Hermite-Hadamard's Inequality and Its Extensions for Conformable Fractional Integrals of Any Order $\alpha > 0$

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Abstract

Recently the authors Abdeljawad and Khalil et al. are introduced a new simple well-behaved definition of the fractional integral called conformable fractional integral [1, 6]. In this article, we establish Hermite-Hadamard's inequalities for conformable fractional integral. We also gave extensions of Hermite-Hadamard type inequalities for conformable fractional integrals.

Keywords: Convex functions, Hermite-Hadamard inequality, Conformable fractional integrals.

1 Introduction

The following definition have an important place in mathematical analysis and inequality theory.

A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex if the inequality

$$f(\lambda u + (1-\lambda) v) \leq \lambda f(u) + (1-\lambda) f(v)$$

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holds for all $u, v \in I$ and $\lambda \in [0,1]$.

This definition has been used in the following inequality that is called Hadamard's inequality or Hermite-Hadamard inequality.

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

Hadamard's inequality is sensitive in terms of Cauchy Mean-Value Theorem for convex functions. Because one can find upper and lower bounds for the mean value of a convex function with Hadamard's inequality. Many researchers have expended efforts to provide new bounds and estimations by using this inequality. On all of these, we will mention Riemann-Liouvile fractional integrals that have beneficial usages in among others.

Definition 1.1 *Let $f \in L_1[a,b]$. The Riemann-Liouvile integrals $J^{\mu}_{a+}f$ and $J^{\mu}_{b-}f$ of order $\alpha > 0$ are defined by*

$$J^{\mu}_{a+}f(x) = \frac{1}{\Gamma(\mu)}\int_a^x (x-t)^{\mu-1}f(t)dt, \qquad x > a$$

and

$$J^{\mu}_{b-}f(x) = \frac{1}{\Gamma(\mu)}\int_x^b (t-x)^{\mu-1}f(t)dt, \qquad x < b$$

respectively where $\Gamma(\mu) = \int_0^\infty e^{-t}u^{\mu-1}du$.. Here $J^0_{a+}f(x) = J^0_{b-}f(x) = f(x)$

In the case of $\mu = 1$, the fractional integral reduces to classical integral.

We define the Beta function [7, p18]:

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \int_0^1 t^{a-1}(1-t)^{b-1}\,dt, \qquad a, b > 0,$$

where $\Gamma(\alpha) = \int_0^\infty e^{-t}u^{\alpha-1}du$ is Gamma function.

Owing to this definition the inequalities that have been obtained by classical integral and derivative have been improved and generalized. Let us consider the following Hadamard type inequalities for Riemann-Liouvile fractional integrals.

Theorem 1.1 *(See [9])Let $f : [a,b] \to \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L_1[a,b]$. If f is a convex function on $[a,b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[J^{\alpha}_{a+}f(b) + J^{\alpha}_{b-}f(a)] \leq \frac{f(a)+f(b)}{2}. \tag{1.2}$$

Theorem 1.2 *(See [2]) Let $f : [a, b] \to \mathbb{R}$ be a positive, twice differentiable function with $a < b$ and $f \in L_1[a, b]$. If f'' is bounded in $[a, b]$. Then we have*

$$\begin{aligned} & \frac{m\alpha}{2(b-a)^\alpha} \int_a^{\frac{a+b}{2}} \left(\frac{a+b}{2} - x\right)^2 [(x-a)^{\alpha-1} + (b-x)^{\alpha-1}]dx \\ \leq \; & \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)] - f\left(\frac{a+b}{2}\right) \\ \leq \; & \frac{M\alpha}{2(b-a)^\alpha} \int_a^{\frac{a+b}{2}} \left(\frac{a+b}{2} - x\right)^2 [(x-a)^{\alpha-1} + (b-x)^{\alpha-1}]dx \end{aligned}$$

and

$$\begin{aligned} & \frac{-M\alpha}{2(b-a)^\alpha} \int_a^{\frac{a+b}{2}} (x-a)(b-x)[(x-a)^{\alpha-1} + (b-x)^{\alpha-1}]dx \\ \leq \; & \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)] - \frac{f(a)+f(b)}{2} \\ \leq \; & \frac{-m\alpha}{2(b-a)^\alpha} \int_a^{\frac{a+b}{2}} (x-a)(b-x)[(x-a)^{\alpha-1} + (b-x)^{\alpha-1}]dx \end{aligned}$$

with $\alpha > 0$, where $m = inf_{t\in[a,b]} f''(t)$, $M = sup_{t\in[a,b]} f''(t)$.

Theorem 1.3 *(See [2]) Let $f : [a, b] \to \mathbb{R}$ be a positive, differentiable function with $a < b$ and $f \in L_1[a, b]$. If $f'(a + b - x) \geq f'(x)$ for all $x \in [a, \frac{a+b}{2}]$. Then the following inequalities for fractional integrals hold*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)] \leq \frac{f(a) + f(b)}{2}$$

with $\alpha > 0$.

Recently several Hermite-Hadamard type inequalities were obtained for various classes of functions using fractional integrals; one may refer to such works as (for example) [10, 5].

In spite of its valuable contribitions to mathematical analysis, the Riemann-Liouvile Fractional integrals have deficiencies. For example the solution of the differential equation that is given as;

$$y^{(\frac{1}{2})} + y = x^{(\frac{1}{2})} + \frac{2}{\Gamma(2.5)} x^{(\frac{3}{2})}, \quad y(0) = 0$$

where $y^{(\frac{1}{2})}$ is the fractional derivative of y of order $\frac{1}{2}$.

The solution of the above differential equation have caused to imagine on a new and simple represention of the definition of fractional derivative. In [6] , Khalil et al. gave a new definition that is called "Conformable fractional derivative". They not only proved further properties of this definitons but also gave the differences with the other fractional derivatives. Besides, another considerable study have presented by Abdeljawad to discuss the basic concepts of fractional calculus. In [1], Abdeljawad gave the following definitions of Right-Left fractional integrals:

Definition 1.2 *Let $\alpha \in (n, n+1]$, $n = 0, 1, 2, ...$ and set $\beta = \alpha - n$. Then the left conformable fractional integral of any order $\alpha > 0$ is defined by*

$$(I_\alpha^a f)(t) = \frac{1}{n!} \int_a^t (t-x)^n (x-a)^{\beta-1} f(x) dx$$

Analogously, the right conformable fractional integral of any order $\alpha > 0$ is defined by

$$({}^b I_\alpha f)(t) = \frac{1}{n!} \int_t^b (x-t)^n (b-x)^{\beta-1} f(x) dx.$$

Notice that if $\alpha = n+1$ then $\beta = \alpha - n = n+1-n = 1$ and hence $(I_\alpha^a f)(t) = (J_{n+1}^a f)(t)$.

We think that a new evoluation is necessary to explain the deficiencies of the previous results by obtained via Riemann-Liouvile fractional integrals.

Let us consider the function f defined as $f : \mathbb{R}^+ \to \mathbb{R}$, $f = x^2 e^x$ which is convex. If we choose this function to provide applications by the previous inequalities that have been obtained by Riemann-Liouvile fractional inequalities, we can see that the inequalities do not hold for $f(x)$. Because, the Riemann-Liouvile derivatives are not valid for product of two functions. The results which are obtained by using the conformable fractional integrals have a wide range of validity.

The aim of this paper is to prove new Hadamard's type inequalities that are valid for all elements of the class of convex functions via conformable fractional integrals. We also obtain extensions of Hadamard's inequality by using the conformable fractional integrals.

2 Hermite-Hadamard's inequalities for conformable fractional integrals

Theorem 2.1 *Let $f : [a, b] \to \mathbb{R}$ be a function with $0 \le a < b$ and $f \in L_1[a, b]$. If f is a convex function on $[a, b]$, then the following inequalities for* *conformable fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}\Gamma(\alpha-n)}[(I_{\alpha}^{a}f)(b)+({}^{b}I_{\alpha}f)(a)] \leq \frac{f(a)+f(b)}{2} \quad (2.1)$$

with $\alpha \in (n, n+1]$.

Proof. Let $x, y \in [a,b]$. If f is a convex function on $[a,b]$,

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2}$$

i.e., with $x = ta + (1-t)b$, $y = (1-t)a + tb$,

$$2f\left(\frac{a+b}{2}\right) \leq f(ta+(1-t)b) + f((1-t)a+tb). \quad (2.2)$$

Multiplying both sides of (2.2) by $\frac{1}{n!}t^n(1-t)^{\alpha-n-1}$, then integrating the resulting inequality with respect to t over $[0,1]$, we get

$$\begin{aligned}
\frac{2}{n!}f\left(\frac{a+b}{2}\right)\int_0^1 t^n(1-t)^{\alpha-n-1}dt &\leq \frac{1}{n!}\int_0^1 t^n(1-t)^{\alpha-n-1}f(ta+(1-t)b)dt \\
&+ \frac{1}{n!}\int_0^1 t^n(1-t)^{\alpha-n-1}f((1-t)a+tb)dt \\
&= \frac{1}{n!}\int_a^b \left(\frac{b-u}{b-a}\right)^n\left(\frac{u-a}{b-a}\right)^{\alpha-n-1}f(u)\frac{du}{a-b} \\
&+ \frac{1}{n!}\int_a^b \left(\frac{u-a}{b-a}\right)^n\left(\frac{b-u}{b-a}\right)^{\alpha-n-1}f(u)\frac{du}{b-a} \\
&= \frac{1}{(b-a)^{\alpha}}[I_{\alpha}^{a}f(b) + {}^{b}I_{\alpha}f(a)].
\end{aligned}$$

Note we have

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}\Gamma(\alpha-n)}[I_{\alpha}^{a}f(b) + {}^{b}I_{\alpha}f(a)]$$

where

$$\int_0^1 t^n(1-t)^{\alpha-n-1}dt = B(n+1, \alpha-n) = \frac{\Gamma(n+1)\Gamma(\alpha-n)}{\Gamma(\alpha+1)}$$

and the first half of the inequality in (2.1) is proved.

Since f is a convex, we have the following inequalities:

$$\begin{aligned} f(ta+(1-t)b) &\leq tf(a)+(1-t)f(b) \\ f((1-t)a+tb) &\leq (1-t)f(a)+tf(b). \end{aligned}$$

Adding these two inequalities, we get

$$f(ta+(1-t)b)+f((1-t)a+tb) \leq f(a)+f(b).$$

Multiplying both sides of the resulting inequality by $\frac{1}{n!}t^n(1-t)^{\alpha-n-1}$ and integrating with respect to t over $[0,1]$, we have

$$\begin{aligned} \frac{1}{(b-a)^\alpha}[I^a_\alpha f(b)+^b I_\alpha f(a)] &\leq \frac{1}{n!}\int_0^1 t^n(1-t)^{\alpha-n-1}[f(a)+f(b)]dt \\ &\leq \frac{1}{n!}B(n+1,\alpha-n)[f(a)+f(b)] \\ &\leq \frac{\Gamma(\alpha-n)}{\Gamma(\alpha+1)}[f(a)+f(b)]. \end{aligned}$$

So we get

$$\frac{\Gamma(\alpha+1)}{(b-a)^\alpha\Gamma(\alpha-n)}[(I^a_\alpha f)(b)+({}^bI_\alpha f)(a)] \leq f(a)+f(b).$$

The proof is completed.

Remark 2.1 *In Theorem 2.1, if we take $\alpha=n+1$, then the inequality (2.1) becomes inequality (1.2) and we don't suppose that f is a positive function which is required in Theorem 1.1.*

3 Extensions of Hermite-Hadamard Inequalities

Theorem 3.1 *Let $f:[a,b]\to\mathbb{R}$ be a twice differentiable function with $a<b$ and $f\in L_1[a,b]$. If f'' is bounded in $[a,b]$, then we have*

$$\begin{aligned} &\frac{m\Gamma(\alpha+1)}{2(b-a)^\alpha\Gamma(\alpha-n)n!}\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-x\right)^2 \\ &\times[(b-x)^n(x-a)^{\alpha-n-1}+(x-a)^n(b-x)^{\alpha-n-1}]dx \\ \leq\ &\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha\Gamma(\alpha-n)}[(I^a_\alpha f)(b)+({}^bI_\alpha f)(a)]-f\left(\frac{a+b}{2}\right) \qquad (3.1) \\ \leq\ &\frac{M\Gamma(\alpha+1)}{2(b-a)^\alpha\Gamma(\alpha-n)n!}\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-x\right)^2, \\ &\times[(b-x)^n(x-a)^{\alpha-n-1}+(x-a)^n(b-x)^{\alpha-n-1}]dx \end{aligned}$$

and

$$\begin{aligned}
&\frac{-M\Gamma(\alpha+1)}{2(b-a)^{\alpha}\Gamma(\alpha-n)n!}\int_{a}^{\frac{a+b}{2}}(x-a)(b-x)\\
&\times[(b-x)^{n}(x-a)^{\alpha-n-1}+(x-a)^{n}(b-x)^{\alpha-n-1}]dx\\
\leq\ &\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}\Gamma(\alpha-n)}[(I_{\alpha}^{a}f)(b)+\ (^{b}I_{\alpha}f)(a)]-\frac{f(a)+f(b)}{2} \qquad (3.2)\\
\leq\ &\frac{-m\Gamma(\alpha+1)}{2(b-a)^{\alpha}\Gamma(\alpha-n)n!}\int_{a}^{\frac{a+b}{2}}(x-a)(b-x)\\
&\times[(b-x)^{n}(x-a)^{\alpha-n-1}+(x-a)^{n}(b-x)^{\alpha-n-1}]dx
\end{aligned}$$

with $\alpha \in (n, n+1]$*, where* $m = inf_{t\in[a,b]}f^{''}(t),\ M = sup_{t\in[a,b]}f^{''}(t)$.

Proof. We first prove (3.1).

$$\begin{aligned}
&\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}\Gamma(\alpha-n)}[I_{\alpha}^{a}f(b)+\ ^{b}I_{\alpha}f(a)]\\
=\ &\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}\Gamma(\alpha-n)}\Big[\frac{1}{n!}\int_{a}^{b}(b-x)^{n}(x-a)^{\alpha-n-1}f(x)dx\\
&+\frac{1}{n!}\int_{a}^{b}(x-a)^{n}(b-x)^{\alpha-n-1}f(x)dx\Big]\\
=\ &\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}\Gamma(\alpha-n)n!}\\
&\times\int_{a}^{b}f(x)[(b-x)^{n}(x-a)^{\alpha-n-1}+(x-a)^{n}(b-x)^{\alpha-n-1}]dx\\
=\ &\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}\Gamma(\alpha-n)n!}\\
&\times\int_{a}^{b}f(a+b-x)[(b-x)^{n}(x-a)^{\alpha-n-1}+(x-a)^{n}(b-x)^{\alpha-n-1}]dx
\end{aligned}$$

So

$$\begin{aligned}
&\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}\Gamma(\alpha-n)}[I_{\alpha}^{a}f(b)+\ ^{b}I_{\alpha}f(a)]\\
=\ &\frac{\Gamma(\alpha+1)}{4(b-a)^{\alpha}\Gamma(\alpha-n)n!}\int_{a}^{b}[f(x)+f(a+b-x)]\\
&\times[(b-x)^{n}(x-a)^{\alpha-n-1}+(x-a)^{n}(b-x)^{\alpha-n-1}]dx \qquad (3.3)
\end{aligned}$$

Then, we obtain

$$\begin{aligned}
&\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}\Gamma(\alpha-n)}[(I_{\alpha}^{a}f)(b)+\ ({}^{b}I_{\alpha}f)(a)]-f\left(\frac{a+b}{2}\right)\\
=\ &\frac{\Gamma(\alpha+1)}{4(b-a)^{\alpha}\Gamma(\alpha-n)n!}\int_{a}^{b}\left[f(x)+f(a+b-x)-2f\left(\frac{a+b}{2}\right)\right]\\
&\times[(b-x)^{n}(x-a)^{\alpha-n-1}+(x-a)^{n}(b-x)^{\alpha-n-1}]dx.
\end{aligned}$$

Since

$$\left[f(x)+f(a+b-x)-2f\left(\frac{a+b}{2}\right)\right][(b-x)^{n}(x-a)^{\alpha-n-1}+(x-a)^{n}(b-x)^{\alpha-n-1}]$$

is symmetric about $x=\frac{a+b}{2}$, we have

$$\begin{aligned}
&\frac{\Gamma(\alpha+1)}{4(b-a)^{\alpha}\Gamma(\alpha-n)n!}\int_{a}^{b}\left[f(x)+f(a+b-x)-2f\left(\frac{a+b}{2}\right)\right]\\
&\times[(b-x)^{n}(x-a)^{\alpha-n-1}+(x-a)^{n}(b-x)^{\alpha-n-1}]dx\\
=\ &\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}\Gamma(\alpha-n)n!}\int_{a}^{\frac{a+b}{2}}\left[f(x)+f(a+b-x)-2f\left(\frac{a+b}{2}\right)\right]\\
&\times[(b-x)^{n}(x-a)^{\alpha-n-1}+(x-a)^{n}(b-x)^{\alpha-n-1}]dx.
\end{aligned}$$

As consequence, we get

$$\begin{aligned}
&\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}\Gamma(\alpha-n)}[(I_{\alpha}^{a}f)(b)+\ ({}^{b}I_{\alpha}f)(a)]-f\left(\frac{a+b}{2}\right)\\
=\ &\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}\Gamma(\alpha-n)n!}\int_{a}^{\frac{a+b}{2}}\left[f(x)+f(a+b-x)-2f\left(\frac{a+b}{2}\right)\right]\\
&\times[(b-x)^{n}(x-a)^{\alpha-n-1}+(x-a)^{n}(b-x)^{\alpha-n-1}]dx. \qquad (3.4)
\end{aligned}$$

Since

$$f(a+b-x)-f\left(\frac{a+b}{2}\right)=\int_{\frac{a+b}{2}}^{a+b-x}f^{'}(t)dt$$

and

$$f\left(\frac{a+b}{2}\right)-f(x)=\int_{x}^{\frac{a+b}{2}}f^{'}(t)dt,$$

we obtain

$$
\begin{aligned}
f(x)+f(a+b-x)-2f\left(\frac{a+b}{2}\right) &= \int_{\frac{a+b}{2}}^{a+b-x} f^{'}(t)dt-\int_{x}^{\frac{a+b}{2}} f^{'}(t)dt \\
&= \int_{x}^{\frac{a+b}{2}} f^{'}(a+b-t)dt-\int_{x}^{\frac{a+b}{2}} f^{'}(t)dt \\
&= \int_{x}^{\frac{a+b}{2}} [f^{'}(a+b-t)-f^{'}(t)]dt. \quad (3.5)
\end{aligned}
$$

Since

$$
f^{'}(a+b-t)-f^{'}(t)=\int_{t}^{a+b-t} f^{''}(y)dy,
$$

then for $t\in\left[a,\frac{a+b}{2}\right]$, we get

$$
m(a+b-2t)\leq f^{'}(a+b-t)-f^{'}(t)\leq M(a+b-2t).
$$

So

$$
\begin{aligned}
\int_{x}^{\frac{a+b}{2}} m(a+b-2t)dt &\leq f(x)+f(a+b-x)-2f(\frac{a+b}{2}) \\
&\leq \int_{x}^{\frac{a+b}{2}} M(a+b-2t)dt.
\end{aligned}
$$

Hence, we get

$$
\begin{aligned}
m\left(\frac{a+b}{2}-x\right)^2 &\leq f(x)+f(a+b-x)-2f\left(\frac{a+b}{2}\right) \\
&\leq M\left(\frac{a+b}{2}-x\right)^2.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\frac{m\Gamma(\alpha+1)}{2(b-a)^{\alpha}\Gamma(\alpha-n)n!}\int_{a}^{\frac{a+b}{2}}\left(\frac{a+b}{2}-x\right)^2 \\
&\times[(b-x)^n(x-a)^{\alpha-n-1}+(x-a)^n(b-x)^{\alpha-n-1}]dx \\
\leq\ &\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}\Gamma(\alpha-n)}[(I_{\alpha}^{a}f)(b)+({}^{b}I_{\alpha}f)(a)]-f\left(\frac{a+b}{2}\right) \\
\leq\ &\frac{M\Gamma(\alpha+1)}{2(b-a)^{\alpha}\Gamma(\alpha-n)n!}\int_{a}^{\frac{a+b}{2}}\left(\frac{a+b}{2}-x\right)^2 \\
&\times[(b-x)^n(x-a)^{\alpha-n-1}+(x-a)^n(b-x)^{\alpha-n-1}]dx,
\end{aligned}
$$

which completes the proof of (3.1).

Now we prove second inequality. From (3.3), we have

$$\begin{aligned}
&\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}\Gamma(\alpha-n)n!}[(I^{a}_{\alpha}f)(b)+\ ({}^{b}I_{\alpha}f)(a)]-\frac{f(a)+f(b)}{2}\\
=\ &\frac{\Gamma(\alpha+1)}{4(b-a)^{\alpha}\Gamma(\alpha-n)n!}\int_{a}^{b}[f(x)+f(a+b-x)-(f(a)+f(b))]\\
&\times[(b-x)^{n}(x-a)^{\alpha-n-1}+(x-a)^{n}(b-x)^{\alpha-n-1}]dx.
\end{aligned}$$

Using

$$[f(x)+f(a+b-x)-(f(a)+f(b))]\,[(b-x)^{n}(x-a)^{\alpha-n-1}+(x-a)^{n}(b-x)^{\alpha-n-1}]$$

is symetric about $x=\frac{a+b}{2}$, we get

$$\begin{aligned}
&\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}\Gamma(\alpha-n)n!}[(I^{a}_{\alpha}f)(b)+\ ({}^{b}I_{\alpha}f)(a)]-\frac{f(a)+f(b)}{2}\\
=\ &\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}\Gamma(\alpha-n)n!}\int_{a}^{\frac{a+b}{2}}[f(x)+f(a+b-x)-(f(a)+f(b))]\\
&\times[(b-x)^{n}(x-a)^{\alpha-n-1}+(x-a)^{n}(b-x)^{\alpha-n-1}]dx. \qquad (3.6)
\end{aligned}$$

Since

$$f(b)-f(a+b-x)=\int_{a+b-x}^{b}f^{'}(t)dt$$

and

$$f(x)-f(a)=\int_{a}^{x}f^{'}(t)dt,$$

we obtain

$$\begin{aligned}
&f(x)+f(a+b-x)-(f(a)+f(b))\\
=\ &\int_{a}^{x}f^{'}(t)dt-\int_{a+b-x}^{b}f^{'}(t)dt\\
=\ &\int_{a}^{x}f^{'}(t)dt-\int_{a}^{x}f^{'}(a+b-t)dt\\
=\ &-\int_{a}^{x}[f^{'}(a+b-t)-f^{'}(t)]dt. \qquad (3.7)
\end{aligned}$$

We also have

$$f^{'}(a+b-t)-f^{'}(t)=\int_{t}^{a+b-t}f^{''}(y)dy.$$

Then for $t \in [a, \frac{a+b}{2}]$, we get

$$m(a+b-2t) \leq f^{'}(a+b-t) - f^{'}(t) \leq M(a+b-2t)$$

Hence

$$\begin{aligned}
-\int_a^x M(a+b-2t)dt &\leq f(x) + f(a+b-x) - (f(a)+f(b)) \\
&\leq -\int_a^x m(a+b-2t)dt.
\end{aligned}$$

That is,

$$\begin{aligned}
-M(x-a)(b-x) &\leq f(x) + f(a+b-x) - (f(a)+f(b)) \\
&\leq -m(x-a)(b-x)
\end{aligned}$$

and

$$\begin{aligned}
& \frac{-M\Gamma(\alpha+1)}{2(b-a)^\alpha \Gamma(\alpha-n)n!} \int_a^{\frac{a+b}{2}} (x-a)(b-x) \\
& \times[(b-x)^n(x-a)^{\alpha-n-1} + (x-a)^n(b-x)^{\alpha-n-1}]dx \\
\leq \; & \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha \Gamma(\alpha-n)}[(I_\alpha^a f)(b) + ({}^bI_\alpha f)(a)] - \frac{f(a)+f(b)}{2} \\
\leq \; & \frac{-m\Gamma(\alpha+1)}{2(b-a)^\alpha \Gamma(\alpha-n)n!} \int_a^{\frac{a+b}{2}} (x-a)(b-x) \\
& \times[(b-x)^n(x-a)^{\alpha-n-1} + (x-a)^n(b-x)^{\alpha-n-1}]dx,
\end{aligned}$$

we have completed the proof.

Remark 3.1 *If the function $f : [a,b] \to \mathbb{R}$ is differentiable with a nondecreasing derivative, then f is convex. In particular, if f is twice differentiable and $f^{''} \geq 0$, then the function is convex. In Theorem (3.1), if $f^{''} \geq 0$, then we obtain inequality (2.1). Moreover if $f^{''} \geq 0$, $\alpha = n+1$ and $n = 0$, we obtain inequality (1.1).*

It is obvious that $f^{''} \geq 0$ impies that $f^{'}$ non-decreasing. Therefore

$$f^{'}(a+b-x) \geq f^{'}(x). \tag{3.8}$$

is holds for all $x \in [a, \frac{a+b}{2}]$. So, we establish the following theorem using inequality of (3.8)

Theorem 3.2 *Let $f : [a,b] \to \mathbb{R}$ be a positive, differentiable function with $a < b$ and $f \in L_1[a,b]$. If $f^{'}(a+b-x) \geq f^{'}(x)$ for all $x \in [a, \frac{a+b}{2}]$. Then the following inequalities for fractional integrals hold*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}\Gamma(\alpha-n)}[(I_{\alpha}^{a}f)(b) + ({}^{b}I_{\alpha}f)(a)] \leq \frac{f(a)+f(b)}{2}.$$

Proof. From (3.4) and (3.5), one has

$$\begin{aligned} & \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}\Gamma(\alpha-n)}[I_{\alpha}^{a}f(b) + {}^{b}I_{\alpha}f(a)] - f\left(\frac{a+b}{2}\right) \\ = & \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}\Gamma(\alpha-n)n!}\int_{a}^{\frac{a+b}{2}}\left[f(x)+f(a+b-x)-2f\left(\frac{a+b}{2}\right)\right] \\ & \times[(b-x)^{n}(x-a)^{\alpha-n-1} + (x-a)^{n}(b-x)^{\alpha-n-1}]dx \\ = & \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}\Gamma(\alpha-n)n!}\int_{a}^{\frac{a+b}{2}}\left[\int_{a}^{\frac{a+b}{2}}[f^{'}(a+b-t)-f^{'}(t)]dt\right] \\ & \times[(b-x)^{n}(x-a)^{\alpha-n-1} + (x-a)^{n}(b-x)^{\alpha-n-1}]dx \\ \geq & 0. \end{aligned}$$

Similarly, from (3.6) and (3.7), one gets

$$\begin{aligned} & \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}\Gamma(\alpha-n)n!}[(I_{\alpha}^{a}f)(b) + ({}^{b}I_{\alpha}f)(a)] - \frac{f(a)+f(b)}{2} \\ = & \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}\Gamma(\alpha-n)n!}\int_{a}^{\frac{a+b}{2}}\left[-\int_{a}^{x}[f^{'}(a+b-t)-f^{'}(t)]dt\right] \\ & \times[(b-x)^{n}(x-a)^{\alpha-n-1} + (x-a)^{n}(b-x)^{\alpha-n-1}]dx \\ \leq & 0. \end{aligned}$$

We have completed the proof.

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