COMMUNICATION

ITERATIVE BEHAVIOUR OF ONE-DIMENSIONAL THRESHOLD AUTOMATA

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This paper presents two results on the dynamical behaviour of some finite binary one-dimensional threshold automata, where the interaction coefficient between two cells \( i, j \) is a decreasing function of \(|i-j|\).

1. Introduction

A finite binary one-dimensional threshold automaton is a 5-tuple \((L, n, \{0, 1\}, b, a)\) where

- \(L \in \mathbb{N}^*\) is the number of cells in the one-dimensional array. Hereafter, the cells are denoted by integers \(i \in \{1, 2, \ldots, L\}\).
- \(n \in \mathbb{N}^*\) is the neighborhood size which means that any cell \(i\) is connected to \(u = \text{Max}\{1, i-n\}, u+1, \ldots, \text{Min}\{L, i+n\}\).
- \(\{0, 1\}\) is the set of states that can be assumed by any cell.
- \(b = (b_1, \ldots, b_L) \in \mathbb{N}^L\) is the threshold vector.
- \(a = (a_{-n}, \ldots, a_0, \ldots, a_n) \in \mathbb{N}^{2n+1}\), where \(a_k, -n \leq k \leq n\) is the interaction coefficient between two cells \(i\) and \(i + k\).

Starting from a configuration \(x^{(0)} = (x_1^{(0)}, \ldots, x_L^{(0)}) \in \{0, 1\}^L\) at time \(t = 0\), where \(x_i^{(0)}\) is the initial state of cell \(i\), the system evolves in a parallel and synchronous manner. The configuration \(x^{(t+1)}\) at time \(t + 1\) is the image of \(x^{(t)}\) under the global transition function \(F\) and it is denoted by \(x^{(t+1)} = F(x^{(t)})\) where

\[
    x_i^{(t+1)} = \begin{cases} 
    1 & \text{if } \sum_{k=-n}^{n} a_k x_{i+k}^{(t)} \geq b_i, \\
    0 & \text{otherwise},
    \end{cases}
\]
We assume the following boundary conditions:

\[ 0 = x_{-n+1}^{(t)} = \cdots = x_0^{(t)} = x_{L+1}^{(t)} = \cdots = x_{L-n}^{(t)} \text{ for any } t \geq 0. \]

2. The Results

Several authors [1, 3, 4, 5, 6, 7, 8] have studied the dynamical behaviour of such structures for particular choices of the interaction coefficients \( a_k, -n \leq k \leq n \). Here we are interested in the context where the interaction coefficient between two cells \( i, j \) is a decreasing function of \( |i-j| \), i.e.

\[ a_0 \geq a_1 \geq a_2 = a_{-2} \geq \cdots \geq a_n = a_{-n} \geq 0. \]

Since \( \{0, 1\}^L \) is finite, any sequence \( \{F^r(x) : r \geq 0\} \) is ultimately periodic of period \( T(x) \geq 1 \), that is to say that \( F^{r+k}(x) \neq F^r(x) \) for \( 0 < k < T(x) \), and there exists \( p(x) \geq 0 \) such that \( F^{r+p(x)}(x) = F^r(x) \) for any \( r \geq p(x) \).

Since the interaction coefficients satisfy the symmetry condition \( a_k = a_{-k} \) for any \( k \), in the automata studied here, we know from [1] that \( T(x) \leq 2 \) for any \( x \in \{0, 1\}^L \). Here we try to discriminate between threshold automata satisfying \( T(x) = 1 \) for any \( x \), and those for which there exists a configuration \( x \) with \( T(x) = 2 \).

Theorem 1. Let \( a_0, a \in \mathbb{N}^* \) and \( a_k = 1 \) for \( 1 \leq |k| \leq n \).

(i) If \( n < \frac{3a - 2}{2} \), then \( \max_x T(x) = 1 \) for any \( L \in \mathbb{N}^* \) and \( b \in \mathbb{N}^L \).

(ii) If \( n \geq 5a - 2 \), then for an appropriate choice of \( L \in \mathbb{N} \) and \( b \in \mathbb{N}^L \), \( \max_x T(x) = 2 \).

Comment. For a fixed \( a_0 \in \mathbb{N}^* \), this theorem gives a localization of a critical neighborhood size \( n \), for which the dynamics of the system can change from a stable to an oscillating behaviour.

Outline of the Proof. (i) Let \( x, y \in \{0, 1\}^L \) be such that \( x \neq y \), \( F(x) = y \) and \( F(y) = x \), and let us denote \( u = x - y \). It can be shown that [8]:

\[ u_i \neq 0 = u_i (au_i + \sum_{k=1}^a (u_{i-k} + u_{i+k})) < 0. \]

Thus, if \( S(i) = \sum_{k=-n}^a u_{i+k} \), then

\[ u_i = 1 \Rightarrow S(i) \leq -a, \]

\[ u_i = -1 \Rightarrow S(i) \geq a. \]

Since, for \( i < j \),

\[ S(j) - S(i) = (S(i+1) - S(i)) + (S(i+2) - S(i+1)) + \cdots + (S(j) - S(j-1)) = (u_{i+1+n} - u_{i-n}) + (u_{i+2+n} - u_{i+1-n}) + \cdots + (u_{j+n} - u_{j-1-n}). \]
it follows from (2) and (3) that:

(4) \( i < j \) and \( u_i \cdot u_j = -1 \Rightarrow j - i \geq a, \)

(5) \( i < j, \ u_i \cdot u_j = -1 \) and \( u_{i-n} - u_{i+1-n} - \cdots - u_{j-1-n} - 0 \Rightarrow j - i \geq 2a. \)

By successive application of properties (1) to (5), it can be shown that \( n \geq \frac{3}{2}a - 2. \) The details can be found in [2].

(ii) For \( a = 2r, \ r > 0, \) and \( n = 5a - 2 = 10r - 2, \) let us take \( L = 37r - 5 \) and let us denote by \( c_{u,v} \) the \( L \)-vector defined by the two parameters \( u, v \) as shown in the following Table 1. Clearly, \( c_{0,1} \neq c_{1,0} \) and \( F(c_{0,1}) - c_{1,0}, F(c_{1,0}) = c_{0,1}. \)

The case \( a = 2r + 1 \) can be treated in a similar way [2].

<table>
<thead>
<tr>
<th>Partial length</th>
<th>6r-1</th>
<th>3r</th>
<th>3r-1</th>
<th>4r</th>
<th>3r-1</th>
<th>4r</th>
<th>3r-1</th>
<th>3r-1</th>
<th>3r-1</th>
<th>4r-1</th>
<th>r</th>
</tr>
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</table>

**Table 1**

<table>
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<tr>
<th>Threshold component</th>
<th>3r-5r</th>
<th>5r</th>
<th>&gt;7r</th>
<th>6r</th>
<th>&gt;7r</th>
<th>6r</th>
<th>&gt;7r</th>
<th>5r</th>
<th>&gt;7r</th>
<th>3r-3r</th>
</tr>
</thead>
</table>

**Theorem 2.** If \( a_k = n - |k| \) for \( k = -n, \ldots, 0, \ldots, n, \) then \( \text{Max}_x T(x) = 1 \) for any \( L \in \mathbb{N} \) and \( b \in \mathbb{N}^L. \)

**Proof.** Let \( C \in M_{L \times L}(R) \) be the matrix defined by

\[
c_{ij} = \begin{cases} a_k & \text{if } -n \leq i - j = k \leq n, \\ 0 & \text{otherwise}. \end{cases}
\]

Let \( x, y \in \{0, 1\}^L \) be such that \( F(x) = y \) and \( F(y) = x, \ x \neq y, \) and let \( u = x - y. \) It follows from Property 1 (see the proof of Theorem 1) that

\[
u_i \neq 0 \Rightarrow u_i \cdot (Cu)_i < 0.
\]

Hence \( u^t C u < 0, \) and this contradicts the fact that \( C \) is positive definite.

**Comment.** Some extensions of Theorems 1,2 to finite two-dimensional structures have been studied in [2].

**References**

