapping. The inverse problem is approached using Newton’s iterative algorithm and results from simulated experiments are presented. With a moderate increase in computational complexity, the method yields superior results compared to those of regularized linear regression and can be implemented to address the nonlinear inverse problem.

keywords: Impedance tomography transform, quadratic regression, Newton’s method

1 Introduction

In Electrical Impedance Tomography (EIT) voltage measurements captured at the boundary of a conductive domain are used to estimate the spatial distribution of its electrical properties. The technique has numerous applications in exploration geophysics [40], environmental monitoring and hydrogeophysics [6], [21], biomedical imaging [2], industrial process monitoring [34], archaeological site assessment [29] and non-destructive testing of materials [33]. Owing to its many applications as indeed its intriguing mathematics, EIT has seen numerous theoretical and computational developments, see for example the the chapter expositions in [1], [22] and [19]. Among its fundamental challenges remain the non-linearity and ill-posedness of the inverse problem, which inevitably compromise the spatial

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resolution of the reconstructed images. From the mathematical prospective, this inverse boundary value problem, formalized by the seminal publication of Calderón [9], presents a number of implications on the existence, uniqueness and numerical stability of the inverse solution [7], [1]. Although the issues of existence and uniqueness can be eradicated under some mild assumptions, see for example [36] for isotropic conductivity fields, the instability causes the problem to be extremely sensitive to inaccuracies and small errors in the data. To alleviate the ill-posedness one usually resorts in implementing some type of regularization strategy that stabilizes the solution [16]. Based on some prior information about the unknown electrical parameters and/or the noise statistics in the measurements, the regularization schemes impose constraints to in order to stabilize the reconstructions. In the typical variational framework for example, regularization methods are often expressed as additive penalty terms augmenting the associated data–misfit function, essentially biasing the solution away from features that are inconsistent with the available a priori information.

The nonlinearity of the inverse problem is manifested in a less disruptive manner, however it has a profound impact on the quantitative information of the solution, e.g. the image. Aside a few notable exceptions, like the d-bar method [23] and the factorization method [15], algorithms that treat the nonlinearity are Newton-type iterative solvers, such as the often used Gauss–Newton (GN) method, that implement local linearization and regularization, essentially exploiting the Fréchet differentiability of the analytic forward operator, to yield a quadratic error function with respect to the unknown parameters [10], [27]. Starting from a feasible guess, one applies a number of linearization–regularization cycles until a convergence is reached in the sense of the discrepancy principle. Analysis on the convergence rates of the GN algorithm and quasi-Newton variants for high-dimensional problems can be found in [4], [16] and [20], and [14]. These results state that convergence is not guaranteed unless a stable Newton direction, descent in the usual case of minimization, is computed at each linearization point. In turn, this relies on the optimal tuning of regularization at each iteration, indeed a delicate and challenging task as the degree of ill-posedness may vary significantly [24]. To rectify this problem and aid convergence line search algorithms can be used, that scale optimally the solution increment in the descent Newton direction [5]. An additional important complication may arise when the typically-neglected linearization error is significantly large, invariably when the linearization point is ‘not close enough’ to the true solution [31]. This implies that a component of the linearized data should not be considered in the fitting process, since local linearization approximation holds true in a rather narrow trust region, and hence to cope with the lack of this information at each iteration one seeks to recover a ‘small’ perturbation of the parameters. In this context, this work contributes:

(i) A nonlinear integral transform as a forward model that maps arbitrarily large, bounded changes in electrical properties to changes in boundary observations. Effectively, this replaces the linear approximation involving the Jacobian of the forward mapping [25]. The transform has a closed form and admits a numerical approximation using the finite element method.

(ii) Exploiting the new model, a high-order misfit function is formulated for the inverse problem in the context of regularized regression. Numerical experiments on the resulting inverse problem have yielded solutions with small image errors and adequate spatial resolution.

As we develop our methodology we address mainly the EIT problem with complex isotropic admittivity and the complete electrode boundary conditions [35]. However, our derivations are not bounded by isotropic or complex property assumptions and thus can be easily shown to hold true for the similar problems of Electrical Resistance and Capacit-
tance Tomography (ERT/ECT) with purely real coefficients in scalar or tensor field material properties [1], [28]. Moreover, we show that the form of the new forward model remains unchanged with the governing elliptic differential equation is addressed in the context of more generalized boundary conditions that resemble more simplistic electrode models conventionally encountered in the geophysical setting [6], [3].

1.1 Notation and paper organization

Consider a simply connected domain \( B \subset \mathbb{R}^d, \ d = 2, 3 \) with Lipschitz smooth boundary \( \partial B \) and a space depended isotropic admittivity function \( \gamma(x, \omega) : B \rightarrow \mathbb{C} \). At an angular frequency \( \omega \geq 0 \), the admittivity can be expressed as

\[
\gamma(x, \omega) = \sigma(x) + i\omega\epsilon(x),
\]

where \( \infty > C_1 > \sigma > c_1 > 0 \) and \( \infty > C_2 > \epsilon \geq 0 \) denote the domain’s electrical conductivity and permittivity respectively for some positive bounding constants \( C_1, C_2, c \). If there are no charges or sources in the interior of \( B \) and the angular frequency of the applied currents is small enough, then Maxwell’s equations describing the electromagnetic fields in the interior of the domain reduce to the elliptic equation

\[
\nabla \cdot [\gamma(x, \omega) \nabla u(x, \omega)] = 0, \quad x \in B,
\]

where \( u \) denotes the scalar electric potential function. Measuring the potential at the accessible parts of the boundary of the domain through a finite number of sensors yields a set of observations \( \zeta \) that are likely to suffer from some type of noise and measurement imprecision \( \eta \). We will assume EIT systems equipped with \( L \) electrodes exciting the domain with a sequence of currents \( I = (I_1, \ldots, I_q) \) all fixed at frequency \( \omega \). In such a case \( \zeta \) is typically a linear combination of the electrode potentials \( U(I_i) = (U_1, \ldots, U_L) \), for \( i = 1, \ldots, q \) at the various current patterns. For an applied current pattern \( I \) we associate an electric potential field \( u \) in \( B \), and an array of electrode potentials \( U \) at \( \partial B \). When required by the context we shall denote their dependence on admittivity and applied current as \( u(\gamma) \) and \( u(I) \), or both as \( u(\gamma, I) \); and respectively \( U(\gamma) \), \( U(I) \) and \( U(\gamma, I) \). The first and second partial derivatives of \( u \) with respect to \( \gamma \) will be denoted by \( \partial_{\gamma} u \) and \( \partial_{\gamma\gamma} u \), a notation adopted for both continuous and discrete spatial functions, while for matrices and vectors the differentiation is to be considered element-wise. The position in \( B \) is specified by the vector \( x \in \mathbb{R}^d \), while the outward unit normal vector at the boundary is quoted as \( n \). Matrices and vector fields are expressed in bold capital letters while vectors and scalar fields in small case regular. For a matrix \( A \), \( a_{ij} \) will denote the \( j \)th row, \( A_{i,j} \) its \((i, j)\)th element and \( A' \) its transposition. For a vector \( v \), \( v_i \) is the \( i \)th element and \( \bar{v} \) is the complex conjugate. The spaces of real and complex numbers are quoted by \( \mathbb{R} \) and \( \mathbb{C} \), while we use \( \mathcal{R}(c) \) to express the real component of the complex argument \( c \).

The paper is organized as follows: We begin with a brief review of the the EIT model equations and associated preliminary concepts and then proceed to formulate the inverse problem commenting on the algorithms addressing the problem through local linearization. The next section is devoted to the derivation of the nonlinear integral transform under the complete electrode model and it’s generalization to the Poisson’s equation with mixed boundary conditions. Further on we consider the high-order regularized regression problem
and approximate the nonlinear system as a quadratic operator equation. Using the finite element we obtain a numerical approximation and subsequently implement Newton’s algorithm to solve the problem. Finally, we present numerical results from simulated studies that demonstrate the advantages of the proposed methodology and we end the paper with the conclusions section.

2 EIT model equations and preliminaries

The complete electrode model in electrical impedance tomography is derived from Maxwell’s time-harmonic equations at the quasi-static limit and describes the electric potential field in the closure of a conductive domain \( B \) with known electrical properties \( \gamma \) and impressed boundary excitation conditions. The model has been extensively reviewed and analyzed in several publications, including [35] where the authors prove the existence and uniqueness of the solution. With reference to figure 1, assuming no charges or current sources in the interior of \( B \), when a current is applied at the boundary, the electric potential \( u \) satisfies the elliptic partial differential equation (1). The applied current, inducing this field, is expressed by the Neumann boundary conditions

\[
\int_{\Gamma_e} ds \quad \gamma(x, \omega) \nabla u(x) \cdot n = I_\ell, \quad x \in \Gamma_e, \ \ell = 1, \ldots, L, \tag{2}
\]

\[
\gamma(x, \omega) \nabla u(x) \cdot n = 0, \quad x \in \partial B \setminus \Gamma_e. \tag{3}
\]

An accurate modeling of the electrodes is critical when comparing experimental measurements to synthetic model predictions. In effect, the voltage measurement recorded at the \( \ell \)’th electrode with contact impedance \( z_\ell \) is given by the Robin condition

\[
U_\ell = u(x) + z_\ell \gamma(x, \omega) \nabla u(x) \cdot n, \quad x \in \Gamma_e, \ \ell = 1, \ldots, L \tag{4}
\]

assuming that the characteristic function of the contact impedance is uniform on each electrode and \( \Re\{z_\ell\} > 0 \). The model admits a unique solution \((u, U)\) upon enforcing the charge conservation principle on the applied currents and a choice of ground is made

\[
\sum_{\ell=1}^L I_\ell = 0, \quad \text{and} \quad \exists \ x_0 \in \overline{B} : u(x_0) = 0. \tag{5}
\]

For the so-called forward or direct problem [1]–[5] we adopt the following essential assumptions [22], [19].

**Assumption 1** (a) The domain \( B \) is simply connected with boundary \( \partial B \) at least Lipschitz continuous.

(b) The electrical admittivity \( \gamma \in L^\infty(B) \) with \( \text{ess inf} \ (\gamma) > c_1 > 0 \).

(c) The potential field \( u \in H^1_0(B) = \{ u \in H^1(B) : u(x_0) = 0 \} \)

(d) The applied currents \( I \) and measured voltages \( \zeta \) belong in the Hilbert spaces of the \( L \), and respectively in dimensional complex vectors \( \mathbb{C}^L \) and \( \mathbb{C}^m \).

We will often refer to the solution \((u, U) \in H^1_0(B) \oplus \mathbb{C}^L \) as the direct solution, and to the problem [1]–[5] as the direct problem. Pertinent to this model is the adjoint forward problem which we define here for completeness [3]. Consider the direct solution under a pair
Figure 1: The domain under consideration $B$ with a number of round surface electrodes $L$ attached at the boundary $\Gamma_e$.

drive current pattern $I^d$ with positive and negative polarity applied at electrodes $e_p$ and $e_n$ respectively. Moreover, let the $k$’th boundary measurement be of the form

$$\zeta_k = U_{e_p} - U_{e_n}, \quad p', n' \in \{1, \ldots, L\},$$

for a pair of electrodes $e_{p'}$ and $e_{n'}$. Then the adjoint field solution $(v, V) \in H^1_0(B) \oplus \mathbb{C}^L$ satisfies the equations

\begin{align}
\nabla \cdot [\gamma(x, \omega) \nabla v(x)] & = 0 \quad x \in B, \\
\overline{\gamma}(x, \omega) \nabla v(x) \cdot n & = 0 \quad x \in \partial B \setminus \Gamma_e, \\
\int_{e_\ell} ds \overline{\gamma}(x, \omega) \nabla v(x) \cdot n & = I^m_\ell \quad x \in \Gamma_e, \\
v(x) + z\ell \overline{\gamma}(x, \omega) \nabla v(x) \cdot n & = V_\ell \quad x \in \Gamma_e, \quad 1 \leq \ell \leq L
\end{align}

where $\overline{\gamma}(x, \omega) = \gamma(x, -\omega)$ is the conjugated admittivity, and $I^m_\ell \in \mathbb{C}^L$ is the adjoint current pattern whose $\ell$’th entry equals to $I^m_\ell = I^d_\ell$, if $\ell = e_p$ or $\ell = e_n$ and zero otherwise. The adjoint solution is unique subject to the constraints in (5).

2.1 Green’s reciprocity

In what follows, we make reference to the reciprocity principle. Originally derived from Maxwell’s laws of electromagnetics, Maxwell’s reciprocity principle has an analogue for irrotational fields known as Green’s reciprocity. In the context of the impedance experiment it states, that if a current intensity $I$ is applied at the boundary of a closed domain between two electrodes, say $P_1 \doteq (e_p, e_n)$, then the potential measured at the boundary through another pair of electrodes $P_2 \doteq (e_{p'}, e_{n'})$ will be equal to the potential measured at $P_1$ if the same current is applied to $P_2$. Impedance data acquisition instruments rely on this principle to avoid making redundant, i.e. linearly dependent, measurements. To see this consider a linear conductive medium $B$ whose admittivity $\gamma$ has a nonzero imaginary component at the
operating non-resonant frequency $\omega$. Suppose we apply a time-harmonic electric current $I^d$ at the boundary of the domain through electrodes $P_1$,

$$I^d(x, t) = J(x, \omega) e^{i\omega t}, \quad x \in \partial B,$$

where $J$ is the current density field. From Maxwell’s laws the electric and magnetic fields $E$, and $H$, within the domain satisfy

$$\nabla \times H(x) = \gamma(x) E(x), \quad x \in \overline{B}.$$  

As the domain is simply connected, using $E(x) = -\nabla u(x)$ and $\nabla \times H(x) = J(x)$ reduces to Ohm’s law

$$J(x) = -\gamma(x) \nabla u(x). \quad (10)$$

Let the magnitude of the applied current be equal to $I_o$, such that $|I^d_{e_p}| = I_o$ and $I^d_{e_m} = -I^d_{e_p}$. Similarly, allow $I^m$ a different current pattern of unit magnitude applied through a different pair of boundary electrodes, say $P_2$, inducing a new electric potential field. We denote the two fields as $u(I^d)$ and $u(I^m)$ to emphasize their dependence on the excitation currents. Taking the normal component of the vector fields in (10) for $I^d$, multiplying with $u(I^m)$ and integrating over the boundary yields

$$\int_{\partial B} ds \ u(I^m) J(I^d) \cdot n = -\int_{\partial B} ds \ \gamma u(I^m) \nabla u(I^d) \cdot n.$$  

At $x \in \partial B$, let $j(x) = J(x) \cdot n$ be the normal component of the boundary current density field, then combining with conditions (2) and (4) the left hand size of the equation above reduces to

$$\int_{\partial B} ds \ u(I^m) j(I^d) = \int_{\Gamma_e} ds \ u(I^m) j(I^d)$$

$$= \sum_{\ell=1}^{L} I^d_{\ell} \int_{\Gamma_e} ds (U_{\ell}(I^m) - z_{\ell} j(I^m))$$

$$= \sum_{\ell=1}^{L} I^d_{\ell} U_{\ell}(I^m) - \sum_{\ell=1}^{L} z_{\ell} I^d_{\ell} I^m$$

$$= I_o \left(U_{e_p}(I^m) - U_{e_m}(I^m)\right),$$

where the last simplification follows as the supports of $I^d$ and $I^m$ are disjoint. Using the Green’s first formula, the right hand side of the same equation can be developed to

$$-\int_{\partial B} ds \ \gamma u(I^m) \nabla u(I^d) \cdot n = -\int_B dx \ \gamma \nabla u(I^m) \cdot \nabla u(I^d),$$

and hence equating the two yields

$$I_o \left(U_{e_p}(I^m) - U_{e_m}(I^m)\right) = -\int_B dx \ \gamma \nabla u(I^d) \cdot \nabla u(I^m).$$

Working similarly for the adjoint field $u(I^m)$ leads to

$$U_{e_p}(I^d) - U_{e_m}(I^d) = -\int_B dx \ \gamma \nabla u(I^m) \cdot \nabla u(I^d),$$
therefore for \( I_o = 1 \) we have the standard form of Green’s reciprocity theorem

\[
U_{ep}(I^m) - U_{en}(I^m) = U_{ep'}(I^d) - U_{en'}(I^d). \tag{11}
\]

An alternative way to formalize this important result is via the complete electrode admittance operator \( A_{\gamma,z} : \mathbb{C}^L \to \mathbb{C}^L \), a complex symmetric matrix that is the discrete equivalent to the Dirichlet-Neumann operator encountered at the analysis of the continuum EIT model \([7]\). For a fixed pair of \( \gamma \in L^\infty(\mathcal{B}) \) and \( z \in \mathbb{R}^L \) this bounded operator maps linearly the electrode potentials to the boundary currents inducing them, \( A_{\gamma,z} U = I \). Let \( \mu_d, \mu_m \in \mathbb{R}^L \) be two vectors of zero sum

\[
\mu_d(\ell) \triangleq \begin{cases} 
1 & \ell = e_p \\
-1 & \ell = e_n,
\end{cases} \quad \mu_m(\ell) \triangleq \begin{cases} 
1 & \ell = e_p' \\
-1 & \ell = e_n',
\end{cases} \tag{12}
\]

and consider the current pattern \( I^d = I_o \mu_d \) where \( \sum_{d=1}^L I^d_\ell = 0 \). By the symmetry of \( A_{\gamma,z} \) the \( k \)’th measurement \( \zeta_k = U_{p'} - U_{n'} \) can now be expressed as

\[
\zeta_k = \mu'_m U(I^d) = \mu'_m A_{\gamma,z}^{-1} I^d = I_o \mu'_m A_{\gamma,z}^{-1} \mu_d = I_o \mu'_d A_{\gamma,z}^{-1} \mu_m = \mu'_d U(I^m),
\]

thus we arrive at the principle \((11)\).

### 2.2 The inverse problem and its linear approximation

The inverse problem of EIT is to reconstruct the properties of the admittivity function \( \gamma \in L^\infty(\mathcal{B}) \) given the operator \( A_{\gamma,z} \). Invariably, this constitutes to determining \( \gamma \) given a finite set of linearly independent current patterns \( (I^1, I^2, \ldots, I^q) \) and their respective electrode potentials \( (U^1, U^2, \ldots, U^q) \). Typically, in EIT measurements one deals with frame(s) of (independent) data \( \zeta \) that arise as linear combinations of the \( U \) vectors. To address this ill-posed problem some prior information on the data noise \( \eta \) and the (spatial) properties of \( \gamma \) is necessitated. To approach this problem one usually considers the nonlinear operator equation

\[
\zeta = \mathcal{E}(\gamma) + \eta, \tag{13}
\]

where \( \mathcal{E} : L^\infty(\mathcal{B}) \to \mathbb{C}^m \). A solution to this problem can be obtained by considering the regularized regression problem

\[
\gamma^* = \arg \min_{\gamma} \{ \| \zeta - \mathcal{E}(\gamma) \|^2 + \mathcal{G}(\gamma) \}, \tag{14}
\]

where \( \mathcal{G} : L^\infty(\mathcal{B}) \to \mathbb{R} \) is a regularization functional. The choice of \( \mathcal{G} \) depends on the a priori knowledge on \( \gamma \), and it usually takes the form of a smoothness enforcing term \([32]\), an \( L^1 \) norm allowing for sparse solutions \([12]\) or a total variation norm that preserves large discontinuities in the electrical properties \([8, 39]\). As the forward operator was proved to be analytic \([7]\), then subject to the differentiability of \( \mathcal{G} \), problem \((14)\) becomes suitable for gradient optimization methods \([16]\). Linearizing \( \mathcal{E} \) locally within a \( d \)-dimensional sphere
\( S_{\gamma_p, \kappa} = \{ \gamma \cdot \| \gamma - \gamma_p \|^2 \leq \kappa^2 \} \), centered at an a priori guess-estimate \( \gamma_p \in L^\infty(\overline{B}) \), yields the Taylor series expansion

\[
\mathcal{E}(\gamma | S_{\gamma_p, \kappa}) = \mathcal{E}(\gamma_p) + \partial_{\gamma} \mathcal{E}(\gamma_p)(\gamma - \gamma_p) + O(\| \gamma - \gamma_p \|^2),
\]  

(15)

where \( \partial_{\gamma} \mathcal{E} : L^\infty(\overline{B}) \rightarrow \mathbb{C}^m \) is the Fréchet derivative of the forward operator and \( \kappa \geq 0 \) can be thought to be the Taylor series convergence radius. Truncating the series to first-order accuracy yields the linearized approximation of \((13)\)

\[
\zeta \simeq \mathcal{E}(\gamma_p) + \partial_{\gamma} \mathcal{E}(\gamma_p)(\gamma - \gamma_p) + \eta, \quad \gamma \in S_{\gamma_p, \kappa},
\]  

(16)

which upon inserting into problem \((14)\) leads to the regularized least-squares problem– that coincides with the first iteration of the regularized GN algorithm, for the optimal admittivity perturbation

\[
\delta \gamma^*_p = \arg \min_{\| \delta \gamma \|^2 \leq \kappa} \left\{ \| \delta \zeta - \partial_{\gamma} \mathcal{E}(\gamma_p) \delta \gamma \|^2 + \mathcal{G}(\delta \gamma) \right\}, \quad \delta \zeta = \zeta - \mathcal{E}(\gamma_p),
\]  

(17)

Subject to the invertibility of the Hessian \( [\partial_{\gamma} \mathcal{E}(\gamma_p)]' \partial_{\gamma} \mathcal{E}(\gamma_p) + \partial_{\gamma} \mathcal{G}(\gamma_p) \), implementing a GN algorithm for \( p = 0, 1, 2 \ldots \) yields a sequence of solutions \( \{ \gamma_0, \gamma_1, \gamma_2, \ldots \} \) that converges to a point in the neighborhood of \( \gamma^* \), subject to the level of noise in the data. Analysis and numerical results on the implementation of GN for the problem \((17)\) can be found in many publications and textbooks on EIT, see for example [38], [16], [24] and [20]. We emphasize that this popular approach, as well as its variants of Levenberg-Marquardt [16] and quasi-Newton schemes [14], rely fundamentally on the local linearization of the forward operator \( \mathcal{E} \), and thus yield a linear regression problem. Moreover, when \( \mathcal{G} \) is quadratic, the resulting cost-objective function to be minimized is quadratic and thus Newton-type methods provide for speedy analytically expressed solutions. The NOSER algorithm proposed in [10] is a typical example of this approach, where the solution is computed after a single regularized GN iteration. Here we propose an alternative approach that leads to high-order regression problems. In this study we address explicitly the quadratic case. In this study we address explicitly the quadratic case. The starting point toward this direction is the nonlinear integral admittivity transform that we derive next.

### 3 Nonlinear integral transform

#### 3.1 Perturbation in power

To derive the nonlinear transform that maps changes in admittivity to those they cause on the observed boundary data we follow an approach of power perturbation. The method, which is due to Lionheart, has been developed in [32] and [31] to treat the real conductivity problem. Here we extend it to the complex admittivity case incorporating also the nonlinear terms arising in the perturbation analysis. With minimal loss of generality we restrict ourselves to the case of real contact impedance. If \( \gamma \) and \( u \) are smooth enough, then applying the divergence theorem on \([1]\) for a test function \( \psi \in H^1_0(B) \) we have

\[
0 = \int_B dx \psi \nabla \cdot \gamma \nabla u = - \int_B dx \gamma \nabla u \cdot \nabla \psi + \int_{\partial B} ds \psi \gamma \nabla u \cdot n.
\]  

(18)
If $\psi$ is set to satisfy the boundary conditions on the applied currents \( \Omega \), the above develops to
\[
\int_B \, dx \, \gamma \nabla u \cdot \nabla \psi = \sum_{\ell=1}^{L} \int_{\Gamma_e} ds \left( \psi - \Psi_\ell \right) \gamma \nabla u \cdot n + \sum_{\ell=1}^{L} I_\ell \Psi_\ell,
\]
where $\Psi \in \mathbb{C}^L$ is a test vector for the electrode potentials. Plugging in the boundary condition on the measurements \( \Omega \) yields the weak form of the forward problem \( \Omega \). Notice that
\[
\nabla \cdot j = 0 \quad \text{on} \quad \Gamma_e = \delta \Gamma_e,
\]
for all $(\psi, \Psi) \in H^1_0(B) \otimes \mathbb{C}^L$. Existence and uniqueness of the weak (variational) solution $(u, U) \in H^1_0(B) \otimes \mathbb{C}^L$ has been proved in \( \Omega \). If $z_\ell > 0$, then substituting $\psi = u$, $\Psi = U$ into the weak form yields the power conservation law
\[
\int_B \, dx \, \gamma |\nabla u|^2 + \sum_{\ell=1}^{L} z_\ell \int_{\Gamma_e} ds \left| \gamma \nabla u \cdot n \right|^2 = \sum_{\ell=1}^{L} I_\ell U_\ell,
\]
which states that the power driven into the domain is either stored as electric potential or dissipated at the contact impedances of the electrodes. Consider now a complex perturbation $\gamma \rightarrow \gamma + \delta \gamma$, causing $u \rightarrow u + \delta u$ in the interior, and $U_\ell \rightarrow U_\ell + \delta U_\ell$, $j \rightarrow j + \delta j$ at the boundary. Recall that the normal component of the current density field at the boundary is $j = \gamma \nabla u \cdot n$, under the new state of the model the volume integral in \( \Omega \) becomes
\[
\int_B \, dx \, (\gamma + \delta \gamma) |\nabla (u + \delta u)|^2 = \int_B \, dx \, \gamma |\nabla u|^2 + \int_B \, dx \, \gamma \nabla u \cdot \nabla \bar{u} + \int_B \, dx \, \gamma \delta u \cdot \nabla \bar{u} + \int_B \, dx \, \gamma |\nabla \delta u|^2 + \int_B \, dx \, \delta \gamma |\nabla (u + \delta u)|^2.
\]
Notice that $\nabla \delta u \cdot \nabla \bar{u} = \nabla u \cdot \nabla \bar{u}$ hence the second and third integrals on the right sum up to $2 \int_B \, dx \, \gamma \mathcal{R}\{\nabla u \cdot \nabla \bar{u}\}$. For $I$ and $z_\ell$ fixed, developing the surface term in \( \Omega \) gives
\[
\sum_{\ell=1}^{L} z_\ell \int_{\Gamma_e} ds \left| j + \delta j \right|^2 = \sum_{\ell=1}^{L} z_\ell \int_{\Gamma_e} ds \left( |j|^2 + j \delta j + \delta j \bar{j} + |\delta j|^2 \right),
\]
hence putting together the power conservation law for the new state of the model and subtracting \( \Omega \) gives
\[
\sum_{\ell=1}^{L} I_\ell \bar{U}_\ell = \int_B \, dx \, \gamma |\nabla \delta u|^2 + \int_{\Omega} dx \, \gamma \nabla u \cdot \nabla \bar{u} + \int_{\Omega} dx \, \delta \gamma |\nabla (u + \delta u)|^2 + \int_B \, dx \, \gamma \nabla \delta u \cdot \nabla \bar{u} + \sum_{\ell=1}^{L} z_\ell \int_{\Gamma_e} ds \, \delta j \bar{j} + \sum_{\ell=1}^{L} z_\ell \int_{\Gamma_e} ds \, j \delta j + \sum_{\ell=1}^{L} z_\ell \int_{\Gamma_e} ds \, |\delta j|^2.
\]
From the weak form \[ \psi = \delta u \] with the second integral above simplifies as
\[
\int_B \mathbf{d}x \gamma \nabla u \cdot \nabla \delta u = \int_{\partial B} \mathbf{d}s \; \mathbf{d}u \gamma \nabla u \cdot \mathbf{n} = \int_{\Gamma_e} \mathbf{d}s \; \mathbf{d}u \delta j = \sum_{\ell=1}^L I_\ell \partial U_\ell - \sum_{\ell=1}^L z_\ell \int_{\Gamma_e} \mathbf{d}s \; \overline{\delta j}j,
\]
thus substituting back into the previous equation gives the perturbed power conservation law
\[
\int_B \mathbf{d}x \gamma |\nabla \delta u|^2 + \int_B \mathbf{d}x \; \delta \gamma |\nabla (u + \delta u)|^2 + \int_B \mathbf{d}x \gamma \nabla \delta u \cdot \nabla \overline{U} = \sum_{\ell=1}^L z_\ell \int_{\Gamma_e} \mathbf{d}s |\delta j|^2 + \int_{\Gamma_e} \mathbf{d}s \; \delta \overline{\delta j} \overline{j} = 0
\] (21)
In \( B \), subtracting \( \nabla \cdot \gamma \nabla u = 0 \) from \( \nabla \cdot (\gamma + \delta \gamma) \nabla (u + \delta u) = 0 \) gives the elliptic equation
\[
\nabla \cdot [\gamma \nabla \delta u + \delta \gamma \nabla (u + \delta u)] = 0 \quad \text{in } B,
\] (22)
and then applying \[ \psi = \overline{\delta u} \] for \( \psi = \overline{\delta u} \) yields
\[
\int_B \mathbf{d}x \gamma |\nabla \delta u|^2 + \int_B \mathbf{d}x \; \delta \gamma |\nabla (u + \delta u)|^2 + \int_B \mathbf{d}x \gamma \nabla \delta u \cdot \nabla \overline{U} = \sum_{\ell=1}^L z_\ell \int_{\Gamma_e} \mathbf{d}s |\delta j|^2 + \int_{\Gamma_e} \mathbf{d}s \; \delta \overline{\delta j} \overline{j} = 0
\]
where the second equality holds true by the definition of the perturbed normal component of boundary current density \( j + \delta j = (\gamma + \delta \gamma) \nabla (u + \delta u) \cdot \mathbf{n} \). Substituting back to (21) yields
\[
\int_B \mathbf{d}x \delta \gamma |\nabla u|^2 + \int_B \mathbf{d}x \; (\gamma + \delta \gamma) \nabla \delta u \cdot \nabla \overline{U} + \sum_{\ell=1}^L z_\ell \int_{\Gamma_e} \mathbf{d}s |\delta j|^2 + \sum_{\ell=1}^L z_\ell \int_{\Gamma_e} \mathbf{d}s \delta \overline{\delta j} \overline{j} = 0
\] (23)
while applying the perturbations to the electrode potential boundary condition \[ \psi \] gives \( \overline{\delta u} = \delta U_\ell - z_\ell \overline{\delta j} \), and therefore the last integral term becomes
\[
\int_{\Gamma_e} \mathbf{d}s \; \delta \overline{U} \overline{\delta j} = \sum_{\ell=1}^L \int_{\Gamma_e} \mathbf{d}s \; (\delta U_\ell - z_\ell \overline{\delta j}) \delta j
\]
\[
= \sum_{\ell=1}^L \delta U_\ell \int_{\Gamma_e} \mathbf{d}s \delta j - \sum_{\ell=1}^L z_\ell \int_{\Gamma_e} \mathbf{d}s \; |\delta j|^2
\]
\[
= - \sum_{\ell=1}^L z_\ell \int_{\Gamma_e} \mathbf{d}s \; |\delta j|^2,
\]
Lemma 3.1 The perturbations in electrical admittivity $\gamma \rightarrow \gamma + \delta \gamma$, and induced electric potential in the interior of the domain $u \rightarrow u + \delta u$ give rise to a perturbation in the boundary current density with vanishing integral

$$\int_{\partial B} ds \, \delta j(x) = 0, \quad x \in \partial B.$$  

Proof From the Neumann boundary condition (2) the current applied at the $\ell$th electrode satisfies

$$I_\ell = \int_{\Gamma_e} ds \, \gamma \nabla u \cdot n = \int_{\Gamma_e} ds \, j.$$  

Keeping $I_\ell$ fixed before and after effecting the perturbations gives

$$I_\ell = \int_{\Gamma_e} ds \, (\gamma + \delta \gamma) \nabla (u + \delta u) \cdot n = \int_{\Gamma_e} ds \, (j + \delta j).$$  

Splitting the last integral, equating the right hand sides of the two equations above, and recalling from (2), that $(j(x) + \delta j(x)) = 0$ for $x \in \partial B \setminus \Gamma_e$ yields the result.

Effectively equation (23) reduces further to

$$\int_B dx \, \delta \gamma |\nabla u|^2 + \int_B dx \, (\gamma + \delta \gamma) \nabla \delta u \cdot \nabla \bar{u} + \sum_{\ell=1}^L z_\ell \int_{\Gamma_e} ds \, \delta j \, \bar{j} = 0, \quad (24)$$  

and using once again the perturbed Robin condition the last integral simplifies further to

$$\sum_{\ell=1}^L z_\ell \int_{\Gamma_e} ds \, \delta j \, \bar{j} = \sum_{\ell=1}^L \delta U_\ell \int_{\Gamma_e} ds \, \bar{j} - \int_{\Gamma_e} ds \, \delta u \cdot \bar{j} = \sum_{\ell=1}^L \bar{T}_\ell \delta U_\ell - \int_{\Gamma_e} ds \, \delta u \cdot \bar{u} \cdot n \, n$$  

Now, consider the adjoint field problem (7)-(9) subject to a current $I^m = \bar{I}$. Then by the properties of the complete electrode admittance operator $A_{\gamma,z}$ it is easy to show that the adjoint solution $v(\bar{\gamma}, I^m)$ coincides with $u(\gamma, I^d)$. Applying the divergence theorem to the adjoint field equation (6) gives

$$\int_B dx \, \bar{\nabla} \cdot \nabla \delta u = \int_{\Gamma_e} ds \, \delta u \bar{\nabla} \cdot n = \int_{\Gamma_e} ds \, \delta u \, \bar{j}.$$  

From the above the perturbed power conservation law finalizes to

$$\sum_{\ell=1}^L \bar{T}_\ell \delta U_\ell = - \int_B dx \, \delta \gamma |\nabla u|^2 - \int_B dx \, \delta \gamma \nabla \delta u \cdot \nabla \bar{u} - \int_B dx \, (\gamma - \bar{\gamma}) \nabla \delta u \cdot \nabla \bar{u}. \quad (25)$$  

Notice that for the purely real conductivity case, i.e. the cases of electrical resistance tomography where $\omega = 0$, the third term to the right hand side vanishes and the above collapses to the formula provided in [31].
Lemma 3.2 If the applied currents are purely real, the perturbed power conservation law \((25)\) simplifies to
\[
\sum_{\ell=1}^{L} I_\ell \delta U_\ell = - \int_{\Omega} dx \delta \gamma \nabla u \cdot \nabla u - \int_{\Omega} dx \delta \gamma \delta u \cdot \nabla u. \tag{26}
\]

Proof Consider applying the diverge theorem to \((22)\) for a test function \(\psi = \bar{u}\) and to the adjoint pde \((6)\) for \(\psi = \delta u\). Then upon subtracting the later from the former yields,
\[
\sum_{\ell=1}^{L} \overline{I}_\ell \delta U_\ell = - \int_{\Gamma_e} ds \int_{\Gamma_e} \nabla u \cdot n - \int_{\Gamma_e} ds \int_{\Gamma_e} u \left( \gamma \nabla \delta u + \delta \gamma \nabla (u + \delta u) \right) \cdot n
= \int_{\Gamma_e} ds \int_{\Gamma_e} \delta u \nabla \delta u \cdot n - \int_{\Gamma_e} ds \int_{\Gamma_e} u \left( \gamma \nabla \delta u + \delta \gamma \nabla (u + \delta u) \right) \cdot n
= \int_{\Gamma_e} ds \int_{\Gamma_e} \left( \delta U_\ell - z_\ell j \right) \nabla \delta u \cdot n - \int_{\Gamma_e} ds \int_{\Gamma_e} u \left( \gamma \nabla \delta u + \delta \gamma \nabla (u + \delta u) \right) \cdot n,
\]
where the last equation is due to lemma \((3.1)\). Similarly, from the diverge theorem to \((22)\) with \(f = u\) and to \((1)\) with \(\psi = \delta u\) one obtains
\[
- \int_{\Omega} dx \delta \gamma \nabla u \cdot \nabla u - \int_{\Omega} dx \delta \gamma \delta u \cdot \nabla u
= \int_{\Gamma_e} ds \int_{\Gamma_e} \nabla u \cdot n - \int_{\Gamma_e} ds \int_{\Gamma_e} u \left( \gamma \nabla \delta u + \delta \gamma \nabla (u + \delta u) \right) \cdot n
= \int_{\Gamma_e} ds \int_{\Gamma_e} \delta u \nabla \delta u \cdot n - \int_{\Gamma_e} ds \int_{\Gamma_e} u \left( \gamma \nabla \delta u + \delta \gamma \nabla (u + \delta u) \right) \cdot n
= \int_{\Gamma_e} ds \int_{\Gamma_e} \left( \delta U_\ell - z_\ell j \right) \nabla \delta u \cdot n - \int_{\Gamma_e} ds \int_{\Gamma_e} u \left( \gamma \nabla \delta u + \delta \gamma \nabla (u + \delta u) \right) \cdot n
= \int_{\Gamma_e} ds \int_{\Gamma_e} \left( \delta U_\ell - z_\ell j \right) \nabla \delta u \cdot n - \int_{\Gamma_e} ds \int_{\Gamma_e} u \left( \gamma \nabla \delta u + \delta \gamma \nabla (u + \delta u) \right) \cdot n
= \sum_{\ell=1}^{L} I_\ell \delta U_\ell.
\]
From the above the result follows in the case where \(I_\ell = I_\ell\), i.e. the imaginary component of the currents is zero.

For simplicity we assume the case of real excitation currents. For a current pattern \(I\), let \(\gamma_p, \gamma \in L^\infty(\overline{B})\), the states of the model before and after the admittivity perturbation so that the change on the potential of the \(\ell\)’th electrode is
\[
\delta U_\ell(I) = U_\ell(\gamma, I) - U_\ell(\gamma_p, I),
\]
and evaluate equation \((25)\) for some pair drive current patterns that satisfy the constraint \((5)\). Let \(\mu_d, \mu_m \in \mathbb{R}^L\) as in \((12)\) some discrete patterns of zero sum, and define the currents
\[
I^d = a\mu_d, \quad I^m = \mu_m, \quad I^c = I^d + I^m.
\]
Suppose the currents are applied to the model with known admittivity \(\gamma_p\), and then to that of the unknown \(\gamma\), giving rise to \(U(\gamma_p, I^t) = A^{-1}_{\gamma_p, z} I^t\), and \(U(\gamma, I^t) = A^{-1}_{\gamma, z} I^t\), from which we compute the difference as
\[
\delta U(I^t) = U(\gamma, I^t) - U(\gamma_p, I^t),
\]
for \( t = \{d, m, c\} \). Based on the linearity of the admittance operator we deduce that

\[
\begin{align*}
\delta U(I^c) &= A_{\gamma,z}^{-1}(I^d + I^m) - A_{\gamma,p,z}^{-1}(I^d + I^m), \\
\delta U(I^d) &= A_{\gamma,z}^{-1}I^d - A_{\gamma,p,z}^{-1}I^d, \quad \text{and} \\
\delta U(I^m) &= A_{\gamma,z}^{-1}I^m - A_{\gamma,p,z}^{-1}I^m.
\end{align*}
\]

Evaluating the left hand side of (25) for the three current patterns yields

\[
\begin{align*}
\sum_{\ell=1}^{L} I_{\ell}^{c}\delta U_{\ell}^{c} - \sum_{\ell=1}^{L} I_{\ell}^{d}\delta U_{\ell}^{d} - \sum_{\ell=1}^{L} I_{\ell}^{m}\delta U_{\ell}^{m} &= I_{o}(\delta U_{e_{p}}^{m} - \delta U_{e_{n}}^{m}) + (\delta U_{e_{p}}^{d} - \delta U_{e_{n}}^{d}).
\end{align*}
\]

It is worth noticing that only \( \delta U^d \) are realistically measurable, since data acquisition occurs only under the direct patterns and borrowing the reciprocity result (11) for \( I_o = 1 \) gives

\[
\begin{align*}
\sum_{\ell=1}^{L} I_{\ell}^{c}\delta U_{\ell}^{c} - \sum_{\ell=1}^{L} I_{\ell}^{d}\delta U_{\ell}^{d} - \sum_{\ell=1}^{L} I_{\ell}^{m}\delta U_{\ell}^{m} &= 2(\delta U_{e_{p}}^{d} - \delta U_{e_{n}}^{d}).
\end{align*}
\]

Expanding the corresponding right hand sides from (26) yields

\[
\begin{align*}
\sum_{\ell=1}^{L} I_{\ell}^{c}\delta U_{\ell}^{c} - \sum_{\ell=1}^{L} I_{\ell}^{d}\delta U_{\ell}^{d} - \sum_{\ell=1}^{L} I_{\ell}^{m}\delta U_{\ell}^{m} &= -2\int_B dx \, \delta\gamma \, \nabla u(I^d) \cdot \nabla u(I^m) - 2\int_B dx \, \delta\gamma \, \nabla \delta u(I^d) \cdot \nabla u(I^m),
\end{align*}
\]

where we have used \( u(\gamma, I^c) = u(\gamma, I^d) + u(\gamma, I^m) \) for the interior fields. Let the \( k \)'th measurement be \( \zeta_k = \mu_m U \) and note that \( u(\gamma_p, I^m) = \overline{\gamma} \), for \( v \) the adjoint fields solution of (6).

In effect, substituting and simplifying yields

\[
\begin{align*}
\delta\zeta_k &= -\int_B dx \, \delta\gamma \, \nabla u(\gamma_p, I^d) \cdot \nabla \overline{\gamma}(\gamma_p, I^m) - \int_B dx \, \delta\gamma \, \nabla \delta u(I^d) \cdot \nabla \overline{\gamma}(\gamma_p, I^m).
\end{align*}
\]

We are now ready to tabulate our main result in the form of the following theorem.

**Theorem 3.3** (*The forward EIT transform*) Consider the complete electrode model of (1) - (5), under the assumptions 2 on a simply connected domain \( B \). Suppose further that the applied currents are purely real and that boundary measurements \( \zeta \in \mathbb{C}^m \) are observed. If \( u \) is the direct solution of this problem and \( v \) the pertinent adjoint vector satisfying (6), then for any prior admittivity guess \( \gamma_p \in L^\infty(E(\gamma_p)) \) with direct solution \( E(\gamma_p) \), the data change \( \delta\zeta = \zeta - E(\gamma_p) \) satisfies

\[
\begin{align*}
\delta\zeta &= -\int_B dx \, \delta\gamma \, \nabla u(\gamma) \cdot \nabla \overline{\gamma}(\gamma_p),
\end{align*}
\]

where \( \delta\gamma = \gamma - \gamma_p \) is the residual vector between the target solution and the initial-prior guess.

**Proof** The result follows immediately by substituting \( \delta u = u(\gamma) - u(\gamma_p) \) for all direct currents \( I^d \) to the integral equation (28), and holds true for all admissible bounded perturbations \( \delta\gamma \). This completes the proof.
3.2 Generalization to Poisson’s equation with mixed boundary conditions

Although this is nowadays the standard model for EIT, our new model formulation in (29) as well as the image reconstruction method to be described next are easily amenable to treat more simplistic electrode models. In particular, we now show that the above result holds true for a more general setting of impedance imaging involving the Poisson equation with Dirichlet and Neumann boundary conditions and point electrodes [3], [21]. In geo-electrical application one usually encounters the model

$$\nabla \cdot \left[ \gamma(x, \omega) \nabla u(x, \omega) \right] = f(x), \quad x \in B,$$  

(30)

with boundary conditions of the form

$$\alpha(x) \gamma(x, \omega) \nabla u(x, \omega) \cdot n + \beta(x) u(x, \omega) = 0, \quad x \in \partial B,$$  

(31)

where $\alpha$ and $\beta$ are functions defined on $\partial B$ and are not simultaneously zero to thoroughly impose the boundary conditions. To consider problems with different types of boundary conditions on different regions of $\partial B$, the functions $\alpha$ and $\beta$ are allowed to be discontinuous. Figure 2 shows a common geophysical problem associated with the model in (30)–(31). In this problem $\Gamma_n$ is the interface between the earth and air where a zero current condition ($\beta = 0$) holds. In the remaining boundary $\Gamma_m = \partial B \setminus \Gamma_n$, the values $\alpha$ and $\beta$ are appropriately chosen to model an infinite half-space [30]. When the sources of current are far from $\Gamma_m$, a zero potential condition ($\alpha = 0$) may be used as an approximation to the infinite half-space [37].

The electric potential measurements are collected through point-wise electrodes, contact impedances of which are effectively zero. The measurement points are $x_\ell$ for $\ell = 1, 2, \ldots, L$ and the measured potential at every point is

$$U_\ell = \int_B dx \, u(x) \delta(x - x_\ell),$$  

(32)

where $\delta(.)$ denotes the Dirac delta function. Consider a perturbation $\gamma \to \gamma + \delta \gamma$ in the additivity causing the potential perturbation $u \to u + \delta u$. Introducing these into (30)–(31)
gives
\[ \nabla \cdot ((\gamma + \delta \gamma) \nabla (u + \delta u)) = f, \quad \text{on } B, \]  
\[ \alpha (\gamma + \delta \gamma) \nabla (u + \delta u) \cdot \mathbf{n} + \beta (u + \delta u) = 0, \quad \text{on } \partial B. \]  
(33)  
(34)

Expanding (32) and (33) and using (30)–(31) to simplify the resulting terms yields
\[ \nabla \cdot (\delta \gamma \nabla u) + \nabla \cdot (\gamma \nabla \delta u) + \nabla \cdot (\delta \gamma \nabla \delta u) = 0, \quad \text{on } B, \]  
\[ \alpha (\delta \gamma \nabla u \cdot \mathbf{n} + \gamma \nabla \delta u \cdot \mathbf{n} + \delta \gamma \nabla \delta u \cdot \mathbf{n}) + \beta \delta u = 0, \quad \text{on } \partial B. \]  
(35)  
(36)

Based on (32) a perturbation in the measurement at \( x_\ell \) can be written as a volume integral
\[ \delta U_\ell = \int_B dx \delta u(x) \delta (x - x_\ell) \]  
(37)

To proceed with finding a closed form for the measurement perturbation \( \delta U_\ell \), it is useful to define \( v \), as the solution to the adjoint system
\[ \nabla \cdot (\nabla v_\ell) = \delta (x - x_\ell), \quad x \in B, \]  
\[ \alpha \nabla v_\ell \cdot \mathbf{n} + \beta v_\ell = 0, \quad x \in \partial B, \]  
(38)  
(39)

from which it is easily inferred that \( \overline{\nabla v_\ell} \) satisfies
\[ \nabla \cdot (\nabla \overline{\nabla v_\ell}) = \delta (x - x_\ell), \quad x \in B, \]  
\[ \alpha \nabla \overline{\nabla v_\ell} \cdot \mathbf{n} + \beta \overline{\nabla v_\ell} = 0, \quad x \in \partial B. \]  
(40)  
(41)

Using (37) and (40) we conclude that the perturbation to the residuals can be written in terms of the adjoint field as
\[ \delta U_\ell = \int_B dx \delta u(x) \nabla \cdot (\gamma \nabla \overline{\nabla v_\ell}). \]  
(42)

The remaining derivation requires extensive use of the following identity derived from Green’s theorem [26] for vector function \( \Psi \) and scalar function \( \psi \)
\[ \int_B dx \Psi \cdot \nabla \psi + \int_B dx \psi \nabla \cdot \Psi = \int_{\partial B} ds \psi \Psi \cdot \mathbf{n}. \]  
(43)

We begin by taking \( \psi = \delta u \) and \( \Psi = \gamma \nabla \overline{\nabla v_\ell} \) in (42) to obtain
\[ \delta U_\ell = -\int_B dx \gamma \nabla \overline{\nabla v_\ell} \cdot \nabla \delta u + \int_{\partial B} ds \gamma \delta u \nabla \overline{\nabla v_\ell} \cdot \mathbf{n}. \]  
(44)

Next using \( \psi = \overline{\nabla v_\ell} \) and \( \Psi = \gamma \nabla \delta u \) in the first term on the right hand side of (44), we have
\[ \delta U_\ell = \int_B dx \overline{\nabla v_\ell} \cdot (\gamma \nabla \delta u) - \int_{\partial B} ds \gamma \overline{\nabla v_\ell} \nabla \delta u \cdot \mathbf{n} + \int_{\partial B} ds \gamma \overline{\nabla v_\ell} \nabla \delta u \cdot \mathbf{n}. \]  
(45)

From (35), \( \nabla \cdot (\gamma \nabla \delta u) = -\nabla \cdot (\delta \gamma \nabla u) - \nabla \cdot (\delta \gamma \nabla \delta u) \) which we use in the first term on the right hand side of (45) to arrive at
\[ \delta U_\ell = -\int_B dx \overline{\nabla v_\ell} \cdot (\delta \gamma \nabla (u + \delta u)) - \int_{\partial B} ds \gamma \overline{\nabla v_\ell} \nabla \delta u \cdot \mathbf{n} + \int_{\partial B} ds \gamma \delta u \nabla \overline{\nabla v_\ell} \cdot \mathbf{n}. \]  
(46)
Appealing once more to (43) with \( \psi = v_\ell \) and \( \Psi = \delta \gamma \nabla (u + \delta u) \) in the first term of (46) gives

\[
\delta U_\ell = \int_B dx \, \delta \gamma \nabla v_\ell \cdot \nabla u + \int_B dx \, \delta \gamma \nabla v_\ell \cdot \nabla \delta u \\
- \int_{\partial B} ds \left( \gamma \nu_\ell \delta u \cdot n + \delta \gamma \nu_\ell \nabla u \cdot n + \delta \gamma \nu_\ell \nabla \delta u \cdot n - \gamma \delta u \nu_\ell \cdot n \right). \tag{47}
\]

We now show that the surface integral term in (47) is zero. For this purpose we multiply both sides of (41) by \( \delta u \) to arrive at

\[
\alpha \gamma \delta u \nabla v_\ell \cdot n + \beta \delta u v_\ell = 0 \tag{48}
\]

Using (36) to replace the term \( \beta \delta u \) in (48) results in

\[- \alpha \left( \gamma \nu_\ell \nabla \delta u \cdot n + \delta \gamma \nu_\ell \nabla \delta u \cdot n + \delta \gamma \nu_\ell \nabla \delta u \cdot n - \gamma \delta u \nu_\ell \cdot n \right) = 0, \quad \text{on } \partial B. \tag{49}\]

The parenthesized expression in (49) is the same as the surface integrand in (47). We partition the boundary \( \partial B \) into \( \Gamma_\alpha \) where \( \alpha \neq 0 \) and \( \partial B / \Gamma_\alpha \) where \( \alpha = 0 \). Clearly (49) results the inside bracket expression to vanish on \( \Gamma_\alpha \). On the remaining surface \( \partial B / \Gamma_\alpha \) that \( \alpha = 0 \), we certainly have \( \beta \neq 0 \) since \( \alpha \) and \( \beta \) may not be simultaneously zero and using this fact in (36) and (41) would result in \( \delta u = 0 \) and \( \nu_\ell = 0 \) which again make the inside bracket term zero. Therefore the surface integral in (47) vanishes both on \( \Gamma_\alpha \) and \( \partial B / \Gamma_\alpha \) and therefore

\[
\delta U_\ell = \int_B dx \, \delta \gamma \nabla v_\ell \cdot \nabla u + \int_B dx \, \delta \gamma \nabla v_\ell \cdot \nabla \delta u, \tag{50}
\]

and thus by substituting for \( \delta u \) in the second term we arrive at the result of the theorem 3.3.

4 High-order regularized regression

Within the \( d \)-dimensional sphere \( S_{\gamma p, \kappa} \), the electric potential field in the interior of the domain admits a Taylor expansion

\[
u(\gamma) = u(\gamma_p) + \partial_\gamma u(\gamma_p) \delta \gamma + O(\|\delta \gamma^2\|)
\]

hence to first-order accuracy this can be approximated by

\[
u(\gamma) \approx \hat{u}(\gamma) = u(\gamma_p) + \partial_\gamma u(\gamma_p) \delta \gamma. \tag{51}
\]

Introducing \( \hat{u} \) in place of \( u \) in the integral equation (29) gives

\[
\delta \zeta \approx - \int_B dx \, \delta \gamma \nabla \left( u(\gamma_p) + \partial_\gamma u(\gamma_p) \delta \gamma \right) \cdot \nabla v(\gamma_p), \quad \text{(52)}
\]

where the first, linear term, involves the definition of the Fréchet derivative of the forward mapping as in [16], [3], [27], and the second nonlinear term the differential operator \( \partial_\gamma u(\gamma_p) \) that provides a measure on local sensitivity of the potential in the interior of the domain to perturbations in electrical properties. From (51), (52), it is trivial to deduce that the linear
approximation of the forward operator $\mathcal{E}$ as in [15], as proposed by Calderón in [9], effectively imposes a zeroth-order Taylor approximation on the electric potential $\hat{u}(\gamma) \approx u(\gamma_p)$. In turn this enforces $\partial_\gamma u$ and higher-order derivatives to vanish everywhere in $\bar{B}$, thus eliminating the nonlinear terms in (28) and (52). Let the linear operator $\partial_\gamma \mathcal{E} = J : L^\infty(\bar{B}) \to \mathbb{C}^m$, and nonlinear, quadratic in $\delta\gamma$, $K : L^\infty(\bar{B}) \to \mathbb{C}^m$ defined by

\begin{equation}
J \delta\gamma \doteq -\int_B dx \delta\gamma \nabla u(\gamma_p) \cdot \nabla \bar{v}(\gamma_p),
\end{equation}

\begin{equation}
K \delta\gamma \doteq -\int_B dx \delta\gamma \nabla \partial_\gamma u(\gamma_p) \delta\gamma \cdot \nabla \bar{v}(\gamma_p)
\end{equation}

then the inverse problem can be formulated in the context of regularized regression based on the nonlinear operator equation

\begin{equation}
\delta\zeta = J \delta\gamma + K \delta\gamma + \eta.
\end{equation}

### 4.1 Numerical approximation

Usually the EIT problem is approached with a numerical approximation method like finite elements, where the governing equations are discretized on a finite dimensional model of the domain, say $B_h(k, n)$ comprising $n$ nodes connected in $k$ elements [32]. For simplicity in the notation we assume linear Lagrangian finite elements and consider element-wise linear and constant basis functions for the support of the electric potential $u$ and conductivity $\gamma$ respectively,

\begin{equation}
\begin{aligned}
    u(x, \omega) &= \sum_{i=1}^n u_i \phi_i, \quad \phi_i : B_h \to \mathbb{R}, \\
    \gamma(x, \omega) &= \sum_{i=1}^k \gamma_i \chi_i, \quad \chi_i : B_h \to \mathbb{R}
\end{aligned}
\end{equation}

where $\{\phi_i\}_{i=1}^n$ and $\{\chi_i\}_{i=1}^k$ the respective bases in $B_h$. For clarity in the notation, we keep $u$ and $\gamma$ as the vectors of coefficients relevant to the respective functions as from now on we deal exclusively the numerical approximation of the problem. On the discrete domain the weak form of the operator equation (55) is approximated by

\begin{equation}
\delta\zeta_i = j_i\delta\gamma + \delta\gamma^T K_i \delta\gamma + \eta_i, \quad i = 1, \ldots, m
\end{equation}

where $\zeta_i \in \mathbb{C}$ is the $i$th measurement, $j_i$ the $i$th row of the Jacobian matrix $J$ that is the discrete form of $\partial_\gamma \mathcal{E}(\gamma_p)$, $K_i \in \mathbb{C}^{k \times k}$ is the $i$th coefficients matrix derived from $K$ in (54), $\eta$ the noise in the $i$th measurement and $\delta\gamma \in \mathbb{C}^k$ the required perturbation in the admittivity coefficients. Let the additive noise be uncorrelated zero-mean Gaussian with diagonal covariance matrix $C_\eta$. If $c_i$ is the positive element of $C_\eta$ define the data misfit function

\begin{equation}
Q(\delta\gamma) = \sum_{i=1}^m c_i^{-1} \left( \delta\zeta_i - j_i\delta\gamma - \delta\gamma^T K_i \delta\gamma \right)^2
\end{equation}

and consider the regularized quadratic regression problem

\begin{equation}
\delta\gamma^* = \arg\min_{\delta\gamma \in \mathbb{C}^k} \xi(\delta\gamma), \quad \xi(\delta\gamma) = \frac{1}{2} \left\{ Q(\delta\gamma) + \alpha G(\delta\gamma) \right\}
\end{equation}
with $\mathcal{G}: \mathbb{C}^k \rightarrow \mathbb{R}$ a convex differentiable regularization term. On the other hand, choosing to neglect the matrices $K_i$ yields the conventional misfit function

$$
\Lambda(\delta \gamma) = \sum_{i=1}^{m} c_i^{-1} (\delta \zeta_i - f_i' \delta \gamma)^2,
$$

(60)

often used in the context of regularized linear regression formulations. As shown in $[1]$ the Jacobian matrix can be computed directly from $[53]$ and $[56]$ using numerical integration as

$$
J_{i,j} = - \int_{B_j} dx \chi_j \sum_{l \in \text{supp}(B_j)} u_l \nabla \phi_l \sum_{l \in \text{supp}(B_j)} \nabla \phi_l, \quad i = 1, \ldots, m, \ j = 1, \ldots, k
$$

(61)

with $v$ the coefficients of the adjoint field solution corresponding to the $i$th measurement, and $\text{supp}(B_j)$ the support of the $j$th element. To derive the respective element of $K_i$ we follow an approach similar to that of Kaipio et al. in $[18]$ that is based on the Galerkin formulation of the problem. For this we choose $\{\phi_1, \ldots, \phi_n\}$ as a test basis for the potentials and by substituting into the variational form of the model we arrive at

$$
\sum_{i=1}^{n} \sum_{j=1}^{n} \left( \int_B \nabla \gamma \phi_i \cdot \nabla \phi_j + \sum_{l=1}^{L} z_l \int_{e_l} ds \phi_i \phi_j \right) u_i = - \sum_{l=1}^{L} z_l \int_{e_l} ds \phi_i U_{e_l} = 0.
$$

Imposing the Neumann conditions for the applied boundary currents yields the additional equations

$$
I_{e_l} = - z_l \sum_{i=1}^{n} \left( \int_{e_l} ds \phi_i \right) u_i + z_l |e_l| U_{e_l}, \quad e_l = 1, \ldots, L,
$$

with $|e_l|$ the area of the $l$th electrode. In matrix form the electric potential expansion coefficients $u \in \mathbb{C}^n$ and the electrode potentials $U \in \mathbb{C}^L$ can be computed by solving the $(n + L) \times (n + L)$ matrix equation

$$
\begin{bmatrix}
A_{11} & A_{12} \\
A'_{12} & A_{22}
\end{bmatrix}
\begin{bmatrix}
u \\
U
\end{bmatrix} =
\begin{bmatrix}0 \\
I
\end{bmatrix},
$$

(62)

where

$$
A_{11,i,j} = \int_B dx \gamma \nabla \phi_i \cdot \nabla \phi_j + \sum_{l=1}^{L} z_l \int_{e_l} ds \phi_i \phi_j, \quad i, j = 1, \ldots, n
$$

$$
A_{12,i,l} = - z_l \int_{e_l} ds \phi_i, \quad i = 1, \ldots, n, \quad l = 1, \ldots, L,
$$

$$
A_{22,l,l} = z_l |e_l|, \quad l = 1, \ldots, L.
$$

For a conductivity $\gamma$ and applied current $I$, let $\begin{bmatrix} u \\
U
\end{bmatrix} = A^{-1}(\gamma) \begin{bmatrix}0 \\
I
\end{bmatrix}$ the solution of (62). Using the matrix differentiation formula, the partial derivatives with respect to the $q$th admittivity element are

$$
\partial_{\gamma_q} \begin{bmatrix} u \\
U
\end{bmatrix} = \partial_{\gamma_q} \begin{bmatrix} A^{-1}(\gamma) \begin{bmatrix}0 \\
I
\end{bmatrix} \end{bmatrix} = - A^{-1}(\gamma) \partial_{\gamma_q} \{A(\gamma)\} A^{-1}(\gamma) I
$$

$$
= - A^{-1}(\gamma) \partial_{\gamma_q} \{A(\gamma)\} \begin{bmatrix} u \\
U
\end{bmatrix}.
$$
where

$$\partial_{\gamma_q} \{A(\gamma)\} = \partial_{\gamma_q} \{A_{11}(\gamma)\} = \int_{B_q} dx \nabla \phi_i \cdot \nabla \phi_j, \quad q = 1, \ldots, k,$$

as only the block $A_{11}$ depends on admittivity. Separating the above as

$$\partial_{\gamma} \left\{ \begin{bmatrix} u \\ U \end{bmatrix} \right\} = \begin{bmatrix} \partial_{\gamma_1} u & \partial_{\gamma_k} U \end{bmatrix},$$

and evaluating the upper part for all elements in the model yields the required matrix in vector concatenation form

$$\partial_{\gamma} u(\gamma) = \begin{bmatrix} \partial_{\gamma_1} u(\gamma) & \partial_{\gamma_2} u(\gamma) & \ldots & \partial_{\gamma_k} u(\gamma) \end{bmatrix},$$

(63)

while $\partial_{\gamma_k} U$ are the elements of the Jacobian matrix $J$. Effectively the element of $K^i$ matrix is given by

$$K^i_{r,j} = -\int_{B_j} dx \psi_j \sum_{l \in supp(B_j)} \nabla \phi_l \partial_{\gamma_r} u_l \sum_{l \in supp(B_j)} \nabla_l \nabla \phi_l, \quad i = 1, \ldots, m, \quad r, j = 1, \ldots, k$$

with $u$ the adjoint field corresponding to the $i$'th measurement and $\partial_{\gamma_r} u$ the derivative of the $i$'th direct field with respect to $\gamma_r$.

### 4.2 Newton’s minimization method

We propose solving the regularized problem (59) using Newton’s minimization method [16]. At a feasible point $\delta \gamma_p$ the minimization cost function $\xi$ is approximated by a second-order Taylor series

$$\hat{\xi}(\delta \gamma) = \xi(\delta \gamma_p) + \partial_{\delta \gamma_1} \xi(\delta \gamma_p)(\delta \gamma - \delta \gamma_p) + \frac{1}{2}(\delta \gamma - \delta \gamma_p)'\partial_{\delta \gamma_1} \xi(\delta \gamma_p)(\delta \gamma - \delta \gamma_p),$$

(64)

where applying first-order optimality conditions $\partial_{\delta \gamma_p} \hat{\xi}(\delta \gamma) = 0$ yields the linear system

$$\partial_{\delta \gamma_p} \hat{\xi}(\delta \gamma_p) = -\partial_{\delta \gamma_1} \hat{\xi}(\delta \gamma_p)(\delta \gamma - \delta \gamma_p).$$

From (58), let the $i$th residual function be

$$r_i(\delta \gamma) = c_i^{-1/2} \left(\delta \zeta_i - \sum_{j=1}^{k} J_{i,j} \delta \gamma_j - \sum_{j=1}^{k} \sum_{l=1}^{k} K^i_{j,l} \delta \gamma_l\right),$$

such that $Q(\delta \gamma) = \|r(\delta \gamma)\|^2$, then the cost gradient $\partial_{\delta \gamma_p} \xi(\gamma_p)$ and Hessian $\partial_{\delta \gamma_p} \hat{\xi}(\delta \gamma_p)$ are expressed as

$$\partial_{\delta \gamma_p} \hat{\xi}(\delta \gamma_p) = \partial_{\delta \gamma} r(\delta \gamma_p)' r(\delta \gamma_p) + \alpha C_\gamma^{-1} \delta \gamma_p,$$

$$\partial_{\delta \gamma_1} \hat{\xi}(\delta \gamma_p) = \partial_{\delta \gamma} r(\delta \gamma_p)' \partial_{\delta \gamma} r(\delta \gamma_p) + \alpha C_\gamma^{-1}$$

for $r(\delta \gamma) = [r_1(\delta \gamma), \ldots, r_m(\delta \gamma)]'$, and assuming a Tikhonov-type regularization function $G(\delta \gamma) = \alpha \delta \gamma^T C_\gamma^{-1} \delta \gamma$, with $C_\gamma^{-1}$ positive semidefinite and $\alpha$ a positive regularization parameter. The Jacobian of the residual $\partial_{\delta \gamma} r(\delta \gamma_p) \in \mathbb{C}^{m \times k}$ is then formed using the vectors

$$\partial_{\delta \gamma_l} r_i(\delta \gamma) = -c_i^{-1/2} J_{i,l} - c_i^{-1/2} \sum_{j=1}^{k} \left( K^i_{l,j} + K^i_{j,l} \right) \delta \gamma_j, \quad l = 1, \ldots, k.$$
evaluated at $\delta \gamma_p$ like

$$\partial_{\delta \gamma} r(\delta \gamma_p) = \left[ \partial_{\delta \gamma} r_1(\delta \gamma_p) \mid \partial_{\delta \gamma} r_2(\delta \gamma_p) \mid \ldots \mid \partial_{\delta \gamma} r_m(\delta \gamma_p) \right]' .$$

If $\partial_{\delta \gamma} \xi(\delta \gamma_p)$ is full rank and positive definite the solution can be computed iteratively using Newton’s algorithm

$$\delta \gamma_{p+1} = \delta \gamma_p - \partial_{\delta \gamma}^{-1} \delta \gamma_p \partial_{\delta \gamma} \xi(\delta \gamma_p), \quad p = 0, 1, 2, \ldots \quad (65)$$

Using standard arguments from the convergence analysis of Newton’s method on convex minimization it is easy to show convergence as in $[16], [24]$ however a convergence in the sense of the discrepancy principle is more appropriate as the data are likely to contain noise $[19]$.

**Corollary 4.1** Initializing the quadratic regression iteration $(65)$ with $\delta \gamma_0 = 0$ yields a first iteration that coincides with the linear regularized regression estimator

$$\delta \gamma_1 = \left( J' C_{\eta}^{-1} J + \alpha C_{\gamma}^{-1} \right)^{-1} J' C_{\eta}^{-1} \delta \zeta \quad (67)$$

**Proof** The proof is by substitution of the residual and its Jacobian at $\delta \gamma_0 = 0$ into the expressions for the gradient and Hessian of the cost function. In particular for $r(\delta \gamma_0) = C_{\eta}^{-1/2} \delta \zeta$ and $\partial_{\delta \gamma} r(\delta \gamma_0) = C_{\eta}^{-1/2} J$, iteration $(65)$ yields the result.

Combining the convergence remarks of $(66)$ with the corollary above, we assert that for $p > 1$ the quadratic regression iterations should converge in a solution whose error does not exceed that of the linear regression problem $(17)$. Suppose now that at a certain iteration $p$ the value of the residual $r(\delta \gamma_p)$ converges to the level of noise $\| \eta \|$. Then according to the discrepancy principle one updates the admittivity estimate as $\gamma_{p+1} = \gamma_p + \delta \gamma_p$ and thereafter the definitions of $J$ and $K_i$, and then proceeds to the next iteration. Effectively, the resulting scheme can be expressed as a Newton-type algorithm.

1. Given data $\zeta \in \mathbb{C}^m$ and finite domain $B_h$ with unknown admittivity parameters $\gamma^* \in \mathbb{C}^k$
2. Set $q = 0$, choose initial admittivity distribution $\gamma_0$,
3. For $q = 1, 2, \ldots$ (Newton exterior iterations)
4. Compute data $\delta \zeta = \zeta - \mathcal{E}(\gamma_q)$, and matrices $J \in \mathbb{C}^{m \times k}$, $K_i \in \mathbb{C}^{k \times k}$, $i = 1, \ldots, m$,
   (a) Set $p = 0$, $\delta \gamma_p = 0$,
   (b) For $p = 1, 2, \ldots$ (Newton interior iterations)
   (c) Compute update $\delta \gamma_{p+1} = \delta \gamma_p - \tau_p \partial_{\delta \gamma}^{-1} \delta \gamma_p \partial_{\delta \gamma} \xi(\delta \gamma_p)$ for $\tau_q > 0$,
   (d) End $p$ iterations, update solution estimate $\gamma_{q+1} = \gamma_q + \tau_p \delta \gamma_p$ for $\tau_p > 0$,
5. End $q$ iterations.
In performing the outer iterations, a complication will likely arise in that a certain update admittivity change $\delta \gamma_p$ may cause the real and/or imaginary components of $\gamma_{q+1}$ to become zero or negative. This of course violates a physical restriction on the electrical properties of the media, and the solution cannot be admitted. For this reason the problem of (59) should be posed as a linearly constrained problem

$$\delta \gamma^* = \arg \min_{\delta \gamma \geq \gamma} \xi(\delta \gamma),$$

at each $\gamma_q$. A convenient heuristic to prevent this complication is by adjusting the step sizes $\tau_q, \tau_p$ until the above inequality is satisfied \[92, \ 38\]. Note also, that the above methodology makes no explicit assumptions on the type of the regularization functional $\mathcal{G}(\gamma)$, aside its differentiability, thus we anticipate it can be also be implemented in conjunction with total variation and $\ell_1$-type regularization \[8\] as well as the level sets method \[11\].

5 Numerical results

For the numerical experiments we focus on the case of two-dimensional admittivity functions, although the extension to three dimensions follows in a trivial way. Consider a rectangular domain $B = [-16, 16] \times [0, -32] \subset \mathbb{R}^2$, with $L = 30$ point electrodes attached on boundary $\Gamma_e$ in a borehole and surface arrangement as shown in figure \[5\]. The domain is assumed to have an unknown target conductivity $\gamma^*$ whose real and imaginary components are functions with respective bounds $1.46 \leq \sigma^* \leq 5.60$ and $0.74 \leq \omega^* \leq 3.90$. To compute the measurements we consider opposite pair drives $I^d$, $d = 1, \ldots, L/2$, yielding a total of $m = 390$ voltage measurements $\zeta \in \mathbb{C}^m$. The forward problem is solved approximately using the finite element method as outlined in the previous section, and to the measurements we add a Gaussian noise signal of zero mean and positive definite covariance matrix $\mathbf{C}_n = 10^{-5} \max |\zeta| \mathbf{I}$, where $\mathbf{I}$ is the identify matrix.

For the forward problem we use a finite dimensional model $B_f$ comprising $n = 1701$ nodes connected in $k = 3144$ linear triangular elements. All other computations are performed on a coarser grid $B_i$ with $n = 564$ nodes and $k = 1038$ elements. The two finite models are nested, hence for any function $\gamma$ approximated on $B_i$ with expansion coefficients $\gamma_i$ there exists a projection $\gamma_f = \Pi \gamma_i$, mapping it onto $B_f$. To reconstruct the synthetic data we assume an initial homogeneous admittivity model $\gamma_0 = 3.90 + 2.40i$ which coincides with the mean value of $\gamma^*$, a methodology adopted from \[17\]. This starting point yields an optimal admittivity change $\delta \gamma^*$ as this is depicted at the top of figure \[5\].

At the initial admittivity guess $\gamma_0$ we approximate the potential $u(\gamma^*)$ with $\hat{u}(\gamma^*)$ using the zeroth-order Taylor series $u(\gamma_0)$ and first-order $u(\gamma_0) + \partial_u u(\delta \gamma_0)(\gamma - \gamma_0)$. The respective normalized approximation errors are illustrated at the top of figure \[3\] next to those of the error in the induced potential gradient as this is involved in the computation of the $\mathbf{K}$ matrices for $i = 1, \ldots, 390$. The results show that the linear approximation sustains a smaller error in both quantities and at all applied current patterns. In the same figure we also plot the measurement perturbations $\delta \zeta = \zeta - \mathcal{E}(\gamma_0)$ versus the linear and the quadratic predictions to demonstrate that the proposed quadratic regression will fit the noisy measurements at a smaller error. In particular, the quadratic and linear misfit cost functions in (58) and (60) attain values $Q(\delta \gamma^*) = 0.057$ and $\Lambda(\delta \gamma^*) = 0.123$ without any noise and $Q(\delta \gamma^*) = 0.062$ and $\Lambda(\delta \gamma^*) = 0.126$ with the additive noise infused in the data.
Figure 3: At the top row, the normalized errors in the electric potential field approximation and its gradient, assuming zeroth-order (dashed line with + markers) and first-order (solid line with × markers) Taylor series approximations of $u(\gamma^*, I_d)$ direct fields. In both cases the errors with the linear approximation are lower. Second row, the quality of the linear and quadratic approximations in predicting the nonlinear change in the boundary data $\delta \zeta$. The solid line denotes $\delta \zeta_i$, the dashed $j'_i \delta \gamma$ and the dotted $j'_i \delta \gamma + \sum_{i=1}^{m} \delta \gamma' K_i \delta \gamma$, over the interval $i = 150, \ldots, 220$. The corresponding data misfit norms are 0.057 for the quadratic approximation $Q(\delta \gamma^*)$ and 0.123 for the linear $A(\delta \gamma^*)$, assuming no additive noise. With the prescribed additive noise these values change to 0.062 and 0.126 respectively.
To reconstruct the admittivity function we implement the proposed iteration (65) using a precision matrix $C^{-1} = R'R$, with $R \in \mathbb{R}^{k \times k}$ is a smoothness enforcing operator. In our numerical experiments we opted to use two different values of the regularization parameter to investigate the performance of the scheme at different levels of regularization. Using $\alpha = 5 \times 10^{-4}$ and $\alpha = 5 \times 10^{-6}$, we run two iterations after which the algorithm converged to an error value just above the noise level. The error reduction is illustrated at the graphs of figure 4 showing a significant reduction in both the misfit error $Q(\delta \gamma_p)$ and the image error $\| \delta \gamma^* - \delta \gamma_p \|$ right after the first iteration. The algorithm was initialized with $\delta \gamma_0 = 0$ hence we can regard $\delta \gamma_1$ as the Tikhonov solution (67) and $\delta \gamma_2$ the quadratic regressor after two iterations (65). To aid convergence a backtracking line search algorithm was used where the optimal Newton steps computed at $\tau_1 = 1$ and $\tau_2 = 0.8$ for both values of $\alpha$. The computational time required to compute $J \in \mathbb{C}^{390 \times 1038}$ was about 0.34 s, each of the $390 \mathbf{K}^i \in \mathbb{C}^{1038 \times 1038}$ matrices took about 4.75 s and the duration of each iteration was about 12 s depending on the line search. The images of the reconstructed admittivity perturbation at each iteration are plotted in figure 5 below the target images for comparison. As the error graphs clearly indicate, the reconstructed images show a profound quantitative improvement in spatial resolution, with the regularized quadratic regression solution $\delta \gamma_2$ to outperform the Tikhonov solution $\delta \gamma_1$.

In solving the nonlinear EIT problem [24], [38], [19] one typically performs a number of GN iterations until convergence is reached in the sense of the discrepancy principle. The above presented results obviously refer to a single (exterior) GN iteration, i.e. $q = 0$, and two interior iterations $p = 1, 2$. Although trivial and certainly beneficial to implement, we choose to present the results from the first nonlinear iteration, claiming that by virtue of a more accurate problem approximation and the convergence properties of the Newton algorithm [4], the quadratic regression solution will sustain a smaller error for any number of GN iterations.

In a case where a number of GN iterations are required, measures should be taken to prevent the admittivity estimates from becoming negative. For real electrical properties, this can be achieved by a logarithmic transformation in the variables [13], which prevents the electrical properties of attaining zero or negative values, without imposing additive constraints to the cost function. Clearly, in the numerical example demonstrated in figure 5 negative electrical parameter values are incurred. This is merely due to the fact that in scaling interior iterations we chose step sizes $\tau_1, \tau_2$ that minimize the value of the objective function, as in the conventional use of linear search algorithms for unconstrained optimization problems [5]. Ultimately, no explicit positivity constraints have been considered in the problem formulation, however this will be the topic of a future inquisition in the context of this framework.

6 Conclusions

This paper proposes a new approach for the inverse impedance tomography problem. Based on a power perturbation approach we derive a nonlinear integral transform relating changes in electrical admittivity to those observed in the respective boundary measurements. This transform was then modified by assuming that the electric potential in the interior of a domain with unknown electrical properties can be approximated by a first-order Taylor expansion centered at an a priori known admittivity. This framework yields a quadratic
regression problem which we then regularized in the usual Tikhonov fashion. Implementing Newton’s iterative algorithm we demonstrate that the method quickly converges to results that outperform those typically computed by applying the Gauss-Newton method on the linearized inverse problem. An extension of this work worth investigating is to consider the regression problem with bound constraints on the admittivity change that enforce positivity on the electrical parameters.

References


Figure 5: Simulated and reconstructed admittivity perturbations. Top row, the target changes in simulated conductivity (left) and permittivity scaled by frequency $\omega$ (right) on $B_f$, and the position of the electrodes. Second row, the respective images from first iteration or linear regression $\delta\gamma_1$, then the second iteration from the quadratic regression on $B_i$, $\delta\gamma_2$. The image errors $\|\delta\gamma^* - \Pi\delta\gamma_p\|$ for $p = 1, 2$, assuming $\delta\gamma_0 = 0$, are respectively 28.06 and 14.20 using $\alpha = 5 \times 10^{-6}$. 

The image errors $\|\delta\gamma^* - \Pi\delta\gamma_p\|$ for $p = 1, 2$, assuming $\delta\gamma_0 = 0$, are respectively 28.06 and 14.20 using $\alpha = 5 \times 10^{-6}$. 

\[ \alpha = 5 \times 10^{-6} \]


