A direct approach with computerized symbolic computation for finding a series of traveling waves to nonlinear equations

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Abstract

A direct approach with computerized symbolic computation is applied to construct a series of traveling wave solutions for nonlinear equations. Compared with most existing symbolic computation methods such as tanh method and Jacobi function method, the proposed method not only gives new and more general solutions, but also provides a guideline to classify the various types of the solution according to some parameters.

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1. Introduction

The tanh method is considered to be one of most straightforwa...
The next crucial step is that we seek the following polynomial solutions of Eq. (1.2), as

\[ u(x,t) = U(z) = \sum_{i=0}^{n} a_i \tanh^i z. \]  

Doing so, we may take advantage of the property that the derivative of \( \tanh z \) is polynomial in \( \tanh z \), i.e. \((\tanh z)' = 1 - \tanh^2 z\). The positive integer \( n \) is determined by balancing the highest order linear term with the nonlinear terms. By substituting (1.3) into Eq. (1.2) and setting all coefficients of powers of \( \tanh z \) to zeros, we obtain a system of algebraic equations from which the parameters \( a_i \) and \( d \) are explicitly obtained. In the fact, viewed as a special case of PDEs, the method readily applies to ODEs.

Recently much research work has been concentrated on the various extensions and applications of the \( \tanh \) method [3–13]. The basic purpose of these papers is to simplify the routine calculation of the method and to find more general traveling wave solutions. Parkes and Duffy mentioned the difficulty of using the \( \tanh \) method by hand for anything but simple PDEs. Therefore, they automated to some degree the \( \tanh \) method using symbolic computation software \textit{Mathematica} [5,6]. Recently, a new algorithm was given by Baldwin, Göktas, Hereman et al. to compute polynomial solutions of nonlinear equations in terms of the Jacobi elliptic functions [13]. We developed a new direct algebraic method which greatly exceeds the applicability of the existing \( \tanh \) methods and Jacobi function method in obtaining a series of traveling wave solutions of nonlinear equations [14].

The aim of this paper is to further extend our above method and provide a specific computer program to handle the proposed method. Our paper is organized as follows. In the following Section 2, an ODE is introduced and several series of fundamental solutions are presented. The detail derivation of the proposed method will be given in Section 3. In Section 4, the applications of the proposed method to nonlinear equations are illustrated. In Section 5, two simple computer programs are provided to solve nonlinear equations.

2. An ODE and its series solutions

We introduce the following ODE

\[ w' = k \sqrt{c_0 + c_1 w + c_2 w^2 + c_3 w^3 + c_4 w^4}, \]  

where \( k = \pm 1 \) and the prime ‘ denotes \( d/dz \). By considering the different values of \( c_0, c_1, c_2, c_3 \) and \( c_4 \), we find that Eq. (2.1) admits many kinds of fundamental solutions which are listed as follows.

Case A. If \( c_3 = c_4 = 0 \), Eq. (2.1) possesses:

two polynomial type solutions

\[ w = k \sqrt{c_0 z}, \quad c_1 = c_2 = 0, \quad c_0 > 0, \]  

and

\[ w = \frac{c_0}{c_1} + \frac{1}{4} c_1 z^2, \quad c_2 = 0, \quad c_1 \neq 0, \]  

a exponential type solution

\[ w = -\frac{c_1}{2c_2} + \exp(k \sqrt{c_2} z), \quad c_0 = \frac{c_1^2}{4c_2}, \quad c_2 > 0, \]  

a triangular type solution

\[ w = \frac{c_1}{2c_2} + \frac{kc_1}{2c_2} \sin(\sqrt{-c_2} z), \quad c_0 = 0, \quad c_2 < 0, \]  

and a hyperbolic type solution

\[ w = -\frac{c_1}{2c_2} + \frac{k c_1}{2c_2} \sinh(2\sqrt{c_2^2}z), \quad c_0 = 0, \quad c_2 > 0. \tag{2.6} \]

Case B. If \( c_3 = c_1 = 0 \), Eq. (2.1) admits:

a bell shaped solitary wave solution

\[ w = \sqrt{-\frac{c_2}{c_4}} \text{sech}(\sqrt{c_2^2}z), \quad c_0 = 0, \quad c_2 > 0, \quad c_4 < 0, \tag{2.7} \]

a kink shaped solitary wave solution

\[ w = k \sqrt{-\frac{c_2}{2c_4}} \tanh\left( \sqrt{-\frac{c_2^2}{2}}z \right), \quad c_0 = \frac{c_2^2}{4c_4}, \quad c_2 < 0, \quad c_4 > 0, \tag{2.8} \]

two triangular type solutions

\[ w = \sqrt{-\frac{c_2}{c_4}} \sec(\sqrt{-c_2^2}z), \quad c_0 = 0, \quad c_2 < 0, \quad c_4 > 0, \tag{2.9} \]

and

\[ w = k \sqrt{-\frac{c_2}{c_4}} \tan\left( \sqrt{-\frac{c_2}{2}}z \right), \quad c_0 = \frac{c_2^2}{4c_4}, \quad c_2 > 0, \quad c_4 > 0, \tag{2.10} \]

a rational type solution

\[ w = -\frac{k}{\sqrt{c_4}z}, \quad c_0 = c_2 = 0, \quad c_4 > 0, \tag{2.11} \]

three Jacobi elliptic doubly periodic type solutions

\[ w = \sqrt{-\frac{c_2m^2}{c_4(2m^2-1)}} \cn\left( \sqrt{\frac{c_2}{2m^2-1}}z \right), \quad c_4 < 0, \quad c_2 > 0, \quad c_0 = \frac{c_2^2m^2(1-m^2)}{c_4(2m^2-1)^2}, \tag{2.12} \]

\[ w = \frac{m^2}{c_4(2-m^2) \dn\left( \sqrt{\frac{c_2}{2-m^2}}z \right)}, \quad c_4 > 0, \quad c_2 > 0, \quad c_0 = \frac{c_2^2(1-m^2)}{c_4(2-m^2)^2}, \tag{2.13} \]

and

\[ w = k \sqrt{-\frac{c_2m^2}{c_4(m^2+1)}} \sn\left( \sqrt{\frac{c_2}{m^2+1}}z \right), \quad c_4 > 0, \quad c_2 < 0, \quad c_0 = \frac{c_2^2m^2}{c_4(m^2+1)^2}, \tag{2.14} \]

where \( m \) is a modulus. The Jacobi elliptic functions are doubly periodical and possess properties of triangular functions:

\[ \sn^2 z + \cn^2 z = 1, \quad \dn^2 z = 1 - m^2 \sn^2 z, \]

\( (\sn z)' = \cn z \dn z, \quad (\cn z)' = -\sn z \dn z, \quad (\dn z)' = -m^2 \sn z \cn z. \)

When \( m \to 1 \), the Jacobi functions degenerate to the hyperbolic functions, i.e.

\[ \sn z \to \tanh z, \quad \cn z \to \sech z. \]

When \( m \to 0 \), the Jacobi functions degenerate to the triangular functions, i.e.

\[ \sn z \to \sin z, \quad \cn z \to \cos z. \]
The more detailed notations for the Weierstrass and Jacobi elliptic functions can be found in Refs. [15,16]. As \( m \to 1 \), the Jacobi doubly periodic solutions (2.12) and (2.13) degenerate to the solitary wave solutions (2.7), and the solution (2.14) degenerates to (2.8).

**Case C.** If \( c_4 = 0 \), Eq. (2.1) admits:

a bell shaped solitary wave solution
\[
w = -\frac{c_2}{c_3} \text{sech}^2 \left( \frac{\sqrt{-c_2 z}}{2} \right), \quad c_0 = c_1 = 0, \quad c_2 > 0, \tag{2.15}\]
a triangular type solution
\[
w = -\frac{c_2}{c_3} \text{sec}^2 \left( \frac{\sqrt{-c_2 z}}{2} \right), \quad c_0 = c_1 = 0, \quad c_2 < 0, \tag{2.16}\]
a rational type solution
\[
w = \frac{1}{c_3 c_2}, \quad c_0 = c_1 = c_2 = 0, \tag{2.17}\]
and a Weierstrass elliptic doubly periodic type solution
\[
w = \wp \left( \frac{\sqrt{c_3}}{2} z, g_2, g_3 \right), \quad c_2 = 0, \quad c_3 > 0, \tag{2.18}\]
where \( g_2 = -4c_1/c_3 \), and \( g_3 = -4c_0/c_3 \) are called invariants of Weierstrass elliptic function.

**Case D.** If \( c_0 = c_1 = 0 \), Eq. (2.1) admits:

a triangular type solution
\[
w = -\frac{c_2}{c_3} \text{sec}^2 \left( \frac{1}{2} \sqrt{-c_2 z} \right) \frac{\sqrt{-c_2 c_4}}{2k \sqrt{c_2 c_4}} \tan \left( \frac{1}{2} \sqrt{-c_2 c_4} \right) + c_3, \quad c_2 < 0, \tag{2.19}\]
and a solitary wave solution
\[
w = \frac{c_2}{2k \sqrt{c_2 c_4}} \left( \frac{1}{2} \sqrt{-c_2 c_4} \right) 1 + \tanh \left( \frac{1}{2} \sqrt{-c_2 c_4} \right) - c_3, \quad c_2 > 0. \tag{2.20}\]

In the case when \( c_4 = 0 \), the solution (2.19) and (2.20) degenerate to the solution (2.15) and (2.16), respectively. As \( c_3 = 2k \sqrt{c_2 c_4} \), the solution (2.20) degenerates to the following solution
\[
w = \frac{1}{2k} \sqrt{\frac{c_2}{c_4}} \left[ 1 + \tanh \left( \frac{1}{2} \sqrt{-c_2 c_4} \right) \right], \tag{2.21}\]
which is the same kind of solution with (2.8).

Compared with the results in Ref. [14], here we further find some new solutions (2.2)–(2.6) and (2.19), (2.20).

### 3. Instruction for the proposed algorithm

In this section, we outline the main steps of our method. The key idea of our method is to take full advantage of an ODE and use its solutions to replace tanh function in the tanh method, which simply proceeds as follows:

**Step 1.** Similar to the tanh method, by using the wave transformation \( u(x,t) = U(z) \), \( z = x + dt \), Eq. (1.1) is reduced to Eq. (1.2). We introduce a new variable \( w = w(z) \), which is a solution of the following ODE
\[
w' = k \sum_{j=0}^{r} c_j w^j, \tag{3.1}\]
Then the derivatives with respect to the variable $z$ become the derivatives with respect to the variable $w$ as:

$$\frac{d}{dz} \rightarrow k \sqrt{\sum_{j=0}^{r} c_j w^j} \frac{d}{dw},$$

(3.2)

$$\frac{d^2}{dz^2} \rightarrow \frac{1}{2} \sum_{j=1}^{r} j c_j w^{j-1} \frac{d}{dw} + \sum_{j=0}^{r} c_j w^j \frac{d^2}{dw^2}, \quad \ldots$$

(3.3)

**Step 2.** By using the new variable $w$, we expand the solution of Eq. (1.2) as

$$u(x,t) = U(z) = \sum_{i=0}^{n} a_i w^i.$$  

(3.4)

In order to determine $n$ and $r$, we may substitute (3.1) into (1.2) and balance the highest derivative term with the nonlinear terms in Eq. (1.2) by using (3.2) and (3.3), we can obtain a relation for $n$ and $r$, from which the different possible values of $n$ and $r$ can be obtained. These values lead to the series expansions of the traveling wave solutions for Eq. (1.1). For example, in the case of KdV equation

$$u_t + 6uu_x + u_{xxx} = 0,$$

we have

$$r = n + 2.$$  

(3.5)

If we take $n = 1$ and $r = 3$ in (3.5), we may use the following series expansion as a solution of KdV equation

$$u = a_0 + a_1 w, \quad w' = k \sqrt{c_0 + c_1 w + c_2 w^2 + c_3 w^3}.$$  

Similarly, if we take $n = 2, r = 4$ in (3.5), we have

$$u = a_0 + a_1 w + a_2 w^2, \quad w' = k \sqrt{c_0 + c_1 w + c_2 w^2 + c_3 w^3 + c_4 w^4}.$$  

**Step 3.** Substituting the expansion (3.4) into Eq. (1.2) and setting the coefficients of all powers like $w^i$ and $w^j \sqrt{\sum_{j=0}^{r} c_j w^j}$ to zero, we will get a system of algebraic equations, from which the constants $d, a_i, c_j$ ($i = 0, 1, \ldots, n, j = 0, 1, \ldots, r$) can be found explicitly.

**Step 4.** From the constants $c_j$ ($j = 0, 1, \ldots, r$) obtained in Step 3 into Eq. (3.1), we can then obtain all the possible solutions. We remark here that the traveling wave solutions of Eq. (3.1) depend on the explicit solvability of Eq. (3.1). The solution of the system of algebraic equations will be getting tedious with the increase of the values of $n$ and $r$. In this case of Eq. (3.1) when $r = 4$, that is, Eq. (2.1) gives a series of fundamental solutions such as polynomial, exponential, solitary wave, rational, triangular periodic, Jacobi and Weierstrass doubly periodic solutions. We consider only the case $r = 4$ in this paper.

**Remark.** As $c_1 = c_3 = 0, c_0 = 1, c_2 = -2, c_4 = 1$, Eq. (2.1) has a solution tanh $z$ and our method reduces to the tanh method [1,2]. As $c_1 = c_3 = 0, c_0 = c_2^2/4, c_4 = 1$, Eq. (2.1) degenerates to a Riccati equation. In this case our proposed method becomes the extended tanh method [11,12]. The cases (2.12)–(2.14) readily cover the results of Jacobi function expansion method [17,18]. In conclusion, our proposed method is a generalization of either the tanh method and Jacobi function method. Producing a system of algebraic and solving it are key procedures and tedious by hand, but they can implemented in a computer with help of a simple computer program in Mathematica.
4. Application to nonlinear equations

In this section, we show how the proposed method established in Section 3 is handled with a simple computer program to solve nonlinear equations.

Example 1. Consider the Calogero KdV equation \[19–21\]

\[u_t + \frac{1}{4}u_{xxx} + \frac{3u_x^3}{8u^2} + \frac{3u_xu_{xx}}{4u} = 0. \quad (4.1)\]

Using the wave transformations \(u = U(z), z = x + dt\), we change Eq. (4.1) to an ODE

\[8dU^2U' + 2U^2U''' + 3U' + 3U'' - 6UU'U'' = 0. \quad (4.2)\]

According to (3.4), we expand the solution of Eq. (4.2) as follows

\[U = \sum_{i=0}^{n} a_i w^i, \quad (4.3)\]

where \(w\) satisfies Eq. (3.1). Balancing the highest derivative terms with nonlinear terms in (4.2) gives

\[3n - 3 + r = 3n - 1 - 2 + r, \quad (4.4)\]

from which we see that \(n\) and \(r\) are arbitrary. We try to take \(r = 4\) and \(n = 2\) and have

\[U = a_0 + a_1 w + a_2 w^2, \quad (4.5)\]

where \(w\) satisfies (2.1).

Substituting (4.5) into (4.2) yields a system of algebraic equations which then gives explicit expressions for parameters \(c, a_0, a_1, a_2, c_0, c_1, c_2, c_3\) and \(c_4\). This process can be implemented in a computer with help of Program 1 in Mathematica (see in Section 5).

From the output of Mathematica, we find two kinds of solutions, namely,

\[c_1 = c_3 = a_1 = 0, \quad c_4 = a_2(c_2 - 2d) \quad \frac{3a_0}{3a_0}, \quad a_2 = \frac{3 + 8d a_0^2 + 8a_0^2 c_2}{12a_0 c_0}, \quad (4.6)\]

with \(a_0, c_0 \neq 0, c_2\) and \(d\) being arbitrary constants,

\[c_4 = a_2 = 0, \quad c_3 = \frac{a_1(c_2 - 8d)}{3a_0}, \quad c_1 = \frac{3 + 8ca_0^2 + 3a_0^2 c_0 + 2a_0^2 c_2}{3a_0 a_1}, \quad (4.7)\]

with \(a_0, a_1 \neq 0, c_0, c_2\) and \(d\) being arbitrary constants.

Since \(c_1 = c_3 = 0\) in (4.6), by using (2.7), (2.9), (2.11) and (2.12), we obtain four kinds of solutions, namely,

- a solitary wave solution

\[u_1 = a_0 - \frac{3a_0 c_2}{c_2 - 2d} \text{sech}^2\left(\sqrt{c_2}z\right), \quad c_2 > 0, \]

- a triangular periodic solution

\[u_2 = a_0 - \frac{3a_0 c_2}{c_2 - 2d} \sec^2\left(\sqrt{-c_2}z\right), \quad c_2 < 0, \]
a rational solution

\[ u_3 = a_0 - \frac{3a_0}{2d} \frac{1}{z^2}, \]

and a Jacobi doubly periodic solution

\[ u_4 = a_0 - \frac{3m^2a_0c_2}{(2m^2 - 1)(c_2 - 2d)} \cn^2 \left( \frac{c_2}{2m^2 - 1}z \right), \]

where \( z = x + dt \).

As \( m \to 1 \), the Jacobi periodic solution \( u_4 \) degenerates to the solitary wave solution \( u_1 \). Setting \( c_0 = 0 \) in (4.7), then by (2.15)–(2.17), the obtained solutions are the same with \( u_1, u_2 \) and \( u_3 \) by using the transformations \( c_2 \to 4c_2 \) and \( a_1 \to a_2 \).

From (2.18) and (4.7), we get a Weierstrass periodic solution

\[ u_5 = a_0 + a_1 \wp \left( \sqrt{-\frac{2a_1d}{3a_0}} z, g_2, g_3 \right), \]

where \( z = x + dt \) and

\[ g_2 = \frac{3 + 3a_1^2c_0 + 8da_0^2}{2a_1^2d}, \quad g_3 = \frac{3a_0c_0}{2a_1d}. \]

We take \( d = c_2/8 \) such that \( c_3 = 0 \) in (4.7). In this way, (4.7) becomes

\[ c_3 = c_4 = a_2 = 0, \quad c_1 = \frac{1 + a_1^2c_0 + a_2^2c_2}{a_0a_1}. \quad (4.8) \]

If we restrict \( c_0 = c_1^2/(4c_2) \) in (4.8), by using (2.4) we obtain an exponential type solution as follows

\[ u_6 = a_0 - \frac{c_1}{2c_2} + a_1 e^{\pm \sqrt{c_2}z}, \]

where \( z = x + c_2t/8 \), and \( 4c_2(a_0a_1c_1 - a_0^2c_2 - 1) - a_1^2c_1^2 = 0 \).

If we restrict \( c_0 = 0 \) in (4.8), by using (2.10) and (2.12) we obtain a triangular solution

\[ u_7 = a_0 - \frac{1 + a_0^2c_2}{2a_0a_1c_2} \left[ 1 \pm \sin \left( \sqrt{-c_2}z \right) \right], \quad c_2 < 0, \]

and a hyperbolic solution

\[ u_8 = a_0 - \frac{1 + a_0^2c_2}{2a_0a_1c_2} \left[ 1 \pm \sinh \left( 2\sqrt{c_2}z \right) \right], \quad c_2 > 0, \]

where \( z = x + c_2t/8 \). The \( u_6 \) and \( u_8 \) are non-localized solutions.

In the fact, Program 1 presented here also can applied to solve other equations or systems. What we only need to do is to replace the \( n \) and \( f \) by new ones. Let’s consider the following system.

**Example 2.** The variant Boussinesq system

\[ h_t + (hu)_x + u_{xxx} = 0, \]

\[ u_t + h_x + uu_x = 0, \quad (4.9) \]

was introduced as a model of water waves [22]. Its Painlevé property, solitary wave solutions, symmetries and conservation laws have been obtained [23–25]. The proposed method will gives a series of new traveling wave solutions for the the equation.
Using transformation $h = H(z), u = U(z), z = x + dt$, we reduce Eq. (4.9) to a system of ODEs
\[ dH' + (HU)' + U''' = 0, \]
\[ dU' + H' + U'' = 0. \]  \hspace{1cm} (4.10)

According to the proposed method, we expand the solutions of Eq. (4.10) as
\[ U = \sum_{i=0}^{n_1} a_i w^i(z), \quad H = \sum_{j=0}^{n_2} b_j w^j(z), \]  \hspace{1cm} (4.11)
where we $w$ satisfies (3.1). Balancing the highest derivative terms with nonlinear terms in (4.10) gives
\[ n_2 = 2n_1, \quad r = n_2 + 2. \]  \hspace{1cm} (4.12)

Therefore we may choose $n_1 = 1, n_2 = 2, r = 4$ and have
\[ U = a_0 + a_1 w, \quad H = b_0 + b_1 w + b_2 w^2, \]  \hspace{1cm} (4.13)
where $w$ satisfies (2.1). From output of Program 2 (see Section 5), we find a solution, namely,
\[ c_1 = 0, \quad a_0 = -d - \frac{b_1}{a_1}, \quad b_0 = -c_2 + \frac{b_1^2}{a_1^2}, \quad b_2 = -\frac{1}{2}a_1^2, \quad c_3 = -b_1, \quad c_4 = \frac{1}{4}a_1^2, \]  \hspace{1cm} (4.14)
where $a_1 \neq 0, c_0, c_1, c_2$ and $d$ are arbitrary constants.

We let $b_1 = 0$ such that $c_3 = 0$ in (4.14). Since $c_4 > 0, c_1 = c_3 = 0$, by using (2.9)–(2.11) and (2.14), we obtain four kinds of solutions, namely, two triangular solutions
\[ u_1 = -d + 2\sqrt{-c_2} \sec(\sqrt{-c_2} z), \quad h_1 = -c_2 + 2c_2 \sec^2(\sqrt{-c_2} z), \quad c_2 < 0, \]
and
\[ u_2 = -d \pm \sqrt{2c_2} \tan(\sqrt{c_2/2} z), \quad h_2 = -c_2 - c_2 \tan^2(\sqrt{c_2/2} z), \quad c_2 > 0, \]
a solitary wave solution
\[ u_3 = -d \pm \sqrt{-2c_2} \tanh(\sqrt{-c_2/2} z), \quad h_3 = -c_2 + c_2 \tanh^2(\sqrt{-c_2/2} z), \quad c_2 < 0, \]
a rational solution
\[ u_4 = -d \pm \frac{2}{x + dt}, \quad h_4 = -c_2 - \frac{2}{(x + dt)^2} \]
and a Jacobi doubly periodic solution
\[ u_5 = -d \pm 2\sqrt{-\frac{c_2 m^2}{m^2 + 1}} \sn\left(\sqrt{-\frac{c_2}{m^2 + 1}} z\right), \quad h_5 = -c_2 + 2\frac{c_2 m^2}{m^2 + 1} \sn^2\left(\sqrt{-\frac{c_2}{m^2 + 1}} z\right), \quad c_2 < 0, \]
where \( z = x + dt \). As \( m \to 1 \), the Jacobi periodic solutions \((h_5, v_5)\) degenerate to the solitary wave solution \((h_3, v_3)\).

From (2.19), (2.20) and (4.14), we obtain a triangular solution

\[
\begin{align*}
  u_6 &= -d - \frac{b_1}{a_1} + a_1 w, \\
  h_6 &= \frac{b_1}{a_1^2} - c_2 + b_1 w - \frac{1}{2} a_1^2 w^2, \\
  \varphi &= -\frac{c_2 \sec^2 \left( \frac{1}{2} \sqrt{-c_2} z \right)}{\pm a_1 \sqrt{-c_2} \tan \left( \frac{1}{2} \sqrt{-c_2} z \right) - b_1},
\end{align*}
\]

and a solitary wave solution

\[
\begin{align*}
  u_7 &= -d - \frac{b_1}{a_1} + a_1 w, \\
  h_7 &= \frac{b_1}{a_1^2} - c_2 + b_1 w - \frac{1}{2} a_1^2 w^2, \\
  \varphi &= \frac{c_2 \text{sech}^2 \left( \frac{1}{2} \sqrt{c_2} z \right)}{\pm a_1 \sqrt{c_2} \tanh \left( \frac{1}{2} \sqrt{c_2} z \right) + b_1}, \quad c_2 > 0,
\end{align*}
\]

where \( z = x + dt \). To our knowledge, these solutions are new and cannot be obtained by tanh method. To show the properties of solitary wave solutions and doubly periodic wave solution. We draw plots for the solutions \((h_3, u_3)\), \((h_5, u_5)\) and \((h_7, u_7)\) (see Figs. 1–3).

In the following, we consider several special equations whose solutions require some kinds of ‘pre-possessing’ techniques.

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**Fig. 1.** The solitary wave solutions \( h_3 \) and \( u_3 \) with parameters \( d = 0.5, c_2 = -0.2 \).

**Fig. 2.** The Jacobi doubly periodic solutions \( h_5 \) and \( u_5 \) with parameters \( d = 0.5, c_2 = -0.2 \).
Example 3. Consider sine-Gordon equation

\[ u_{xt} = \sin u. \]  \hspace{1cm} (4.15)

Since the sine function in Eq. (4.15) brings some difficulty to apply our method, we let

\[ u = 4 \arctan v, \quad v = V(z), \quad z = x + dt, \]  \hspace{1cm} (4.16)

then

\[ \sin u = \frac{2v(1 - v^2)}{(1 + v^2)^2}. \]  \hspace{1cm} (4.17)

Substituting (4.16) and (4.17) into Eq. (4.15), we obtain

\[ d(1 + V^2)V'' - 2dVV'V^2 + V^3 - V = 0. \]  \hspace{1cm} (4.18)

We expand the solution of Eq. (4.18) as

\[ V = \sum_{i=0}^{n} a_i w^i. \]  \hspace{1cm} (4.19)

Then balancing the \( V^2 V'' \) with \( V V'^2 \) in (4.19), we find that \( n \) and \( r \) are arbitrary. We take \( n = 2, r = 4 \) and have

\[ V = a_0 + a_1 w + a_2 w^2, \]  \hspace{1cm} (4.20)

where \( w \) satisfies Eq. (2.1).

We use Program 1 with replacing \( U \) by \( V \), the equation in \( f \) by (4.18) and \( 3n - 3 + r \) in \( de \) by \( 3n - 2 + r \), then output of Mathematica gives the following solutions

\[ u_1 = 4 \arctan \left\{ a_1 \exp \left( \frac{\pm z}{\sqrt{d}} \right) \right\} \]

and

\[ u_2 = \pm 4 \arctan \left\{ \sqrt{2} \tanh \left( \frac{z}{2\sqrt{-d}} \right) \right\}, \]

where \( z = x + dt \).

Example 4. Schrödinger–Boussinesq equation [26, 27]

\[ iu_t = u_{xx} + uv, \]
\[-v_t + u_{xx} + (v^2)_{xx} - v_{xxxx} = (|u|^2)_{xx}. \] \hspace{1cm} (4.21)
By considering transformations \( u = e^{\theta} U(z), \ v = V(z), \ \theta = px + qt, \ z = x + dt, \) from system (4.20) we obtain the relation \( d = 2p \) and a coupled nonlinear ordinary differential equations
\[
(q - p^2)U + U V + U'' = 0,
(1 - 4p^2)V'' + (V^2)' - (U^2)'' - V''' = 0.
\]
We expand the solution of Eq. (4.16) as
\[
U = a_0 + a_1w + a_2w^2, \quad V = b_0 + b_1w + b_2w^2,
\]
where \( w \) satisfies Eq. (2.1).

We use Program 2 with replacing \( U, H \) by (4.22), equations in \( f, g \) by (4.21) and \( n_1 - 3 + r, 2n_1 - 1 \) by \( n_1 - 2 + r, n_2 - 4 + 2r \), respectively, then from output we may get a series of solutions as follows

**a solitary wave solution**
\[
\begin{align*}
u_1 &= \pm 6\sqrt{2}e^{\theta} \, \text{sech}^2(\sqrt{c_2}z), \\
v_2 &= b_0 - 6c_2 \, \text{sech}^2(\sqrt{c_2}z), \quad c_0 = 0, \quad c_2 > 0,
\end{align*}
\]

**a triangular solution**
\[
\begin{align*}
u_2 &= \pm 6\sqrt{2}e^{\theta} \, \text{sec}^2(\sqrt{-c_2}z), \\
v_2 &= b_0 - 6c_2 \, \text{sec}^2(\sqrt{-c_2}z), \quad c_0 = 0, \quad c_2 < 0,
\end{align*}
\]

**a rational solution**
\[
\begin{align*}
u_3 &= \pm 6\sqrt{2}e^{\theta}, \\
v_3 &= b_0 - \frac{6}{z}, \quad c_0 = c_2 = 0,
\end{align*}
\]

**a Jacobi doubly periodic solution**
\[
\begin{align*}
u_4 &= \pm \sqrt{2}(2c_2 - \delta) \pm \frac{6\sqrt{2}c_2m^2 e^{\theta}}{2m^2 - 1} \text{cn}^2\left(\frac{c_2}{\sqrt{2m^2 - 1}}z\right), \\
v_4 &= b_0 - \frac{6c_2m^2}{2m^2 - 1} \text{cn}^2\left(\frac{c_2}{\sqrt{2m^2 - 1}}z\right), \quad c_0 = c_2(1 - m^2),
\end{align*}
\]
where \( z = x + 2pt, \ \theta = px + qt \) with \( p, q \) and \( \delta \) being given by
\[
p^2 = \frac{1}{4}(1 + 2b_0 + 4c_2 - 4\delta), \quad q = \frac{1}{4}(1 - 2b_0 - 4c_2 - 4\delta), \quad \delta = \sqrt{2b_2c_0 + 4c_2},
\]
and a Weierstrass periodic solution
\[
\begin{align*}
u_5 &= \pm e^{\theta} \left[\sqrt{b_1c_1} - \sqrt{2b_1}\wp(\sqrt{-b_1/6z}, g_2, g_3)\right] \\
v_5 &= b_0 + b_1\wp(\sqrt{-b_1/6z}, g_2, g_3), \quad c_2 = 0, \quad b_1 > 0,
\end{align*}
\]
where \( g_2 = -6c_1/b_1, \ g_3 = -6c_0/b_1, \ z = x + 2pt, \ \theta = px + qt \) with \( p \) and \( q \) being given
\[
p^2 = \frac{1}{4}(1 + 2b_0 + c_2 - 2\delta), \quad q = \frac{1}{4}(1 - 2b_0 - c_2 - 4\delta),
\]
\[
\delta = \sqrt{2b_1c_1 + c_2^2}.
\]
As \( m \to 1 \), the Jacobi periodic solution \((u_4, v_4)\) degenerates to the solitary wave solution \((u_1, v_1)\).
5. Sample programs

*Programming language used:* Mathematica

**Typical running time:** The total run time is from less a second to minutes depending on the complexity of algebraic system. Tested on P3 Toshiba 1800.

**Program 1:**

```mathematica
In[1] := Here is the Mathematica programs of Example 1 run:
In[2] := n = 2
In[3] := r = 4
out[2] := 2
out[3] := 4
In[4] := According to (3.4), we expand the solution of Eq. (4.2) into polynomial form in w satisfying Eq. (3.1). Then balancing the highest derivative terms with nonlinear terms in (4.2) will lead to the relation (4.4), from which we may choose n = 2 and r = 4.
In[5] := U = Sum[a[i]*w[z] / i, { i, 0, n}]
In[6] := Since n = 2, we may assume that the solution U we are looking for is of the following form.
In[7] := Q = (Sum[c[j]*w[z] / j, { j, 0, r}] / (1/2))
In[8] := s = [w[z] -> k*Q]
out[8] := {w[z] -> k*Q}
In[9] := The purpose of this output is to replace the first order derivative of w[z] by k Q in the process of calculation to change Eq. (4.2) into polynomial form in w and Q.
In[10] := f = Collect[Coefficient[Together[Expand[(8*d*U + 2*D[U, z] + 2U + 2*D[D[U, z]] + s*z)/s*y]], { U, z}] /. Q, w[z]]
out[10] := and out[11] := Here the purpose of these two outputs is to get a system of algebraic equations.

\[
\begin{align*}
+ 6k^3a[0]^2a[2]c[1] + 2k^3a[0]^2a[1]c[2], \\
+ 60k^3a[0]a[1]a[2]c[4], \\
\end{align*}
\]
Program 2:

In[1] := Here is the Mathematica programs of Example 2 run:
In[2] := n1 = 1
In[3] := n1 = 2
In[4] := r = 4
out[2] := 1
out[3] := 2
out[4] := 4
In[5] := We expand the solution of Eq. (4.9) in the form (4.11) with $w$ satisfying Eq. (3.1). Then balancing the highest derivative terms with nonlinear terms in (4.10) will lead to the relations (4.12), from which we may choose $n1 = 1, n2 = 2$ and $r = 4$.
In[6] := U = Sum[a[i]*w[z] ∨ i, Financial Table[{i, 0, n1}]]
In[7] := H = Sum[b[j]*w[z] ∨ j, Financial Table[{j, 0, n2}]]
In[8] := Since $n1 = 2$ and $n2 = 2$, we may assume that the solution $U$ we are looking for is the above forms of output.
In[9] := Q = (Sum[c[j]*w[z] ∨ j, Financial Table[{j, 0, r}]] / (1/2))
In[10] := s = (w[z] → k * Q)
In[11] := The purpose of this output is to replace the first order derivative of $w[z]$ by $k Q$ in the process of calculation to change Eq. (4.2) into polynomial form in $w$ and $Q$.
In[12] := f = Collect[Coefficient[
Together[Expand[(d * D[H, z] + D[H * U, z] + D[D[D[U, z], s, z], s, z])/s]], Q, w[z]]
In[13] := g = Collect[Coefficient[
In[14] := sy1 = Append[Coefficient[f, w[z], 0], Coefficient[f, Table[w[z] ∨ i, Financial Table[{i, n1 - 3 + r}]]]]
In[15] := sy2 = Append[Coefficient[g, w[z], 0], Coefficient[g, Table[w[z] ∨ i, Financial Table[{i, 2n1 - 1}]]]]
In[16] := sy = Flatten[sy1, sy2]
out[12] := to out[16] := Here the purpose of these four outputs is to get a system of algebraic equations.
{k[a][1]b[0] + d[k][1][b][1] + k[a][0]b[1][1] + k[3] a[1][c][2],
2 k[a][1]b[1][1] + 2 d[k][b][2][2] + 2 k[a][0][b][2][1] + 3 k[3] a[1][c][3],
3 k[a][1][b][2][1] + 6 k[3] a[1][c][4],
d[k][a][1][1] + k[a][0][a][1][1] + k[b][1],
    k[a][1][1] + 2 k[b][2][1]}
In[17] := var = Flatten[d, Table[b[j], Financial Table[{j, 0, n2}]], Table[c[j], Financial Table[{j, 0, r}]]]
\begin{verbatim}
In[17] := {d, b[0], b[1], b[2], c[0], c[1], c[2], c[3], c[4]}

In[18] := The output gives the parameters to be determined below.

In[19] := Reduce[sy == 0 /. k -> 1, var]

out[19] := The output is solutions of the above system (see the corresponding results (4.14) in Example 2).
\end{verbatim}

In summary, we have applied an unified algebraic method with symbolic computation to construct a series of traveling wave solutions of general nonlinear evolution equations. Except the equations considered in this paper, the proposed method also is readily applicable to a large variety of other nonlinear equations including classical KdV, MdV, Jaulent–Miodek, BBM, modified BBM, Benjamin Ono, Kawachra, variant Boussinesq, Schrödinger, Klein–Gordon, sine-Gordon, sinh-Gordon, (2 + 1)-dimensional KP, (2 + 1)-dimensional Gardner, coupled Schrödinger–KdV and coupled Ito equations.

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