A deluxe FETI-DP algorithm for a hybrid staggered discontinuous Galerkin method for H(curl)-elliptic problems

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SUMMARY

Convergence theories and a deluxe dual and primal finite element tearing and interconnecting algorithm are developed for a hybrid staggered DG finite element approximation of H(curl) elliptic problems in two dimensions. In addition to the advantages of staggered DG methods, the basis functions of the new hybrid staggered DG method are all locally supported in the triangular elements, and a Lagrange multiplier approach is applied to enforce the global connections of these basis functions. The interface problem on the Lagrange multipliers is further reduced to the resulting problem on the subdomain interfaces, and a dual and primal finite element tearing and interconnecting algorithm with an enriched weight factor is then applied to the resulting problem. Our algorithm is shown to give a condition number bound of $C \log \frac{H}{h}$, independent of the two parameters, where $H/h$ is the number of triangles across each subdomain. Numerical results are included to confirm our theoretical bounds. Copyright © 2013 John Wiley & Sons, Ltd.

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1. INTRODUCTION

In this paper, a dual and primal finite element tearing and interconnecting (FETI-DP) algorithm is developed for a fast and stable solution of a new hybrid staggered DG (HSDG) method applied to H(curl) problems. Staggered DG (SDG) methods were first introduced in [1–3] for wave propagation problems. The idea was subsequently applied to other problems, such as convection-diffusion equations [4], electromagnetic problems [5–9], Stokes equations [10], and multiscale wave simulations [11,12]. Similar to [5], the advantages of using SDG for H(curl) problems are the preservation of the structures of the differential operators, the local conservation property, and the optimal convergence. In particular, the discretizations of the two curl operators by the SDG method are adjoint to each other, and the null space of the discrete curl operator is exactly the gradients. Moreover, the divergence condition is automatically satisfied in an appropriate weak sense.

Despite the aforementioned advantages, the implementation of SDG methods requires careful numbering of the supports of the basis functions. To allow an easier implementation of the SDG method, the HSDG method is introduced in this paper. One distinctive advantage of the HSDG method is that the basis functions are all locally supported in the triangular elements, hence, the numbering can be easily performed. The fact that the basis functions are totally discontinuous requires some global couplings. In particular, we will use the Lagrange multiplier approach to enforce the global connection of the basis functions. This idea is also related to the hybridized DG methods [13–16]. In fact, it is shown that the SDG method is the limit of a hybridized DG method.

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However, the new HSDG method considered in this paper is different from any hybridized DG method in the way that no penalty parameter is involved in the scheme.

Fast solvers for SDG methods have been recently developed; see [18] for an overlapping Schwarz type method and [19] for a balancing DD by constraints algorithm. To address a fast solver for the discrete system resulting from the HSDG discretization, we will see that the structure of the system naturally allows the use of a FETI-DP DD algorithm [20, 21], where the original algebraic equation is decoupled into subdomain problems, and the continuity of the solution on the subdomain interfaces is enforced strongly at the selected set of primal unknowns and weakly at the remaining unknowns on the interface by using Lagrange multipliers. The resulting equation on the Lagrange multipliers is then solved iteratively combined with a preconditioner. The FETI-DP algorithm is known to be one of the most scalable DD methods for structural mechanical problems and to be the dual form of BDDC algorithms [22–25].

In our HSDG formulation, for a given triangle, we subdivide it into six triangles to form a refined mesh. Based on this refined mesh, two types of finite element functions are introduced, and they have staggered continuity property across the triangle edges in the refined mesh. The second-order problem is written into a first-order system by introducing a new unknown, \( q = \alpha(x)\nabla \times \mathbf{u} \), and the first-order system is then approximated by an SDG method with two unknowns, \( q_h \) and \( u_h \), using the two types of finite element functions. In the formulation, additional unknowns \( \lambda \) are introduced to approximate the flux condition on the triangle boundaries. After forming discrete problems in the triangles and then eliminating \( q_h \) and \( u_h \), the reduced system on \( \lambda \) is obtained. The unknowns \( \lambda \) can be interpreted as Lagrange multipliers, which enforce tangential continuity of \( u_h \) across the triangle boundaries.

In developing the FETI-DP algorithm, we first form a subdomain partition where each subdomain is a connected union of triangles from the mesh before the aforementioned refinement process. We then reduce the equation on \( \lambda \) into the equation on the subdomain interface unknowns \( \lambda_\Gamma \) by eliminating the unknowns of \( \lambda \) interior to each subdomain. Construction of the coarse component in our FETI-DP algorithm is different from that in the standard FETI-DP algorithm. Here, by observing the relation between Dirichlet and Neumann types of local problems, we define primal unknowns \( \lambda_\Pi \) across the subdomain interface. We then form the coarse component of our FETI-DP algorithm by applying change of basis to the resulting equation on \( \lambda_\Gamma \) and by taking out the block corresponding to the primal unknowns \( \lambda_\Pi \). After eliminating the primal unknowns \( \lambda_\Pi \), the resulting equation on the remaining unknowns \( \lambda_\Lambda \) will be solved iteratively combined with a preconditioner. We emphasize that the way we formulate the FETI-DP algorithm for the discrete system on \( \lambda \), which are unknowns on the triangle boundaries, is new, and this idea can be generalized to other discrete systems arising from hybridized DG methods [14, 16]. We also refer BDDC and FETI-DP algorithms applied to hybrid discretization [26] and DG discretization [27–30].

In addition, to deal with the two coefficients \( \alpha(x) \) and \( \beta(x) \) appearing in our model problem, we introduce an enriched weight factor in the preconditioner following the ideas in [31–33]. Equipped with the new weight factor, we proved a condition number bound of our FETI-DP algorithm to be \( C(1 + \log(H/h))^2 \) with the constant \( C \), independent of coefficients \( \alpha(x) \) and \( \beta(x) \), and any mesh parameters. Here, \( H/h \) denotes the number of triangles in each subdomain. Our theoretical bound is also confirmed by numerical experiments with various choices of \( \alpha(x) \) and \( \beta(x) \).

This paper is organized as follows. In Section 2, the staggered mesh and the HSDG method will be presented. The optimal convergence of our discretization will be proved in Section 3. In Sections 4 and 5, a new FETI-DP algorithm will be constructed, and bounds of its condition numbers will be analyzed. Finally, in Section 6, numerical examples will be presented to confirm the convergence rates and condition number bounds.

## 2. THE HYBRID STAGGERED DISCONTINUOUS GALERKIN METHOD

We consider a model \( H(\text{curl}) \)-elliptic problem in a bounded two-dimensional domain \( \Omega \):

\[
\nabla \times (\alpha(x)\nabla \times \mathbf{u}(x)) + \beta(x)\mathbf{u}(x) = \mathbf{f}(x), \quad \forall x \in \Omega,
\]

(2.1)
where \( u \cdot t_{\partial \Omega} = 0 \) on \( \partial \Omega \) and \( \alpha(x) \geq \alpha_0 > 0 \) and \( \beta(x) \geq \beta_0 > 0 \) with \( \alpha_0 \) and \( \beta_0 \) constants. The solution \( u(x) \) lies in the space \( H_0(\text{curl}; \Omega) \), which is defined as

\[
H_0(\text{curl}; \Omega) = \left\{ v \in \left[ L^2(\Omega) \right]^2 : \nabla \times v \in L^2(\Omega) \text{ and } v \cdot t_{\partial \Omega} = 0 \text{ on } \partial \Omega \right\}.
\]

The two coefficient functions \( \alpha(x) \) and \( \beta(x) \) can be discontinuous, and they will be assumed to be changing moderately inside each subdomain \( \Omega_i \) of a non-overlapping partition of \( \Omega \). Thus, for simplicity, we assume \( \alpha(x)|_{\Omega_i} = \alpha_i \) and \( \beta(x)|_{\Omega_i} = \beta_i \), where \( \alpha_i \) and \( \beta_i \) are positive constants. The subdomain partition will be introduced later in Section 4. In addition, for a domain \( K \), \( t_K \) is the unit tangent on \( \partial K \), and it is related to the unit normal \( n_K = (n_1, n_2) \) by the relation \( t_K = (-n_2, n_1) \).

When the definition of \( K \) is obvious, we write \( t = t_K \) to simplify notations.

To derive with our HSDG method, we introduce an auxiliary variable \( q = \alpha \nabla \times u \) and rewrite the problem (2.1) into the following first-order form:

\[
\begin{align*}
\alpha^{-1} q - \nabla \times u &= 0, & \text{in } \Omega, \\
\nabla \times q + \beta u &= f, & \text{in } \Omega, \\
\quad u \cdot t &= 0, & \text{on } \partial \Omega.
\end{align*}
\]

We remark that, for a two-dimensional vector \( v = (v_1, v_2) \), we have \( \nabla \times v = \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \), and for a scalar-valued function \( q \), we have \( \nabla \times q = \left( \frac{\partial q}{\partial y}, -\frac{\partial q}{\partial x} \right) \). In the following, we will introduce two finite element spaces for the approximations of \( u \) and \( q \). The construction will be based on forming a refined mesh on the given initial triangulation and then enforcing partial continuity on the finite element functions.

Let \( T \) be an initial triangulation of the domain \( \Omega \) and \( F \) be the set of all edges of this initial triangulation. We then divide each triangle \( \tau \) in \( T \) into six triangles by connecting the centroid \( v \) of the triangle to the three vertices and the mid-points of the three edges in the way shown in Figure 1. From this construction, there are six new edges, and we divide these edges into two categories. We use the notation \( F_u \) to denote all edges connecting the centroid and the vertices of a triangle in the initial triangulation. In addition, we use the notation \( F_q \) to denote all edges connecting the centroid and the mid-points of the edges of a triangle in the initial triangulation. Notice that, after the aforementioned subdivision process, we obtain a new and finer triangulation that is denoted by \( T_s \). Also, every edge in \( F \) is divided into two sub-edges, and we still use the same notation \( F \) to denote the resulting set of subdivided edges.

We define the following two discontinuous finite element spaces:

\[
S_h := \left\{ \psi : \psi|_\tau \in P^k(\tau), \quad \forall \tau \in T_s \text{ and } [\psi]|_e = 0, \quad \forall e \in F_q \right\},
\]

Figure 1. The triangulation \( T_s \): solid lines on the boundary are edges in \( F \), solid lines in the interior represent edges in \( F_u \), and dotted lines represent edges in \( F_q \).
and

\[ V_h := \left\{ v : v \in \left[ P^k(\tau) \right]^2, \quad \forall \tau \in T_s \text{ and } [v \cdot t]|_e = 0, \quad \forall e \in F_u \right\}, \]

where \( P^k(\tau) \) is the set of polynomials of degree up to \( k \) in the triangle \( \tau \). In the aforementioned definition, the jump \([\psi]\) across each \( e \) is defined as

\[ [\psi]|_e := \psi_1 t_1 + \psi_2 t_2 \]

where \( \psi_i = \psi|_{\tau_i} \), \( \tau_1 \) and \( \tau_2 \) are the two triangles in \( T_s \) sharing \( e \), and \( t_i \) is the unit tangent vector to the edge \( e \) defined anti-clockwise with respect to \( \tau_i \). Similarly, the jump \([v \cdot t]\) on each \( e \) is defined as

\[ [v \cdot t]|_e := v_1 \cdot t_1 + v_2 \cdot t_2. \]

In particular, the functions in \( S_h \) are polynomials of degree less than or equal to \( k \) in each triangle in \( T_s \), and they are continuous on edges in \( F_q \). The vector fields in \( V_h \) are vector polynomials of degree less than or equal to \( k \) in each triangle in \( T_s \), and they are tangentially continuous on edges in \( F_u \). Notice that, the basis functions in \( S_h \) and \( V_h \) are supported in the triangles of the initial triangulation \( T \). More precisely, they are supported in exactly two triangles in the refined triangulation \( T_s \); see Figures 2 and 3. This localized and staggered properties of the basis functions are the key advantages of our HSDG method.

![Figure 2. The region \( R(e) \) for an edge \( e \) in \( F_q \).](image)

![Figure 3. The region \( S(e) \) for an edge \( e \) in \( F_u \).](image)
We define the following norm for the space $S_h$:

$$\|\psi\|_S^2 := \int_\Omega \alpha^{-1} \psi^2 \, dx. \tag{2.3}$$

In the space $V_h$, we define two norms, the discrete $H(\text{curl}; \Omega)$-seminorm and the discrete $H(\text{curl}; \Omega)$-norm, by

$$|v|_{V}^2 := \int_\Omega \alpha(\nabla \times v)^2 \, dx + \sum_{e \in \mathcal{F}_u} h_e^{-1} \int_e \alpha [v \cdot t]^2 \, d\sigma, \tag{2.4}$$

$$\|v\|_{V}^2 := \int_\Omega \alpha |v|^2 \, dx + \|v\|_V^2, \tag{2.5}$$

where $h_e$ is the length of $e$, and the integral of $(\nabla \times v)^2$ in (2.4) is defined elementwise. We also recall that, by definition, $v \in V_h$ has a continuous tangential component across each edge $e \in \mathcal{F}_u$. Furthermore, we assume that the coefficient functions $\alpha(x)$ and $\beta(x)$ are positive constants in each triangle of the initial triangulation $T$.

For $e \in \mathcal{F}_u$, let $R(e)$ be the union of two triangles in $T_s$ sharing $e$, and for $e \in \mathcal{F}_u$, we let $S(e)$ be the union of two triangles in $T_s$ sharing $e$ (Figures 2 and 3). By the definitions of $S_h$ and $V_h$, the space $S_h$ is $H^1$-conforming in each $R(e)$, and the space $V_h$ is $H(\text{curl})$-conforming in each $S(e)$. We emphasize that the two spaces $S_h$ and $V_h$ are locally conforming on staggered meshes and that this property will provide a flux condition in our HSDG formulation without introducing any stabilization tensors nor artificial flux terms.

Recall that the functions in the spaces $S_h$ and $V_h$ can be discontinuous across edges in $\mathcal{F}$ of the initial triangulation, that is, the six edges on the boundary in Figure 1. To enforce the continuity, we introduce the following Lagrange multiplier space:

$$\Lambda_h := \left\{ \mu : \mu \in P^k(e), \quad \forall e \in \mathcal{F} \right\}.$$ 

The norm in this space is defined as

$$\|\mu\|_{\Lambda}^2 := \sum_{e \in \mathcal{F}} \int_e \mu^2 \, d\sigma.$$

Next, we will derive the HSDG method. We multiply the first equation in (2.2) by a test function $\psi \in S_h$ and integrate over $R(e)$, where $\psi$ is $H^1$-conforming,

$$\int_{R(e)} \alpha^{-1} q \psi \, dx - \int_{R(e)} (\nabla \times u) \psi \, dx = 0.$$ 

Integrating by parts,

$$\int_{R(e)} \alpha^{-1} q \psi \, dx - \int_{R(e)} u \cdot (\nabla \times \psi) \, dx - \int_{\partial R(e)} (u \cdot t_\kappa) \psi \, d\sigma = 0.$$ 

Here, $t_\kappa$ denotes the anti-clockwise unit tangent to an edge $\kappa$ in $\partial R(e)$, where $\tau$ is a subtriangle in $R(e)$ containing the edge $\kappa$. Summing over all $R(e)$, we obtain

$$\int_\Omega \alpha^{-1} q \psi \, dx - b^*(u, \psi) = 0, \quad \forall \psi \in S_h$$

where

$$b^*(u, \psi) := \int_\Omega u \cdot (\nabla \times \psi) \, dx + \sum_{e \in \mathcal{F}_u} \int_e u \cdot [\psi] \, d\sigma + \sum_{e \in \mathcal{F}_u} \int_{\partial R(e) \cap \mathcal{F}} (u \cdot t_\kappa) \psi \, d\sigma.$$
We multiply the second equation in (2.2) by a test function \( v \in \mathcal{V}_h \), integrate over \( S(e) \), and apply integration by parts to obtain
\[
\int_{\partial S(e)} q(\nabla \times v) \, d\sigma - \int_{S(e)} q(v \cdot t) \, d\sigma + \int_{S(e)} \beta u \cdot v \, dx = \int_{S(e)} f \cdot v \, dx, \quad \forall v \in \mathcal{V}_h.
\]
On each \( \kappa \in \partial S(e) \cap \mathcal{F} \), we approximate \( q \) by introducing a unknown \( \lambda \in \Lambda_h \). Summing the aforementioned over all \( S(e) \), we obtain
\[
b(q, v) - \sum_{\kappa \in \mathcal{F}} \int_{\kappa} \lambda [v \cdot t] \, d\sigma + \int_{\Omega} \beta u \cdot v \, dx = \int_{\Omega} f \cdot v \, dx
\]
where
\[
b(q, v) := \int_{\Omega} q(\nabla \times v) \, dx - \sum_{e \in \mathcal{F}_q} \int_{e} q[v \cdot t] \, d\sigma.
\]
In addition, we enforce the continuity of \( u \cdot t \) over \( e \in \mathcal{F} \):
\[
\sum_{e \in \mathcal{F}} \int_{e} [u \cdot t] \mu \, d\sigma = 0, \quad \forall \mu \in \Lambda_h.
\]
We note that the aforementioned integral condition can be replaced by the pointwise condition at each DOF on \( e \).

We define
\[
J(u, \mu) := \sum_{e \in \mathcal{F}} \int_{e} [u \cdot t] \mu \, d\sigma.
\]

Then, we state the HSDG method: Find \( u_h \in \mathcal{V}_h, q_h \in \mathcal{S}_h \) and \( \lambda_h \in \Lambda_h \) such that
\[
\int_{\Omega} \sigma^{-1} q_h \psi \, dx - b^*(u_h, \psi) = 0, \quad \psi \in \mathcal{S}_h, \quad \tag{2.6}
\]
\[
b(q_h, v) + \int_{\Omega} \beta u_h \cdot v \, dx - J(v, \lambda_h) = \int_{\Omega} f \cdot v \, dx, \quad v \in \mathcal{V}_h, \quad \tag{2.7}
\]
\[
J(u_h, \mu) = 0, \quad \mu \in \Lambda_h. \quad \tag{2.8}
\]

Notice that, using integration by parts, it is easy to see that
\[
b^*(u, q) = b(q, u), \quad \forall u \in \mathcal{V}_h, q \in \mathcal{S}_h.
\]

Moreover, we have the following continuity condition:
\[
b(q, u) \leq L \| u \|_V \| q \|_S, \quad \forall u \in \mathcal{V}_h, q \in \mathcal{S}_h
\]
where \( L > 0 \) is independent of the mesh size.

Let \( \mathcal{U}_h \) be the standard conforming finite element space with piecewise polynomial of degrees up to \( k + 1 \) with respect to the triangulation \( \mathcal{T}_g \). Let \( \phi \in \mathcal{U}_h \). Then, we have \( \nabla \phi \in \mathcal{V}_h \) with the continuity condition that \( [(\nabla \phi) \cdot t]_e = 0 \) for all \( e \in \mathcal{F} \cap \mathcal{F}_u \cap \mathcal{F}_q \). Thus, from (2.7), we have the following weak divergence condition:
\[
\int_{\Omega} \beta u_h \cdot (\nabla \phi) \, dx = \int_{\Omega} f \cdot (\nabla \phi) \, dx. \quad \tag{2.9}
\]

When \( f \) is in \( H(\text{div}; \Omega) \) and is divergence free, (2.9) shows that the numerical solution \( u_h \) is also divergence free in a weak sense.
To form the algebraic equation of our HSDG method, we define several notations. In the following, \( u, q, \) and \( \lambda \) denote the vectors of DOF of \( u_h, q_h, \) and \( \lambda_h, \) respectively. We then have the following algebraic system:

\[
\begin{align*}
M_{\alpha^{-1}} q - B^T u &= 0, \\
B q + N_{\beta} u - J^T \lambda &= f, \\
J u &= 0,
\end{align*}
\]  

(2.10)

where \( J \) denotes a signed Boolean matrix that enforces pointwise continuity condition on \( u \) across each triangle edges in \( \mathcal{F} \), and \( f \) is a vector defining the right-hand side of (2.7). Eliminating \( q \), we obtain

\[
BM_{\alpha^{-1}} B^T u + N_{\beta} u - J^T \lambda = f, \\
J u &= 0.
\]

Notice that one can solve the aforementioned equations for \( u \) and \( \lambda \). To develop our FETI-DP solvers, we further eliminate \( u \) to obtain the equation on \( \lambda \):

\[
J \left( BM_{\alpha^{-1}} B^T + N_{\beta} \right)^{-1} J^T \lambda = -J \left( BM_{\alpha^{-1}} B^T + N_{\beta} \right)^{-1} f.
\]

We remark that our FETI-DP algorithm will be developed for fast solutions of the aforementioned equation on \( \lambda \), which is similar to the resulting matrix equation arising in hybridized DG methods [14].

### 3. CONVERGENCE THEORY

In this section, we prove the convergence of the HSDG method defined in (2.6)–(2.8). First, we note that (2.8) implies that the tangential component of \( u_h \) is continuous on edges in \( \mathcal{F} \). We define \( \mathcal{V}_h^0 \subset \mathcal{V}_h \) by

\[
\mathcal{V}_h^0 := \{ v : v \in \mathcal{V}_h \text{ and } [ v \cdot t ]_e = 0, \quad \forall e \in \mathcal{F} \},
\]

which consists of functions in \( \mathcal{V}_h \) with the additional continuity on all edges in \( \mathcal{F} \). Then, it is clear that (2.6)–(2.8) is equivalent to finding \( (u_h, q_h) \in \mathcal{V}_h^0 \times S_h \) such that

\[
\int_{\Omega} \alpha^{-1} q_h \psi \, dx - b^*(u_h, \psi) = 0, \quad \psi \in S_h,
\]  

(3.11)

\[
b(q_h, v) + \int_{\Omega} \beta u_h \cdot v \, dx = \int_{\Omega} f \cdot v \, dx, \quad v \in \mathcal{V}_h^0.
\]  

(3.12)

To prove the convergence of (3.11)–(3.12), we will need the following two lemmas. The first lemma gives interpolation errors, and the second lemma gives the inf-sup condition.

**Lemma 3.1**

There exist interpolation operators \( \Pi_S : H^{1/2}(\Omega) \rightarrow S_h \) and \( \Pi_V : H_0(\text{curl}; \Omega) \rightarrow \mathcal{V}_h^0 \) such that

\[
b(q - \Pi_S q, v) = 0, \quad \forall v \in \mathcal{V}_h^0 \text{ and } b^*(v - \Pi_V v, q) = 0, \quad \forall q \in S_h.
\]  

(3.13)

Moreover, the following estimates hold for \( v \in [H^{\sigma+1}(\Omega)]^2 \) and \( q \in H^\sigma(\Omega) \) with \( \sigma \geq 1/2, \)

\[
\| q - \Pi_S q \|_{L^2(\Omega)} \leq C h^{\min\{k+1, \sigma\}} |q|_{H^\sigma(\Omega)},
\]

\[
\| v - \Pi_V v \|_{L^2(\Omega)} \leq C h^{\min\{k+1, \sigma+1\}} |v|_{[H^{\sigma+1}(\Omega)]^2},
\]

\[
\| \nabla \times (v - \Pi_V v) \|_{L^2(\Omega)} \leq C h^{\min\{k, \sigma\}} |v|_{[H^{\sigma+1}(\Omega)]^2}.
\]  

(3.14)
Proof
Let \( \tau \in T_s \). We can then define \( \Pi_S q \) by the following DOF:

\[
\int_e (q - \Pi_S q) \ p_k \ d\sigma = 0, \quad e \in F_q \cap \partial \tau, \quad \forall \ p_k \in P^k(e), \\
\int_\tau (q - \Pi_S q) \ p_{k-1} \ dx = 0, \quad \forall \ p_{k-1} \in P^{k-1}(\tau).
\]

Notice that there is exactly one edge on \( \partial \tau \) that belongs to \( F_q \). Thus, the aforementioned conditions define a unique \( \Pi_S q \in S_h \), and by the definition of \( b \), one can easily check that \( b(q - \Pi_S q, v) = 0 \) for all \( v \in V_h^0 \). On the other hand, we can define \( \Pi_V v \) by the following DOF:

\[
\int_e ((v - \Pi_V v) \cdot t) \ p_k \ d\sigma = 0, \quad e \in (F_u \cup F) \cap \partial \tau, \quad \forall \ p_k \in P^k(e), \\
\int_\tau (v - \Pi_V v) \ p_{k-1} \ dx = 0, \quad \forall \ p_{k-1} \in P^{k-1}(\tau)^2.
\]

Notice that for \( \tau \in T_s \), there is exactly one edge on \( \partial \tau \) that belongs to \( F_u \) and exactly one edge on \( \partial \tau \) that belongs to \( F \). Thus, the aforementioned conditions define a unique \( \Pi_V v \in V_h^0 \). One can easily check that by the definition of \( b^* \), we have \( b^*(v - \Pi_V v, q) = 0 \) for all \( q \in S_h \). Finally, the estimates in (3.14) can be proved by the fact that the operators \( \Pi_S \) and \( \Pi_V \) preserve polynomials of degree \( k \) as well as the standard finite element inverse inequality.

Lemma 3.2
For any \( v \in V_h^0 \), there exists \( \psi \in S_h \) such that

\[
\gamma |v|_V \leq \sup_{\psi \in S_h} \frac{b(\psi, v)}{\|\psi\|_S}
\]  

(3.15)

where \( \gamma \) is a positive constant independent of the mesh size.

Proof
Let \( v \in V_h^0 \). We then define \( \psi_1 \in S_h \) by

\[
\psi_1 = \frac{1}{h_e} [v \cdot t], \quad e \in F_q, \\
\int_\tau \psi_1 \ p_{k-1} \ dx = 0, \quad \forall \ p_{k-1} \in P^{k-1}(\tau), \quad \forall \ \tau \in T_s.
\]

Moreover, we define \( \psi_2 \in S_h \) by \( \psi_2|_\tau = p_\tau(\nabla \times v) \) where \( \tau \in T_s \) and \( p_\tau \) is a linear function in \( \tau \) with zero value on \( e \) in \( F_q \) and \( |p_\tau| \leq 1 \) on \( \tau \). Notice that \( p_\tau \) is well-defined because there is exactly one edge on \( \partial \tau \) that belongs to \( F_q \). We take \( \psi = \alpha(\psi_1 + \psi_2) \). Then, we have

\[
b(\psi, v) = b(\alpha \psi_1, v) + b(\alpha \psi_2, v) = \sum_{e \in F_q} \frac{1}{h_e} \int_e \alpha [v \cdot t]^2 \ d\sigma + \sum_{\tau \in T_s} \int_\tau p_\tau \alpha(\nabla \times v)^2 \ dx
\]

\[
\geq \sum_{e \in F_q} \frac{1}{h_e} \int_e \alpha [v \cdot t]^2 \ d\sigma + C \sum_{\tau \in T_s} \int_\tau \alpha(\nabla \times v)^2 \ dx \geq C |v|_V^2.
\]

It is clear that \( \|\psi_2\|_S \leq |v|_V \). Moreover, by a finite dimensional argument, we have \( \|\psi_1\|_{L^2(\tau)} \leq C h_e \int_e (\psi_1)^2 \ d\sigma \), for all \( \tau \in T_s \) and thus \( \|\alpha \psi_1\|_S \leq C |v|_V \). This completes the proof of (3.15).

Now, we state and prove the main theorem:
Theorem 3.3
Let \( (u, q) \in [H^\sigma + 1(\Omega)]^2 \times H^\sigma(\Omega) \) be the solution of (2.2) with \( \sigma \geq 1/2 \) and let \( (u_h, q_h) \) be the solution of (2.6)–(2.8). Then, we have

\[
\| \Pi_V u - u_h \|_V^2 \leq C \left( \int_\Omega \alpha^{-1}(\Pi_S q - q)^2 \, dx + \int_\Omega \beta \| \Pi_V u - u \|^2 \, dx \right). \tag{3.16}
\]

Moreover, we have

\[
\int_\Omega \beta \| u - u_h \|^2 \, dx + h^2 \| u - u_h \|_V^2 \leq C h^{\min(2k + 2, 2\sigma)} \left( \beta^* \| u \|_{[H^\sigma + 1(\Omega)]^2}^2 + \alpha^* \| q \|_{H^\sigma(\Omega)}^2 \right). \tag{3.17}
\]

where \( \beta^* = \max(\beta_i) \) and \( \alpha^* = \max(\alpha_i) \).

Proof
We assume that the solution \( (u, q) \) of (2.2) satisfies

\[
\int_\Omega \alpha^{-1}(q - q_h)^2 \psi \, dx - b^*(u - u_h, \psi) = 0, \quad \psi \in S_h, \tag{3.18}
\]

\[
b(q, v) + \int_\Omega \beta u \cdot v \, dx = \int_\Omega f \cdot v \, dx, \quad v \in \gamma^0_h. \tag{3.19}
\]

Subtracting (3.11) from (3.18) and (3.12) from (3.19), we obtain

\[
\int_\Omega \alpha^{-1}(q - q_h)^2 \psi \, dx - b^*(u - u_h, \psi) = 0, \quad \psi \in S_h,
\]

\[
b(q - q_h, v) + \int_\Omega \beta(u - u_h) \cdot v \, dx = 0, \quad v \in \gamma^0_h.
\]

Using the properties of \( \Pi_S \) and \( \Pi_V \), we have

\[
\int_\Omega \alpha^{-1}(q - q_h)^2 \psi \, dx - b^*(\Pi_V u - u_h, \psi) = 0, \quad \psi \in S_h, \tag{3.20}
\]

\[
b(\Pi_S q - q_h, v) + \int_\Omega \beta(u - u_h) \cdot v \, dx = 0, \quad v \in \gamma^0_h. \tag{3.21}
\]

Taking \( \psi = \Pi_S q - q_h \) in (3.20) and \( v = \Pi_V u - u_h \) in (3.21) and adding the resulting equations, we obtain

\[
\int_\Omega \alpha^{-1}(\Pi_S q - q_h)^2 \, dx + \int_\Omega \beta \| \Pi_V u - u_h \|^2 \, dx \leq \int_\Omega \alpha^{-1}(q - q_h)^2 \, dx + \int_\Omega \beta \| \Pi_V u - u \|^2 \, dx. \tag{3.22}
\]

On the other hand, by the inf-sup condition (3.15),

\[
\gamma \| \Pi_V u - u_h \|_V \leq \sup_{\psi \in S_h} \frac{b(\psi, \Pi_V u - u_h)}{\| \psi \|_S} = \sup_{\psi \in S_h} \frac{b(\psi, u - u_h)}{\| \psi \|_S} = \sup_{\psi \in S_h} \int_\Omega \alpha^{-1}(q - q_h)^2 \psi \, dx.
\]

Thus, we have \( \gamma \| \Pi_V u - u_h \|_V \leq \| q - q_h \|_S \). Hence, using (3.22) and the triangle inequality \( \| q - q_h \|_S \leq \| q - \Pi_S q \|_S + \| \Pi_S q - q_h \|_S \), we obtain (3.16). Furthermore, the estimate (3.17) can be proved by using the triangle inequality and the interpolation error bounds (3.14) as well as the trace inequality for the jump terms. \( \square \)
4. A DUAL AND PRIMAL FINITE ELEMENT TEARING AND INTERCONNECTING ALGORITHM

We recall the resulting linear system of our HSDG method:

\[ J \left( BM_{a-1}^{-1} B^T + N_\beta \right)^{-1} J^T \lambda = -J \left( BM_{a-1}^{-1} B^T + N_\beta \right)^{-1} f, \]  

(4.23)

where \( \lambda \) is defined on \( \mathcal{F} \), the set of edges of triangles in \( \mathcal{T}_s \), which belong to the initial edges before the subdivision process. We will develop a FETI-DP algorithm for a fast solution of the aforementioned linear system.

We assume that the two coefficients \( \alpha(x) \) and \( \beta(x) \) are piecewise constant on the initial triangulation \( \mathcal{T} \). We then introduce a subdomain partition \( \{ \Omega_i \} \), where each \( \Omega_i \) is a connected union of triangles in \( \mathcal{T} \) such that \( \alpha(x) \) and \( \beta(x) \) change moderately inside each \( \Omega_i \). We therefore assume that \( \alpha(x) = \alpha_i \) and \( \beta(x) = \beta_i \) in each \( \Omega_i \) with \( \alpha_i \) and \( \beta_i \) positive constants. We introduce \( \Gamma \) to denote the set of subdomain interfaces,

\[ \Gamma = \bigcup_{ij} \Gamma_{ij}, \]

where \( \Gamma_{ij} = \partial \Omega_i \cap \partial \Omega_j \).

To proceed with our FETI-DP algorithm, we will form two matrices for each subdomain \( \Omega_i \) by assembling Dirichlet and Neumann problems in each initial triangle that belongs to \( \Omega_i \). The two matrices will then be used as building blocks of our algorithm. For each initial triangle \( \tau \in \mathcal{T} \cap \Omega_i \), we introduce \( \partial \tau \) to denote the boundary of \( \tau \). We note that each \( \tau \) in \( \mathcal{T} \) consists of six subtriangles as in Figure 1 and thus \( \partial \tau \) contains six edges in \( \mathcal{F} \). We define local bilinear forms in \( \tau \),

\[ b^*_\tau(u, \psi) := \int_\tau u \cdot \nabla \psi \, dx + \sum_{e \in \mathcal{F}_\mathcal{T} \cap \tau} \int_e u \cdot [\psi] \, d\sigma + \int_{\partial \tau} u \cdot t \, \tau \psi \, d\sigma, \]

and

\[ b_\tau(q, v) := \int_\tau q \nabla \cdot v \, dx - \sum_{e \in \mathcal{F}_\mathcal{T} \cap \tau} \int_e q [v \cdot t] \, d\sigma. \]

Note that \( b^*_\tau(u, \psi) \) and \( b_\tau(q, v) \) are restrictions of \( b^* (u, \psi) \) and \( b(q, v) \) to the triangle \( \tau \), respectively. Thus, the adjoint property also holds

\[ b^*_\tau(v, \psi) = b_\tau(\psi, v). \]

For each \( \tau \) in \( \mathcal{T} \), the initial triangulation, we introduce

\[ S_\mathcal{H}(\tau) := \{ \psi \rvert_\tau : \psi \in S_\mathcal{H} \}, \]

\[ V_\mathcal{H}(\tau) := \{ v \rvert_\tau : v \in V_\mathcal{H} \}, \]

\[ V^D_\mathcal{H}(\tau) := \{ v \in V_\mathcal{H}(\tau) : v \cdot t = 0 \text{ on } \partial \tau \}. \]

We then consider the following two problems in each \( \tau \):

**Dirichlet Problem** For a given \( u_\tau^D \cdot t = R_\tau u \mathcal{F} \) on \( \partial \tau \), find \( (q^D_\tau, u^D_\tau) \in S_\mathcal{H}(\tau) \times V_\mathcal{H}(\tau) \) such that

\[
\frac{1}{\alpha_i} (q^D_\tau, \psi)_\tau - b^*_\tau (u^D_\tau, \psi) = 0 \quad \forall \psi \in S_\mathcal{H}(\tau),
\]

\[
b_\tau(q^D_\tau, v) + \beta_i (u^D_\tau, v)_\tau = 0 \quad \forall v \in V^D_\mathcal{H}(\tau).
\]
**Neumann Problem** For a given $\lambda_{\partial \tau} = R_{\tau} \lambda_{\partial \tau}$ on $\partial \tau$, find $(q_{\tau}^N, u_{\tau}^N) \in S_h(\tau) \times V_h(\tau)$ such that

$$
\frac{1}{\alpha_i} (q_{\tau}^N, \psi)_\tau - b_{\tau}^i (u_{\tau}^N, \psi) = 0 \quad \forall \psi \in S_h(\tau),
$$

$$
b_{\tau} (q_{\tau}^N, \mathbf{v}) + \beta_i (u_{\tau}^N, \mathbf{v})_\tau - \int_{\partial \tau} f(\lambda_{\partial \tau}) \mathbf{v} \cdot t_\tau \, d\sigma = 0 \quad \forall \mathbf{v} \in V_h(\tau).
$$

Here, $R_{\tau} u_{\tau}$ and $R_{\tau} \lambda_{\partial \tau}$ denote the restrictions of $u_{\tau}$ and $\lambda_{\partial \tau}$ to $\partial \tau$, respectively. We note that in the second equation of Neumann Problem, $f(\lambda_{\partial \tau})$ satisfies

$$
\int_{\partial \tau} f(\lambda_{\partial \tau}) \mathbf{v} \cdot t_\tau \, d\sigma = \langle \lambda_{\partial \tau}, R_{\tau} (\mathbf{v} \cdot t_\tau) \rangle,
$$

where $(\lambda, \mathbf{v})$ is the standard vector inner product. The $f(\lambda_{\partial \tau})$ is just a symbolic notation and thus never appears in the implementation. We note that $t_\tau$ denotes the anti-clockwise direction of a tangent vector to the given triangle $\tau$. We introduce the notation $t$ to denote a fixed direction of a tangent vector to the given triangle edge. Using the notation $t$, we obtain the identity

$$
\langle \lambda_{\partial \tau}, R_{\tau} (\mathbf{v} \cdot t) \rangle = \langle \lambda_{\partial \tau}, J_{\tau} (\mathbf{v} \cdot t) \rangle,
$$

where $J_{\tau}$ is the signed Boolean matrix, which implements the pointwise continuity of $J(u, \mu) = 0$, in other words, $J_{\tau}$ is the restriction of $J$ in (2.10) to $\tau$.

We introduce local matrices $M_{\tau}, B_{\tau},$ and $N_{\tau}$ corresponding to $(q, \psi)_\tau, b_{\tau}(\psi, \mathbf{v})$, and $(u, \mathbf{v})_\tau$, respectively. Let

$$
A_{\tau} = \alpha_i B_{\tau} M_{\tau}^{-1} B_{\tau}^T + \beta_i N_{\tau}.
$$

This matrix is further ordered into

$$
A_{\tau} = \begin{pmatrix} A_{\tau,II} & A_{\tau,IB} \\ A_{\tau,BI} & A_{\tau,BB} \end{pmatrix},
$$

where $I$ and $B$ denote blocks corresponding to unknowns interior to $\tau$ and unknowns on the $\partial \tau$, respectively. By using the aforementioned block matrices, we obtain the Schur complement matrix to the Dirichlet problem,

$$
D_{\partial \tau} = A_{\tau,BB} - A_{\tau,BI} A_{\tau,II}^{-1} A_{\tau,IB}
$$

and a Neumann-to-Dirichlet matrix

$$
N_{\partial \tau} = D_{\partial \tau}^{-1}
$$

which is related to the Neumann problem and gives the tangential component of $u_{\tau}^N$ on $\partial \tau$ for the given Neumann data $\lambda_{\partial \tau}$, that is,

$$
R_{\tau} (u_{\tau}^N, t_\tau) = N_{\partial \tau} (J_{\tau}^T \lambda_{\partial \tau}).
$$

We introduce

$$
E_{\tau}(\mathbf{v}) := \alpha_i \int_{\tau} |\nabla \times \mathbf{v}|^2 \, dx + \alpha_i \sum_{e \in F_\tau \cap \tau} h_e^{-1} \int_{e} [\mathbf{v} \cdot t]^2 \, ds + \beta_i \int_{\tau} |\mathbf{v}|^2 \, dx
$$

and we observe that

$$
\langle D_{\partial \tau} u_{\partial \tau}, u_{\partial \tau} \rangle = \min_{\mathbf{v} \in V_h(\tau), \mathbf{v} \cdot t = u_{\partial \tau} \text{ on } \partial \tau} E_{\tau}(\mathbf{v})
$$

and

$$
\langle N_{\partial \tau} \lambda_{\partial \tau}, \lambda_{\partial \tau} \rangle = \max_{\mathbf{v} \in V_h(\tau), \mathbf{v} \cdot t = u_{\partial \tau} \text{ on } \partial \tau} \frac{\langle u_{\partial \tau}, \lambda_{\partial \tau} \rangle}{E_{\tau}(\mathbf{v})}.
$$
Using these two matrices, we will assemble local problem matrices of our FETI-DP algorithm. The theories and algorithm that we will develop mostly depend on the aforementioned two identities.

Let $u_{f,i}$ and $\lambda_{f,i}$ be restrictions of $u_f$ and $\lambda_f$ to each subdomain $\Omega_i$. By assembling $N_{\Omega_i}$ and $D_{\Omega_i}$ for all $\tau$ in $\Omega_i \cap T$, we introduce

$$N_i = \sum_{\tau \in \Omega_i \cap T} J_{\tau,i} N_{\Omega_i} J_{\tau,i}^T$$

and

$$D_i = \sum_{\tau \in \Omega_i \cap T} R_{\tau,i} D_{\Omega_i} R_{\tau,i},$$

where $J_{\tau,i}^T$ is the signed restriction of $\lambda_{f,i}$ on each $\tau$, and $R_{\tau,i}$ is the restriction of $u_{f,i}$ on each $\tau$. These two matrices $N_i$ and $D_i$ will be used as building blocks in our FETI-DP algorithm. For the unknown $u_{f,i}$, we order it into

$$u_{f,i} = \begin{pmatrix} u_{f,i}^{(i)} \\ u_{\Gamma,i}^{(i)} \end{pmatrix},$$

where $I$ denotes the unknowns of the interior edges, and $\Gamma$ denotes the unknowns of the edges on the subdomain interface, $\Gamma_i = \partial \Omega_i \setminus \partial \Omega$. According to the ordering, we obtain the matrices $D_i$ and $N_i$ as

$$D_i = \begin{pmatrix} D_{f,i}^{(ii)} & D_{f,i}^{(if)} \\ D_{f,i}^{(fi)} & D_{f,i}^{(ff)} \end{pmatrix}, \quad N_i = \begin{pmatrix} N_{f,i}^{(ii)} & N_{f,i}^{(if)} \\ N_{f,i}^{(fi)} & N_{f,i}^{(ff)} \end{pmatrix}.$$

After the ordering and then eliminating the interior unknowns, we calculate Schur complements of $D_i$ and $N_i$ and denote them by $S^{D,i}$ and $S^{N,i}$:

$$S^{D,i} = D_{f,i}^{(ff)} - D_{f,i}^{(if)} \left( D_{f,i}^{(ff)} \right)^{-1} D_{f,i}^{(fi)},$$

and

$$S^{N,i} = N_{f,i}^{(ff)} - N_{f,i}^{(if)} \left( N_{f,i}^{(ff)} \right)^{-1} N_{f,i}^{(fi)}.$$

We note that after eliminating unknowns in $\lambda$ interior to each subdomain, the equation in (4.23) can be written as

$$F_\Gamma \lambda_\Gamma = d_\Gamma$$

(4.26)

and the matrix $F_\Gamma$ can be obtained by assembling $S^{N,i}$,

$$F_\Gamma = \sum_i J_{\Gamma,i} S^{N,i} J_{\Gamma,i}^T,$$

where $J_{\Gamma,i}^T$ is the signed restriction of $\lambda_\Gamma$ on $\Gamma_i$. The right-hand side $d_\Gamma$ can be assembled in a similar manner. Our FETI-DP algorithm will be developed for solving (4.26). We introduce

$$E_i(\mathbf{v}) := \sum_{\tau \in \Omega_i \cap T} E_\tau(\mathbf{v}) = \alpha_i \int_{\Omega_i} |\nabla \times \mathbf{v}|^2 \, dx + \alpha_i \sum_{e \in \mathcal{E}_i \cap \Omega_i} h_e^{-1} \int_e |\mathbf{v} \cdot \mathbf{t}|^2 \, ds + \beta_i \int_{\Omega_i} |\mathbf{v}|^2 \, dx$$

(4.27)

and

$$\mathcal{V}_h(\Omega_i, \mathcal{F}) := \{ \mathbf{v} \mid \Omega_i : \mathbf{v} \in \mathcal{V}_h(\Omega) \text{ and } [\mathbf{v} \cdot \mathbf{t}]_e = 0 \quad \forall e \in \mathcal{F} \cap \Omega_i \}. \quad (4.28)$$
By the construction of $S^{D,i}$ and $S^{N,i}$ combined with (4.24) and (4.25), we observe that
\[
\langle S^{D,i} u_{\Gamma_i}, u_{\Gamma_i} \rangle = \min_{\nabla \in V_\partial(\Omega_i, \mathcal{F}), \nabla \cdot u = u_{\Gamma_i} \text{ on } \Gamma_i} E_i(\nabla),
\]
and
\[
\langle S^{N,i} \lambda_{\Gamma_i}, \lambda_{\Gamma_i} \rangle = \max_{\nabla \in V_\partial(\Omega_i, \mathcal{F}), \nabla \cdot u = u_{\Gamma_i} \text{ on } \Gamma_i} \frac{\langle u_{\Gamma_i}, \lambda_{\Gamma_i} \rangle}{E_i(\nabla)}
\]
and we thus obtain that
\[
S^{N,i} = (S^{D,i})^{-1}
\]
and
\[
F_\Gamma = \sum_i J_{\Gamma_i} S^{N,i} J_{\Gamma_i}^T = \sum_i J_{\Gamma_i} (S^{D,i})^{-1} J_{\Gamma_i}^T.
\] (4.29)

The aforementioned property is a consequence of (4.24), (4.25), and the elimination of unknowns interior to $\Omega_i$ from the assembled matrices $D_i$ and $N_i$. The identities in (4.29) will be mainly used in the design and analysis of the FETI-DP algorithm.

For the construction of the coarse component in our FETI-DP algorithm, we transform the unknown $u_\Gamma$ on each $\Gamma_{ij}$ such that
\[
u_{\Gamma_i}^{j} = T_{ij} \left( \begin{array}{c} u_{\Gamma_i}^{j}, \Pi \\ u_{\Gamma_i}^{j}, A \end{array} \right),
\]
where $u_{\Gamma_i}^{j}, A$ is average-free on $\Gamma_{ij}$, and $u_{\Gamma_i}^{j}, \Pi$ is identical to the average of $u_\Gamma$ over $\Gamma_{ij}$. Related to the change of basis on $u_\Gamma|_{\Gamma_{ij}}$, we define the following change of basis on the Lagrange multipliers $\lambda_\Gamma|_{\Gamma_{ij}}$:
\[
\lambda_\Gamma|_{\Gamma_{ij}} = (T_{ij})^{-1} \left( \begin{array}{c} \lambda_{\Gamma_i}^{j}, \Pi \\ \lambda_{\Gamma_i}^{j}, A \end{array} \right),
\]
where $\lambda_{\Gamma_i}^{j}, \Pi$ is the primal unknown on the interface $\Gamma_{ij}$.

We here provide more details about the change of basis on $\lambda_\Gamma|_{\Gamma_{ij}}$. Let $T_\Gamma = \text{diag}(T_{ij})$ and $T_i = \text{diag}_{j \in n(i)}(T_{ij})$, where $n(i)$ is the set of neighboring subdomain indices, which intersect $\Omega_i$ on an edge of $\Omega_i$. We observe that
\[
T_\Gamma^{-1} F_\Gamma (T_\Gamma^{-1})^{-1} = \sum_i T_i^{-1} J_{\Gamma_i} S^{N,i} J_{\Gamma_i}^T (T_i^{-1})^{-1} = \sum_i J_{\Gamma_i} T_i^{-1} S^{N,i} (T_i^{-1})^{-1} J_{\Gamma_i}.
\]
We note that in the previous text, the commuting property $T_\Gamma^{-1} J_{\Gamma_i} = J_{\Gamma_i} T_i^{-1}$ holds because of the block structures of $T_\Gamma$ and the signed Boolean matrix $J_{\Gamma_i}$. Thus, the change of basis can be performed locally in each subdomain boundary. We introduce the transformed matrices
\[
\hat{S}^{D,i} = T_i^{-1} S^{D,i} T_i, \quad \hat{S}^{N,i} = T_i^{-1} S^{N,i} (T_i^{-1})^{-1}.
\]
By using
\[
S^{N,i} = (S^{D,i})^{-1},
\]
we also have the following identity:
\[
T_\Gamma^{-1} F_\Gamma (T_\Gamma^{-1})^{-1} = \sum_i J_{\Gamma_i} \hat{S}^{N,i} J_{\Gamma_i}^T = \sum_i J_{\Gamma_i} \left( \hat{S}^{D,i} \right)^{-1} J_{\Gamma_i}^T, \quad (4.30)
\]
which shows that the suggested change of basis $(T_\Gamma^{-1})^{-1}$ preserves the identity in (4.29).
After the change of unknowns using \((T_i^T)^{-1}\), we obtain the resulting equation on \(\hat{\lambda}_\Gamma = (T_i^T \lambda_\Gamma)\)

\[
\hat{F}_\Gamma \hat{\lambda}_\Gamma = \hat{d}_\Gamma,
\]

where

\[
\hat{F}_\Gamma = \sum_i J_{\Gamma i} \hat{S}_{\Delta i} \hat{J}_{\Delta i}^T, \quad \hat{d}_\Gamma = T_i^{-1} d.
\]

Based on the change of basis, we order the two matrices into

\[
\hat{S}^{N,i} = \begin{pmatrix} \hat{S}^{N,i}_\Pi & \hat{S}^{N,i}_\Delta \\ \hat{S}^{N,i}_\Delta^\Pi & \hat{S}^{N,i}_\Delta \end{pmatrix}, \quad \hat{S}^{D,i} = \begin{pmatrix} \hat{S}^{D,i}_\Pi & \hat{S}^{D,i}_\Delta \\ \hat{S}^{D,i}_\Delta^\Pi & \hat{S}^{D,i}_\Delta \end{pmatrix}.
\] (4.31)

In a similar manner, we order \(\hat{F}_\Gamma\) and \(\hat{\lambda}_\Gamma\) into

\[
\hat{F}_\Gamma = \begin{pmatrix} \hat{F}_\Pi \Delta & \hat{F}_\Pi \Delta \\ \hat{F}_\Delta \Pi & \hat{F}_\Delta \Delta \end{pmatrix}, \quad \hat{\lambda}_\Gamma = \begin{pmatrix} \hat{\lambda}_\Pi \\ \hat{\lambda}_\Delta \end{pmatrix}
\]

and eliminate the unknowns \(\hat{\lambda}_\Pi\) to obtain the equation on \(\hat{\lambda}_\Delta\):

\[
F_{DP} \hat{\lambda}_\Delta = \hat{d}_{DP},
\] (4.32)

where

\[
F_{DP} = \hat{F}_\Delta \Delta - \hat{F}_\Delta \Pi \hat{F}_\Pi^{-1} \hat{F}_\Pi \Delta
\]

and

\[
d_{DP} = \hat{d}_\Delta - \hat{F}_\Delta \Pi \hat{F}_\Pi^{-1} \hat{d}_\Pi.
\]

We note that application of \(F_{DP}\) to the vector \(\hat{\lambda}_\Delta\) can be performed locally by using block matrices of \(\hat{S}^{N,i}\) in (4.31), except the calculation of \(\hat{F}_\Pi^{-1} \hat{F}_\Pi \Delta\), which amounts to solving a global coarse problem. In other words, we have

\[
\hat{F}_\Delta \Delta = \sum_i J_{\Delta i} \hat{S}_{\Delta i} \hat{J}_{\Delta i}^T, \quad \hat{F}_\Pi \Delta = \sum_i J_{\Pi i} \hat{S}_{\Pi i} \hat{J}_{\Pi i}^T, \quad \hat{F}_\Pi \Pi = \sum_i J_{\Pi i} \hat{S}_{\Pi i} \hat{J}_{\Pi i}^T,
\]

where \(J_{\Delta i}^T\) and \(J_{\Pi i}^T\) are the blocks of \(J_{\Delta i}^T\) corresponding to unknowns \(\hat{\lambda}_\Delta\) and \(\hat{\lambda}_\Pi\), respectively. The construction of our coarse problem is different from that in the standard FETI-DP algorithm [20]. In our case, the coarse problem is built on the primal unknowns \(\hat{\lambda}_\Pi\), which are selected from the Lagrange multipliers \(\lambda\) by applying the change of basis on each \(\Gamma_{ij}\).

We will see that such a change of basis results in a scalable bound of condition numbers of our FETI-DP algorithm.

We will solve (4.32) combined with the following Dirichlet preconditioner:

\[
M^{-1} = \sum_i J_{\Delta i} D_{\Delta i} J_{\Delta i}^T, \quad (4.33)
\]

where the weight factor \(D_{\Delta i}\) is defined on each subdomain interface \(F (= \partial \Omega_i \cap \partial \Omega_j)\) as follows:

\[
D_{\Delta i} \big|_F = \left(S_F^{(i)} + S_F^{(j)}\right)^{-1} S_F^{(j)}, \quad D_{\Delta j} \big|_F = \left(S_F^{(i)} + S_F^{(j)}\right)^{-1} S_F^{(i)}.
\] (4.34)
Here, $S_F^{(i)}$ is the block matrix of $\hat{S}_{\Delta,\Delta}^{D,i}$ corresponding to the unknowns in $F$. This enriched weight factor was first introduced in [31] and successfully resolved the discontinuity of the two parameters $\alpha_i$ and $\beta_i$ across the subdomain interface; see also [32].

We end this section by a brief summary of the proposed FETI-DP algorithm. We stress that the algorithm is built based on Dirichlet and Neumann problems at the triangle level, and they are then assembled into subdomain matrices to form $T_i$ and $N_i$. The two matrices are further reduced to block matrices of $O$ and $i$ defined on the subdomain interface unknowns after eliminating interior unknowns in each subdomain. After applying change of basis on $O$ and $i$, respectively, we obtain the transformed matrices $O$ and $i$. After ordering the matrices $O$ and $i$ into dual and primal unknowns, we first assemble the coarse problem matrix $F_{\Pi\Pi}$ by assembling the blocks of $O$ to the primal unknowns, and we then calculate $F_{DP}$ by using block matrices of $O$ and $F_{\Pi\Pi}$. Application of the preconditioner $M^{-1}r$ can be performed by using the weight factors $D_{\Delta,i}$ and the block matrices of $O$. In our algorithm, to deal with the two discontinuous parameters $\alpha_i$ and $\beta_i$, an enriched weight factor is introduced as in (4.34).

5. ESTIMATE OF CONDITION NUMBER

In this section, we will analyze an estimate of the condition number of $M^{-1}F_{DP}$. As in the standard FETI-DP algorithm, we introduce $W$ a space of vectors defined on the subdomain interfaces, which are partially coupled at the primal unknowns across the subdomain interfaces. For $\bar{u} = (u_{\Pi}, u_{\Delta}) \in W$, $\bar{u}$ is coupled at the primal unknowns $u_{\Pi}$ and decoupled at the remaining unknowns $u_{\Delta}$. We call $u_{\Delta}$ dual unknowns. We assemble the matrices $O$ at the primal unknowns $u_{\Pi}$ to obtain the following ordered matrix defined on $W$:

$$O = \left(\begin{array}{ll}
O_{\Pi\Pi} & O_{\Pi\Delta} \\
O_{\Delta\Pi} & O_{\Delta\Delta}
\end{array}\right),$$

where all the block matrices are assembled from the blocks of the local matrices of $O$ in (4.31). We note that in our algorithm for $u$, the edge averages of the tangential component of $u$ are selected as the primal unknowns:

$$u_{\Pi}|_{F} = \frac{\int_{F} u_h \cdot t \, ds}{\int_{F} 1 \, ds}$$

and for $\lambda$, its primal unknowns are selected so that the change of basis preserves the identity in (4.30).

From now on, for simplicity, we will use the notation $\lambda$, instead of $\hat{\lambda}_{\Delta}$, for the Lagrange multipliers to the dual unknowns $u_{\Delta}$ and use the notation $\Lambda$ for the set of Lagrange multipliers $\lambda$. The identity

$$\hat{F}_{\Gamma} = \sum_{i} J_{\Gamma,i} \left(\hat{S}_{\Delta}^{D,i}\right)^{-1} J_{\Gamma,i}^T,$$

and the elimination of the primal unknowns $\hat{\lambda}_{\Pi}$ given that the matrix $F_{DP}$ satisfies

$$\langle F_{DP}, \lambda \rangle = \max_{\bar{u} \in \hat{W}} \frac{\langle \bar{u}, \hat{J}^T \lambda \rangle}{\langle \bar{u}, \bar{u} \rangle},$$

(5.35)

where $\hat{J}^T$ is the extension of $J_{\Delta}^T$ to $\hat{W}$

$$\hat{J}^T = \left(\begin{array}{c}
0 \\
J_{\Delta}^T
\end{array}\right)$$

and $J_{\Delta} u_{\Delta}$ calculates the jump on the decoupled unknowns $u_{\Delta}$ as in the standard FETI-DP algorithms.
We introduce a weight matrix defined for unknowns in $\tilde{W}$:

$$
\tilde{D} = \begin{pmatrix}
0 & 0 \\
0 & D_{\Delta\Delta}
\end{pmatrix}.
$$

where $D_{\Delta\Delta}$ is a block diagonal matrix with $D_{\Delta,i}$ as its blocks. We then observe that

$$
\tilde{j} \tilde{D}^T \tilde{j}^T = I \text{ on } \Lambda.
$$

By (5.35), we obtain

$$
F_{DP} = \tilde{j} \left( \tilde{S}^D \right)^{-1} \tilde{j}^T
$$

and by using the aforementioned notations, we can rewrite (4.33) into

$$
M^{-1} = \tilde{j} \tilde{D}^T \tilde{S}^D \tilde{D} \tilde{j}^T.
$$

**Lower bound estimate** The lower bound estimate is straightforward by the following calculation:

$$
\lambda^T F_{DP} \lambda = \lambda^T F_{DP} \tilde{j} \tilde{D}^T \tilde{j}^T \lambda
= \lambda^T F_{DP} \tilde{j} \tilde{D}^T \left( \tilde{S}^D \right)^{1/2} \left( \tilde{S}^D \right)^{-1/2} \tilde{j}^T \lambda
\leq \left( \left( \left( \tilde{S}^D \right)^{1/2} \tilde{D} \tilde{j}^T F_{DP} \lambda \right)^T \tilde{S}^D \tilde{D} \tilde{j}^T F_{DP} \lambda \right)^{1/2}
\left( \left( \tilde{S}^D \right)^{-1/2} \tilde{j}^T \lambda \right)^T \left( \tilde{S}^D \right)^{-1/2} \tilde{j}^T \lambda
\leq \left( \lambda^T F_{DP} M^{-1} F_{DP} \lambda \right)^{1/2} \left( \lambda^T F_{DP} \lambda \right)^{1/2}.
$$

We therefore obtain the lower bound bounded below by one. In our numerical experiments, we can also observe that our algorithm always gives the minimum eigenvalue one.

**Upper bound estimate** For the upper bound estimate, we introduce the weighted jump operator

$$
P_D = \tilde{D} \tilde{j}^T \tilde{j}
$$

and follows

$$
\lambda^T F_{DP} \lambda = \lambda^T \tilde{j} \left( \tilde{S}^D \right)^{-1} \tilde{j}^T \tilde{D} \tilde{j} \left( \tilde{S}^D \right)^{-1} \tilde{j}^T \lambda
= \lambda^T \tilde{j} \left( \tilde{S}^D \right)^{-1} P_D \tilde{S}^D P_D \left( \tilde{S}^D \right)^{-1} \tilde{j}^T \lambda
= \left( \tilde{S}^D P_D \left( \tilde{S}^D \right)^{-1} \tilde{j}^T \lambda, P_D \left( \tilde{S}^D \right)^{-1} \tilde{j}^T \lambda \right)
\leq \left( \tilde{S}^D \left( \tilde{S}^D \right)^{-1} \tilde{j}^T \lambda, \tilde{S}^D \left( \tilde{S}^D \right)^{-1} \tilde{j}^T \lambda \right)
\frac{\left( \tilde{S}^D P_D \left( \tilde{S}^D \right)^{-1} \tilde{j}^T \lambda \right)^2}{\left( \tilde{S}^D \left( \tilde{S}^D \right)^{-1} \tilde{j}^T \lambda \right)^2}
\leq \| P_D \|_{\tilde{S}^D}^2 \lambda^T F_{DP} \lambda.
$$

We therefore obtain the upper bound bounded above by $\| P_D \|_{\tilde{S}^D}^2$. In the following, an estimate for the upper bound of $\| P_D \|_{\tilde{S}^D}^2$ will be carried out.

**Lemma 5.4**

We obtain

$$
\langle \tilde{S}^D P_D \tilde{w}, P_D \tilde{w} \rangle \leq C \sum_{F \subset \partial \Omega_i} \sum_{i} \left| \tilde{S}^D_F w_{\Delta,F}^{(i)} \right| w_{\Delta,F}^{(i)}
$$

where $C$ is a positive constant independent of $\alpha_i$, $\beta_i$ and any mesh parameters, and $w_{\Delta,F}^{(i)}$ is the part of dual unknowns of $\tilde{w}$ on a subdomain interface $F$ in $\partial \Omega_i$.

**Proof**

We consider

$$
P_D \tilde{w} \big|_{F,i} = D_{i,F} \left( w_{\Delta,F}^{(i)} - w_{\Delta,F}^{(j)} \right),
$$
where $D_{i,F} = D_{\Delta i | F}$ in (4.34), and we have used that $\tilde{w}$ is coupled at the primal unknowns on each interface $F$. We then obtain that

$$\{ \tilde{S} D P_{D} \tilde{w}, \tilde{w} \} \leq C \sum_{i} \sum_{F \subset \Omega_{i}} \left( S^{(i)}_{F} D_{i,F} \left( w^{(i)}_{\Delta F} - w^{(j)}_{\Delta F} \right), D_{i,F} \left( w^{(i)}_{\Delta F} - w^{(j)}_{\Delta F} \right) \right).$$

We finally obtain the resulting estimate

$$\left( S^{(i)}_{F} D_{i,F} \left( w^{(i)}_{\Delta F} - w^{(j)}_{\Delta F} \right), D_{i,F} \left( w^{(i)}_{\Delta F} - w^{(j)}_{\Delta F} \right) \right) \leq C \left( \left( S^{(i)}_{F} D_{i,F} w^{(i)}_{\Delta F}, D_{i,F} w^{(j)}_{\Delta F} \right) + \left( S^{(j)}_{F} D_{i,F} w^{(j)}_{\Delta F}, D_{i,F} w^{(j)}_{\Delta F} \right) \right)$$

where we have used

$$D_{i,F}^{T} S^{(i)}_{F} D_{i,F} \leq S^{(i)}_{F}, \quad D_{i,F}^{T} S^{(j)}_{F} D_{i,F} \leq S^{(j)}_{F},$$

which can be obtained by the carefully designed weight factors in (4.34).

To proceed our analysis, we introduce the restriction of $V_{h}$ to each subdomain $\Omega_{i}$ and use the notation

$$V_{h}(\Omega_{i}) := V_{h} | \Omega_{i}. $$

For $v \in V_{h}(\Omega_{i})$, $v \cdot t$ is continuous on the edges in $F_{u} \cap \Omega_{i}$, but it can be discontinuous on the edges in $F_{q} \cap \Omega_{i}$. In addition, we introduce $V_{h}(\Omega_{i}, F)$; see (4.28) for the definition and $V_{h}^{C}(\Omega_{i})$.

$$V_{h}^{C}(\Omega_{i}) := \{ v \in V_{h}(\Omega_{i}) : v \cdot t \text{ is continuous on } \forall e \in \left( F_{q} \bigcup F \right) \cap \Omega_{i} \}.$$  

Here, for $v$ in $V_{h}(\Omega_{i}, F)$, its tangential component is continuous on each edges in $F_{u}$ and $F$, while it can be discontinuous on each edges in $F_{q}$. For $v$ in $V_{h}^{C}(\Omega_{i})$, its tangential component is continuous on all the edges in $F \cup F_{u} \cup F_{q}$, thus $v$ is curl conforming in $\Omega_{i}$.

We recall the energy $E_{i}(v)$ in (4.27) for $v$ in $V_{h}(\Omega_{i})$. For a given $w^{(i)}$ on $\partial \Omega_{i}$, we define $S H_{i}^{h} \left( w^{(i)} \right)$ by the minimum energy extension $E_{i}(u)$ of $u \cdot t = w^{(i)}$ on $\partial \Omega_{i}$ with respect to the space $V_{h}(\Omega_{i}, F)$. Functions in $V_{h}^{C}(\Omega_{i}, F)$ have continuous tangential components on edges in $F$ and $F_{u}$. By the definition of the matrix $\tilde{S} D F$, the Schur complement matrix of $D_{i}$ that is assembled by $D_{\partial F_{u}}$, we obtain that

$$\left( \tilde{S} D F w^{(i)}, w^{(i)} \right) = E_{i} \left( S H_{i}^{h} \left( w^{(i)} \right) \right),$$

because $S H_{i}^{h} \left( w^{(i)} \right)$ minimizes the energy with respect to the given $w^{(i)}$. Similarly for $S^{(j)}_{F}$, the block matrix of $\tilde{S} D F$ to the unknowns on a subdomain edge $F$, we obtain that

$$\left( S^{(i)}_{F} w^{(i)}_{\Delta F}, w^{(j)}_{\Delta F} \right) = E_{i} \left( S H_{i}^{h} \left( R^{F}_{F} w^{(i)} \right) \right),$$

where $R^{F}_{F}$ is the zero extension of $w^{(i)}_{\Delta F}$ to $\partial \Omega_{i}$.

In our algorithm, we assume that each subdomain is a union of coarse triangles, and thus each subdomain edge is an edge of a coarse triangle. For such a coarse triangulation $T_{H}(\Omega_{i})$, we introduce the lowest order curl-conforming space $V_{h}^{C}(\Omega_{i})$. For $v_{i} \in V_{h}^{C}(\Omega_{i})$ we introduce an interpolation to $V_{h}^{C}(\Omega_{i})$ by

$$\pi_{H}(v_{i}) = \sum_{F \in T_{H}(\Omega_{i})} F(v_{i}) \phi_{F}.$$
where \( \mathcal{F}_H(\Omega_i) \) is the set of edges in the coarse triangulation \( \mathcal{T}_H(\Omega_i) \), \( \phi_F \) is the basis function corresponding to an edge \( F \), and

\[
F(v_i) = \frac{\int_F v_i \cdot t \, ds}{\int_F \phi_F \cdot t \, ds}.
\]

By the assumption that subdomain edges are again edges of the coarse triangulation, we obtain that \( \pi_H(v_i) \) preserves averages of \( v_i \cdot t \) on each subdomain edge, that is,

\[
\int_F \pi_H(v_i) \cdot t \, ds = \int_F v_i \cdot t \, ds, \quad \forall F \subset \partial \Omega_i. \tag{5.36}
\]

For \( v_i \) in \( \mathcal{V}_h(\Omega_i, \mathcal{F}) \), we introduce an interpolation to the curl-conforming space \( \mathcal{V}_h^C(\Omega_i) \), which is defined as

\[
\pi_h(v_i) \cdot t = v_i \cdot t, \quad \forall e \in (\mathcal{F} \cup \mathcal{F}_w) \cap \Omega_i
\]

and

\[
\pi_h(v_i) \cdot t = \frac{v_i |_{\tau_1} \cdot t + v_i |_{\tau_2} \cdot t}{2}, \quad \forall e \in \mathcal{F}_q \cap \Omega_i.
\]

Here, \( \tau_1 \) and \( \tau_2 \) are triangles sharing an edge \( e \), and \( t \) is a unit tangent of a given edge \( e \).

**Lemma 5.5**

For \( w^{(i)} \), we obtain that

\[
\left\langle S^{(i)}_{\Delta,F} w^{(i)}_{\Delta,F}, w^{(i)}_{\Delta,F} \right\rangle \leq C \left( 1 + \log \frac{H}{h} \right)^2 \left\langle \hat{S}^{D,i} w^{(i)}, w^{(i)} \right\rangle,
\]

where \( C \) is a positive constant independent of \( \alpha_i, \beta_i \), and any mesh parameters, and \( w^{(i)}_{\Delta,F} \) is the part of dual unknowns of \( w^{(i)} \) on a subdomain interface \( F \) in \( \partial \Omega_i \).

**Proof**

For \( w^{(i)}_{\Delta,F} \), we recall the identity

\[
\left\langle S^{(i)}_{\Delta,F} w^{(i)}_{\Delta,F}, w^{(i)}_{\Delta,F} \right\rangle = E_i \left( S^{h(\pi_h \circ \pi_H)(v_i)}_{\Delta,F} \left( R^T_F w^{(i)} \right) \right).
\tag{5.37}
\]

Let \( v^{NC}_{i} = S^h_{\mathcal{H}_i}(w^{(i)}) \) in \( \mathcal{V}_h(\Omega_i, \mathcal{F}) \). For \( v^{NC}_{i} \), we introduce \( v_i = \pi_h(v^{NC}_{i}) \), the interpolation of \( v^{NC}_{i} \) to the curl-conforming space \( \mathcal{V}_h^C(\Omega_i) \). Similarly, for \( v_i \), we introduce \( \pi_H(v_i) \), the interpolation to the curl-conforming coarse space \( \mathcal{V}_h^C(\Omega_i) \). Then, \( v_i \) satisfies that

\[
v_i \cdot t = v^{NC}_{i} \cdot t \quad \text{on} \ F \subset \partial \Omega_i
\]

and thus

\[
(v_i - \pi_H(v_i)) \cdot t = w^{(i)}_{\Delta,F} \quad \text{on} \ F \subset \partial \Omega_i.
\]

We note that by (5.36) edge averages of \( \pi_H(v_i) \cdot t \) over \( F \) are the same as the primal unknowns of \( w^{(i)} \) in our algorithm. By the aforementioned identity, we have

\[
E_i \left( S^h_{\mathcal{H}_i} \left( R^T_F w^{(i)} \right) \right) \leq E_i \left( S^{h(C)(\pi_h \circ \pi_H)(v_i)}) \cdot t) \right),
\tag{5.38}
\]

where \( S^{h(C)(\pi_h \circ \pi_H)}(R^T_F w) \) is the energy minimization \( E_i(v) \) to the curl-conforming space \( \mathcal{V}_h^C(\Omega_i) \) with \( v \cdot t = w \) on \( F \) and \( v \cdot t = 0 \) on \( \partial \Omega_i \setminus F \).

Following similarly to the proofs in [32, Lemma 4.8], we can obtain that

\[
E_i \left( S^{h(C)(\pi_h \circ \pi_H)(v_i)}) \cdot t) \right) \leq C \left( 1 + \log \frac{H}{h} \right)^2 E_i(v_i).
\tag{5.39}
\]
where $C$ is a constant independent of $\alpha_i, \beta_i$, and any mesh parameters. Here, we note that in the aforementioned estimate, all the functions are in curl-conforming finite element spaces, and thus the proofs in [32] can be applied straightforwardly without any difficulty. Following similarly as in [19, Lemma 6], for the interpolation $v_i = \pi_h(v_i^{NC})$, we can obtain

$$E_i(v_i) \leq C E_i(v_i^{NC}).$$

(5.40)

By (5.37) to (5.40) combined with $E_i(v_i^{NC}) = \left(\tilde{S}^{D,j}w^{(j)}, w^{(j)}\right)$, we complete the proof.

By Lemmas 5.4 and 5.5, we obtain an upper bound estimate:

**Theorem 5.6**
The operator $P_D$ satisfies

$$\left(\tilde{S}^{D}P_D\tilde{w}, P_D\tilde{w}\right) \leq C \left(1 + \log \frac{H}{h}\right)^2 \left(\tilde{S}^{D}\tilde{w}, \tilde{w}\right), \quad \forall \tilde{w} \in \tilde{W}$$

and thus

$$\|P_D\|^2_{\tilde{S}^{D}} \leq C \left(1 + \log \frac{H}{h}\right)^2,$$

where $C$ is a positive constant independent of $\alpha_i, \beta_i$, and any mesh parameters.

### 6. NUMERICAL EXAMPLES

To test the convergence of our HSDG method, we first consider a model problem with a known exact solution. The computational domain is $\Omega = (0, 1)^2$, and for a model problem with $\alpha = \beta = 1$, the exact solution is given by

$$u(x, y) = \left(\begin{array}{c}
\sin \pi x \\
\sin \pi y \\
x(1-x) \\
y(1-y)
\end{array}\right).$$

For the model problem, we apply our FETI-DP algorithm and present its performance and behavior of $L^2$-errors depending on the mesh size $h$. The domain is divided into uniformly square subdomain partitions, and each subdomain is partitioned into uniform triangles. In our HSDG formulation, the uniform triangulation is further divided following the subdivision process described in Section 2. In the CG iteration, the iteration is stopped when the relative residual norm is reduced to a factor of $10^6$. In the following, $N_d = 4^2$ means that the domain is partitioned into $4 \times 4$ uniformly square subdomains, $H$ denotes the subdomain size, and $h$ denotes the size of triangles in the initial triangulation before the subdivision process.

In Tables I and II, the iteration counts, the two extreme eigenvalues, and the corresponding $L^2$-errors, $\|u - u_h\|_0$, are reported depending on $H/h$ and $N_d$ for both $P_0$ and $P_1$ cases. The $L^2$-errors

<table>
<thead>
<tr>
<th>$H/h$</th>
<th>Iter</th>
<th>$\lambda_{\min}$</th>
<th>$\lambda_{\max}$</th>
<th>$|u - u_h|_0$</th>
<th>order</th>
<th>$\lambda_{\min}$</th>
<th>$\lambda_{\max}$</th>
<th>$|u - u_h|_0$</th>
<th>order</th>
</tr>
</thead>
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<td>1</td>
<td>6</td>
<td>1.01</td>
<td>1.22</td>
<td>6.4163e-1</td>
<td></td>
<td>1.00</td>
<td>1.81</td>
<td>2.1260e-1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>8</td>
<td>1.00</td>
<td>1.62</td>
<td>3.2335e-1</td>
<td>0.99</td>
<td>1.00</td>
<td>2.47</td>
<td>5.7665e-2</td>
<td>1.88</td>
</tr>
<tr>
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<td>9</td>
<td>1.00</td>
<td>2.18</td>
<td>1.6169e-1</td>
<td>1.00</td>
<td>1.00</td>
<td>3.26</td>
<td>1.4978e-2</td>
<td>1.94</td>
</tr>
<tr>
<td>8</td>
<td>11</td>
<td>1.00</td>
<td>2.88</td>
<td>8.0745e-2</td>
<td>1.00</td>
<td>1.00</td>
<td>4.17</td>
<td>3.8133e-3</td>
<td>1.97</td>
</tr>
<tr>
<td>16</td>
<td>12</td>
<td>1.00</td>
<td>3.71</td>
<td>4.0333e-2</td>
<td>1.00</td>
<td>1.00</td>
<td>5.63</td>
<td>9.6176e-4</td>
<td>1.99</td>
</tr>
</tbody>
</table>

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Table II. Performance of the dual and primal finite element tearing and interconnecting algorithm for the problem with uniform $\alpha = \beta = 1$ by increasing $N_d$ with a fixed local problem $H/h = 4$: Iter (number of iterations), $\lambda_{\min}$ (minimum eigenvalues), $\lambda_{\max}$ (maximum eigenvalues), and $\|u - u_h\|_0$ ($L^2$ errors).

<table>
<thead>
<tr>
<th>$N_d$</th>
<th>Iter</th>
<th>$\lambda_{\min}$</th>
<th>$\lambda_{\max}$</th>
<th>$|u - u_h|_0$</th>
<th>order</th>
</tr>
</thead>
<tbody>
<tr>
<td>$4^2$</td>
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<td>1.00</td>
<td>2.18</td>
<td>1.6169e-1</td>
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</tr>
<tr>
<td></td>
<td>11</td>
<td>1.00</td>
<td>3.26</td>
<td>1.4978e-2</td>
<td></td>
</tr>
<tr>
<td>$8^2$</td>
<td>13</td>
<td>1.00</td>
<td>2.58</td>
<td>8.0745e-2</td>
<td>1.00</td>
</tr>
<tr>
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<td>3.89</td>
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<td>1.97</td>
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<td>9.6176e-4</td>
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<tr>
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<td>1.00</td>
<td>4.18</td>
<td>2.4148e-5</td>
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</tr>
</tbody>
</table>

Table III. Performance of the dual and primal finite element tearing and interconnecting algorithm for the problem with varying $\alpha_i$ and with $\alpha_w = \beta_w = \beta_b = 1$ in the 4 by 4 checker board subdomain partition: $P_0$ case (upper table) and $P_1$ case (lower table).

<table>
<thead>
<tr>
<th>$H/h = 1$</th>
<th>$H/h = 2$</th>
<th>$H/h = 4$</th>
<th>$H/h = 8$</th>
<th>$H/h = 16$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_i$</td>
<td>Iter</td>
<td>Cond</td>
<td>Iter</td>
<td>Cond</td>
</tr>
<tr>
<td>$10^{-3}$</td>
<td>7</td>
<td>1.38</td>
<td>9</td>
<td>1.56</td>
</tr>
<tr>
<td>$10^{-2}$</td>
<td>7</td>
<td>1.30</td>
<td>10</td>
<td>1.63</td>
</tr>
<tr>
<td>$10^{-1}$</td>
<td>7</td>
<td>1.25</td>
<td>9</td>
<td>1.70</td>
</tr>
<tr>
<td>$1$</td>
<td>7</td>
<td>1.25</td>
<td>10</td>
<td>1.71</td>
</tr>
<tr>
<td>$10^1$</td>
<td>7</td>
<td>1.22</td>
<td>9</td>
<td>1.71</td>
</tr>
<tr>
<td>$10^2$</td>
<td>7</td>
<td>1.25</td>
<td>9</td>
<td>1.72</td>
</tr>
<tr>
<td>$10^3$</td>
<td>7</td>
<td>1.26</td>
<td>9</td>
<td>1.71</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$H/h = 1$</th>
<th>$H/h = 2$</th>
<th>$H/h = 4$</th>
<th>$H/h = 8$</th>
<th>$H/h = 16$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_i$</td>
<td>Iter</td>
<td>Cond</td>
<td>Iter</td>
<td>Cond</td>
</tr>
<tr>
<td>$10^{-3}$</td>
<td>10</td>
<td>1.67</td>
<td>13</td>
<td>2.16</td>
</tr>
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<td>1.81</td>
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</tr>
<tr>
<td>$10^{-1}$</td>
<td>10</td>
<td>1.92</td>
<td>13</td>
<td>2.63</td>
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<tr>
<td>$1$</td>
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<td>1.94</td>
<td>12</td>
<td>2.66</td>
</tr>
<tr>
<td>$10^1$</td>
<td>10</td>
<td>1.94</td>
<td>12</td>
<td>2.66</td>
</tr>
<tr>
<td>$10^2$</td>
<td>10</td>
<td>1.94</td>
<td>12</td>
<td>2.66</td>
</tr>
<tr>
<td>$10^3$</td>
<td>10</td>
<td>1.94</td>
<td>12</td>
<td>2.66</td>
</tr>
</tbody>
</table>

follow $O(h)$ for the $P_0$ case and $O(h^2)$ for the $P_1$ case, which are optimal for the given degree of the polynomials. By increasing $H/h$, the condition numbers follow the growth $C(1 + \log(H/h))^2$, and by increasing $N_d$, the condition numbers and iteration counts show good scalability, that is, independent of $N_d$.

In the following, we consider a model problem with varying coefficients $\alpha(x)$ and $\beta(x)$, which are piecewise positive constant in each subdomain $\Omega_i$. For that model problem, we consider the exact solution $u(x) = 0$. In our FETI-DP algorithm, we start the CG method with a random nonzero initial and stop the iteration when the relative residual norm is reduced to $10^{-6}$.

In Tables III and IV, we present iteration counts and condition numbers of the algorithm for each $P_0$ and $P_1$ cases varying $\alpha_i$ and $\beta_i$ for a given 4 by 4 checker board partition of $\Omega$ (Figure 4). For the white subdomains, $\alpha_w = \beta_w = 1$, and for the black subdomains, $\alpha_b = \alpha_i$ and $\beta_b = 1$ (in Table III) or $\alpha_b = 1$ and $\beta_b = \beta_i$ (in Table IV). We observe that the condition numbers and iteration counts seem to be robust to the choice of $\alpha_i$ and to become smaller for both $\beta_i$ decreasing and increasing away from one. For a fixed $\alpha_i$ or $\beta_i$, the behavior of condition numbers follows $C(1 + \log(H/h))^2$ as increasing $H/h$.
Table IV. Performance of the dual and primal finite element tearing and interconnecting algorithm for the problem with varying $\beta_b = \beta_i$ and with $\alpha_b = \alpha_w = \beta_w = 1$ in the 4 by 4 checker board subdomain partition: $P_0$ case (upper table) and $P_1$ case (lower table).

<table>
<thead>
<tr>
<th>$\beta_i$</th>
<th>$H/h = 1$</th>
<th>$H/h = 2$</th>
<th>$H/h = 4$</th>
<th>$H/h = 8$</th>
<th>$H/h = 16$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^{-3}$</td>
<td>Iter</td>
<td>Cond</td>
<td>Iter</td>
<td>Cond</td>
<td>Iter</td>
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<tr>
<td>10^{-3}</td>
<td>3</td>
<td>1.00</td>
<td>4</td>
<td>1.00</td>
<td>5</td>
</tr>
<tr>
<td>10^{-2}</td>
<td>4</td>
<td>1.01</td>
<td>5</td>
<td>1.03</td>
<td>6</td>
</tr>
<tr>
<td>10^{-1}</td>
<td>5</td>
<td>1.09</td>
<td>7</td>
<td>1.26</td>
<td>9</td>
</tr>
<tr>
<td>1</td>
<td>7</td>
<td>1.25</td>
<td>10</td>
<td>1.71</td>
<td>12</td>
</tr>
<tr>
<td>10^{1}</td>
<td>5</td>
<td>1.08</td>
<td>7</td>
<td>1.24</td>
<td>8</td>
</tr>
<tr>
<td>10^{2}</td>
<td>4</td>
<td>1.01</td>
<td>4</td>
<td>1.03</td>
<td>5</td>
</tr>
<tr>
<td>10^{3}</td>
<td>3</td>
<td>1.00</td>
<td>3</td>
<td>1.00</td>
<td>4</td>
</tr>
</tbody>
</table>

Figure 4. A checker board subdomain partition when $N_d = 4^2$: $\Omega_b$ (black squares) and $\Omega_w$ (white squares).

In Table V, we present iteration counts and condition numbers increasing $N_d$ with a fixed local problem size for various distributions of $(\alpha_i, \beta_i)$. Here, we consider two types of subdomains, $\Omega_b$ and $\Omega_w$ with $(\alpha_b, \beta_b) = (1000, 1)$ and $(\alpha_w, \beta_w) = (1, 1000)$, respectively. We call $\Omega_b$ a curl-dominated subdomain and $\Omega_w$ a mass-dominated subdomain. For a given uniform subdomain partition $N_d$, we test our algorithm by choosing $\Omega_b$ and $\Omega_w$ randomly and by the checker board distribution of $\Omega_b$ and $\Omega_w$. For the checker board distribution, we observe robust condition numbers and iteration counts increasing $N_d$. In addition, even for the random distribution of $\Omega_b$ and $\Omega_w$, the iteration counts and condition numbers present only slight increase.

In Table VI, we choose $\alpha_i = 10^{10}$ and $\beta_i = 10^{10}$ for each subdomain $\Omega_i$ in the 4 by 4 uniform subdomain partition by generating $r_i$ and $s_i$ randomly in $(-3, 3)$. By increasing $H/h$, the performance of our algorithm is presented with iteration counts and condition numbers for each $P_0$ and $P_1$ cases. For all the test cases, we observe that the condition numbers follow $C(1 + \log(H/h))^2$ as increasing $H/h$, and the performance of our algorithm seems to be robust to the choice of test sets with highly heterogeneous coefficients.
Table V. Performance of the dual and primal finite element tearing and interconnecting algorithm for the problem increasing $N_d$ with a fixed $H/h = 4$ for randomly chosen $(\alpha_i, \beta_i)$ from curl and mass-dominated cases and for checker board distribution of curl and mass-dominated cases: random (randomly chosen) and checker (checker board distribution).

<table>
<thead>
<tr>
<th>$N_d$</th>
<th>$\mathcal{V}_h$ by $P_0$</th>
<th>$\mathcal{V}_h$ by $P_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Random</td>
<td>Checker</td>
</tr>
<tr>
<td>$4^2$</td>
<td>10</td>
<td>1.97</td>
</tr>
<tr>
<td>$8^2$</td>
<td>13</td>
<td>2.20</td>
</tr>
<tr>
<td>$16^2$</td>
<td>14</td>
<td>2.23</td>
</tr>
<tr>
<td>$32^2$</td>
<td>16</td>
<td>2.51</td>
</tr>
</tbody>
</table>

Table VI. Performance of the dual and primal finite element tearing and interconnecting algorithm for the problem with $\alpha_i = 10^{i+1}$ and $\beta_i = 10^i$ in the 4 by 4 subdomain partition by choosing $r_i$ and $s_i$ randomly in $(-3, 3)$: $P_0$ case (upper table) and $P_1$ case (lower table).

<table>
<thead>
<tr>
<th>$H/h = 1$</th>
<th>$H/h = 2$</th>
<th>$H/h = 4$</th>
<th>$H/h = 8$</th>
<th>$H/h = 16$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{V}_h$</td>
<td>Iter</td>
<td>Cond</td>
<td>Iter</td>
<td>Cond</td>
</tr>
<tr>
<td>Set 1</td>
<td>7</td>
<td>1.19</td>
<td>9</td>
<td>1.60</td>
</tr>
<tr>
<td>Set 2</td>
<td>7</td>
<td>1.24</td>
<td>9</td>
<td>1.51</td>
</tr>
<tr>
<td>Set 3</td>
<td>8</td>
<td>1.27</td>
<td>10</td>
<td>1.60</td>
</tr>
<tr>
<td>Set 4</td>
<td>6</td>
<td>1.16</td>
<td>9</td>
<td>1.41</td>
</tr>
<tr>
<td>Set 5</td>
<td>8</td>
<td>1.31</td>
<td>10</td>
<td>1.71</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$H/h = 1$</th>
<th>$H/h = 2$</th>
<th>$H/h = 4$</th>
<th>$H/h = 8$</th>
<th>$H/h = 16$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{V}_h$</td>
<td>Iter</td>
<td>Cond</td>
<td>Iter</td>
<td>Cond</td>
</tr>
<tr>
<td>Set 1</td>
<td>10</td>
<td>1.77</td>
<td>12</td>
<td>2.35</td>
</tr>
<tr>
<td>Set 2</td>
<td>10</td>
<td>1.60</td>
<td>12</td>
<td>2.10</td>
</tr>
<tr>
<td>Set 3</td>
<td>11</td>
<td>1.79</td>
<td>13</td>
<td>2.48</td>
</tr>
<tr>
<td>Set 4</td>
<td>10</td>
<td>1.52</td>
<td>12</td>
<td>1.97</td>
</tr>
<tr>
<td>Set 5</td>
<td>11</td>
<td>1.90</td>
<td>13</td>
<td>2.56</td>
</tr>
</tbody>
</table>

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REFERENCES


