On graphs determining links with maximal number of components via medial construction

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Abstract

Let $G$ be a connected plane graph, $D(G)$ be the corresponding link diagram via medial construction, and $\mu(D(G))$ be the number of components of the link diagram $D(G)$. In this paper, we first provide an elementary proof that $\mu(D(G)) \leq n(G) + 1$, where $n(G)$ is the nullity of $G$. Then we lay emphasis on the extremal graphs, i.e. the graphs with $\mu(D(G)) = n(G) + 1$. An algorithm is given firstly to judge whether a graph is extremal or not, then we prove all extremal graphs can be obtained from $K_1$ by applying two graph operations repeatedly. We also present a dual characterization of extremal graphs and finally we provide a simple criterion on structures of bridgeless extremal graphs.

Key words: plane graph, link diagram, component number, extremal characterization

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1 Introduction

In this paper, the graphs considered allow multiple edges and loops. For any graph $G$, let $p(G), q(G)$ and $k(G)$ be the number of vertices, edges and con-
A graph is planar if it can be embedded in the plane, that is, it can be drawn on the plane so that no two edges intersect. A plane graph is a particular plane embedding of a planar graph. The different embeddings of a planar graph correspond to different plane graphs and they are all isomorphic to the abstract planar graph. Note that the nullity of a connected plane graph is equal to the number of bounded faces of the plane graph according to the well-known Euler formula. A signed graph is a graph with each edge labeled with a sign (+ or −); if it is also a plane graph, we call it a signed plane graph. A graph is said to be trivial if it is an isolated vertex without any edges.

A knot is a simple closed curve in Euclidean 3-space $\mathbb{R}^3$, i.e. an embedding of $S^1$ into $\mathbb{R}^3$. A link is the disjoint union of finite number of knots, each knot is called a component of the link. We denote by $\mu(L)$ the number of components of the link $L$. We take the convention that knot is a one-component link. In classical knot theory, one only considers tame links, that is, we can always think of closed curves as closed polygon curves. Although links live in Euclidean 3-space, we can always represent them by link diagrams, that is, regular projections with a short segment of the underpass curve cut at each double point of the projection.

There is a one-to-one correspondence between link diagrams and signed plane graphs via medial construction. We will give a brief exposition of the correspondence, and for the details and examples, see [1].

Given a non-trivial connected plane graph $G$, its medial graph $M(G)$ is defined as follows, see Chapter 17 of [8]. The vertices of $M(G)$ are the edges of $G$. Each face $F = e_1, \cdots, e_r$ of length $r$ in $G$ determines $r$ edges

$$\{e_i e_{i+1} : 1 \leq i \leq r - 1\} \cup \{e_r e_1\}$$

of $M(G)$. In this definition, a loop $e$ that bounds a face is viewed as a face of length one, and so determines one edge of $M(G)$, which is a loop on $e$. If $G$ has an edge adjacent to a vertex of valency one, then the face containing that edge is viewed as having two consecutive occurrences of $e$ and so once again there is a loop on $e$. If $G$ is trivial, its medial graph is defined to be a simple closed curve surrounding the vertex (strictly, it is not a graph). If a plane graph $G$ is not connected, its medial graph $M(G)$ is defined to be the disjoint union of the medial graphs of its connected components.

Given a signed plane graph $G$, we first draw its medial graph $M(G)$. To turn $M(G)$ into a link diagram $D(G)$, we turn the vertices of $M(G)$ into crossings by defining a crossing to be over or under according to the sign of the edge as shown in Fig. 1. Conversely, given a link diagram $D$, shade it
as in a checkerboard so that the unbounded face is unshaded. Note that a
link diagram can be viewed as a 4-regular plane graph and a 4-regular plane
graph is 2-face-colorable, see Exercise 9.6.1 of [2]. Hence such a shading of D
is always possible. We then associate D with a signed plane graph G(D) as
follows: For each shaded face F, take a vertex v_F, and for each crossing at
which F_1 and F_2 meet, take an edge v_{F_1}v_{F_2} and give the edge a sign also as
shown in Fig. 1.

Fig. 1. The correspondence between a crossing and a signed edge.

The following two facts will be obvious from the correspondence between
signed plane graphs and link diagrams.

(1) The number of components of the link diagram (i.e. the number of com-
ponents of the link it represents) corresponding to a signed plane graph
is irrelevant to the signs of the edges of the graph. Hence, we will neglect
the signs of the signed plane graph later.

(2) A connected plane graph and its dual graph correspond to the same
medial graph, thus the numbers of components corresponding to a plane
graph and its dual graph are the same.

In the figures that appear in the whole paper, we will use solid lines to represent
the edges of plane graphs and dashed lines to represent the curves of their
corresponding link diagrams.

The correspondence between link diagrams and signed plane graphs has been
known for about one hundred years. Indeed, it provides a method of studying
links using graphs. Originally it was used to construct a table of link diagrams
of all links starting with graphs with a relatively small number of edges and
then increasing the number of edges. In the late 1980s, the correspondence
was used to obtain a relation between Jones polynomial [3] in knot theory and

One of the first problems in studying links by using graphs via the corre-
spondence may be determining the number of components of the link diagram
corresponding to a plane graph via parameters of graphs. In this paper, we
restrict ourself to connected plane graphs and study the number of compo-
nents of their corresponding link diagrams. In section 2, we will survey the
known results in this aspect. In section 3, we provide an upper bound for
this number. Then we will lay emphasis on studying the extremal graphs,
i.e. the graphs which reach the upper bound. An algorithm is given to judge
whether a graph is extremal in section 4. We prove all extremal graphs can
be obtained from $K_1$ by applying two graph operations repeatedly in section 5. In section 6, we present a dual characterization of extremal graphs, and in the last section 7, we obtain a theorem, which characterizes the structure of bridgeless extremal graphs. We also obtain some simple necessary conditions for a bridgeless connected plane graph to be extremal in Section 7.

All proofs in the paper require only elementary knowledge of graph theory.

2 Some known results

The number of components of the link diagram corresponding to the plane graph $G$ is also known as the number of straight-ahead walks of the medial graph of $G$ [7], or the number of left-right cycles of the plane graph $G$, see Chapter 17 of [8].

The Tutte polynomial [4] $T_G(x, y)$ of a graph $G$ contains a great deal of information about the graph, see Chapter 10 of [9] for a survey. It also plays an important role in determining the number of components of link diagrams. One has the following result [10]:

**Theorem 2.1** Let $G$ be a connected plane graph, $T_G(x, y)$ be the Tutte polynomial of $G$ and $\mu(D(G))$ be the number of components of the link diagram $D(G)$ corresponding to $G$. Then $T_G(-1, -1) = (-1)^q(G) (-2)^{\mu(D(G))}-1$.

In [11], Mphako studied the number $T_G(-1, -1)$ and obtained the component numbers of link diagrams whose corresponding graphs are fans, wheels and wheels with $q$ consecutive spokes missing. She also studied the component numbers of link diagrams corresponding to 2-sums of graphs.

Another result on $\mu(D(G))$ is related to the Laplacian matrix of the graph $G$. The Laplacian matrix $L(G)$ of a loopless graph $G$ is defined as the matrix $L(G) = D(G) - A(G)$, where $D(G)$ is a diagonal matrix consisting of the degree of vertex $v_i$ of the graph $G$ in its $ii$-th entry, and $A(G)$ is the adjacency matrix of $G$. According to Theorem 17.3.5 and Lemma 14.15.3 of [8], we have:

**Theorem 2.2** Let $G$ be a loopless plane graph, $L(G)$ be the Laplacian matrix of $G$ and $\mu(D(G))$ be the number of components of the link diagram $D(G)$ corresponding to $G$. Then $\mu(D(G))$ is equal to the co-rank of $L(G)$ (over $\mathbb{Z}_2$).

From Theorem 2.1 or 2.2, we can deduce that the number of components of a link formed from a plane graph does not depend on the embedding but depends on the abstract graph.
We also mention that, in [7], Pisanski, Tucker and Žitnik determined the component numbers of the link diagrams corresponding to wheels, prisms and antiprisms (Theorem 1). They also proved that the number of components of a link formed from a plane graph does not depend on the embeddings (Theorem 4). We point out that Theorems 3 and 6 in their paper can be obtained via graphical Reidemeister moves. See Section 2 of [12] for the details of graphical Reidemeister moves.

A pair of edges of \( G \) is called a **parallel pair** if the pair of edges have the same endvertices; a pair of edges of \( G \) is called a **series pair** if it is not a parallel pair and both edges are incident with the same vertex of degree 2. By contracting an edge we mean deleting the edge and identifying its endvertices. Suppose \( e \) is an edge of \( G \). We denote by \( G - e \) and \( G/e \) the plane graph obtained from \( G \) by deleting and contracting the edge \( e \), respectively. The following simple lemma which appeared in [11] and also can be easily obtained via graphical Reidemeister move II will be used in the paper.

**Lemma 2.3** Let \( G \) be a plane graph. If \( e \) and \( f \) are a series pair of \( G \), then \( \mu(D(G/e/f)) = \mu(D(G)) \); and if \( e \) and \( f \) are a parallel pair of \( G \), then \( \mu(D(G - e - f)) = \mu(D(G)) \).

In the next section, we will provide an upper bound for the number of components of links formed from a planar graph.

### 3 An upper bound

Let \( G \) be a plane graph. In this section, we shall show that \( \mu(D(G)) \leq n(G) + 1 \).

**Lemma 3.1** Let \( G_i \) be a plane graph with \( v_i \in V(G_i) \) for \( i = 1, 2 \). Let \( G \) be the plane graph obtained from \( G_1 \) and \( G_2 \) by identifying \( v_1 \) and \( v_2 \). Then \( \mu(D(G)) = \mu(D(G_1)) + \mu(D(G_2)) - 1 \). In particular, adding loops to a plane graph does not change the number of components of the corresponding link diagram.

**Proof.**

![Fig. 2. One vertex cut.](image)
In Fig. 2, note that the two different components \( \alpha_1 \) and \( \alpha_2 \) will be connected to be one component. Hence, the lemma holds.

Note that in Lemma 3.1, \( D(G) \) is actually the connected sum of \( D(G_1) \) and \( D(G_2) \) and loops of graphs correspond to the nugatory crossings of link diagrams, and so it holds clearly. Similarly, if \( G \) is a connected plane graph with blocks \( B_1, B_2, \ldots, B_k \), we have \( \mu(D(G)) = \sum_{i=1}^{k} \mu(D(B_i)) - (k - 1) \).

**Lemma 3.2** If \( T \) is a tree embedded in the plane, then \( \mu(D(T)) = 1 \).

**Proof.** \( T \) has \( q \) blocks, each block is an edge which is not a loop. Note that the link diagram corresponding to an edge has one component. We have \( \mu(D(T)) = q - (q - 1) = 1 \).

Actually, the above two lemmas can both be obtained from Theorem 2.1 by using the knowledge of Tutte polynomial.

**Lemma 3.3** Let \( G \) be a connected plane graph, \( G + e \) be the plane graph obtained from \( G \) by adding a new edge \( e \) connecting two vertices (not necessarily distinct) on a same face of \( G \). Then \( \mu(D(G)) - 1 \leq \mu(D(G + e)) \leq \mu(D(G)) + 1 \).

**Proof.** Case 1. If \( e \) is a loop, then \( \mu(D(G + e)) = \mu(D(G)) \) by Lemma 3.1, the lemma holds.

Case 2. Suppose \( e \) is not a loop, (see Fig. 3). There are two cases.

1. If \( \alpha_1 \) and \( \alpha_2 \) belong to different components of \( D(G) \), then \( \mu(D(G + e)) = \mu(D(G)) - 1 \).
2. If \( \alpha_1 \) and \( \alpha_2 \) belong to the same component of \( D(G) \). There are two cases again.
   1. Along the component, if the order of the four endpoints of the two short arcs \( \alpha_1 \) and \( \alpha_2 \) is \( A, B, C, \) and \( D \), then \( \mu(D(G + e)) = \mu(D(G)) \); and
(b) if the order of the four endpoints of the two short arcs $\alpha_1$ and $\alpha_2$ is $A, B, D,$ and $C$, then $\mu(D(G + e)) = \mu(D(G)) + 1$.

The lemma holds for both subcases. This completes the proof of the lemma.

**Remark.** Let $G$ be a connected plane graph. If $e$ is not a bridge of $G$, then $\mu(D(G-e)) - 1 \leq \mu(D(G)) \leq \mu(D(G-e)) + 1$. If $e$ is a bridge, then $\mu(D(G)) = \mu(D(G-e)) - 1$.

**Theorem 3.4** Let $G$ be a connected plane graph. Then $1 \leq \mu(D(G)) \leq \min\{r(G) + 1, n(G) + 1\}$.

**Proof.** It is clear that $1 \leq \mu(D(G))$. Let $G^*$ be the dual of $G$. Then $\mu(D(G)) = \mu(D(G^*))$ and $r(G) = n(G^*)$. To prove Theorem 3.4, it suffices to prove $\mu(D(G)) \leq n(G) + 1$ holds for any connected plane graph.

Let $T$ be a spanning tree of the connected plane graph $G$. Note that $T$ has $p(G) - 1$ edges, thus $G$ can be obtained from $T$ by adding $n(G) = q(G) - p(G) + 1$ edges one by one. By Lemma 3.2, we know that $\mu(D(T)) = 1$, and by Lemma 3.3, we know that adding one edge will increase the number of components by at most one. Thus the upper bound holds.

**Remark 1.** We point out that Theorem 3.4 can be deduced from Theorem 2.2. Our proof is an elementary one without any linear algebra.

**Remark 2.** There is also a direct proof of $\mu(D(G)) \leq r(G) + 1$. Let $G$ be a connected plane graph, and $T$ be its spanning tree which has $r(G)$ edges. After contracting all edges of $T$, $G$ becomes a graph consisting of one vertex with some loops whose corresponding link diagram has one component. Note that contracting one edge decreases the number of components of the corresponding link diagram by at most one (for details, see Lemma 6.1). Hence, $\mu(D(G)) \leq r(G) + 1$.

We call a connected plane graph $G$ an extremal graph if the equality $\mu(D(G)) = n(G) + 1$ holds. We will concentrate on extremal graphs in the following sections. Now we prove some other bounds of $\mu(D(G))$.

**Corollary 3.5** Let $G$ be a connected plane graph with edge number $q(G)$. Then $\mu(D(G)) \leq \frac{q(G)}{2} + 1$.

**Proof.** By Theorem 3.4,

$$\mu(D(G)) \leq \min\{r(G) + 1, n(G) + 1\}$$
\[ \leq \frac{1}{2}(r(G) + 1 + n(G) + 1) \]
\[ = \frac{1}{2}(p(G) + q(G) - p(G) + 2) \]
\[ = \frac{q(G)}{2} + 1. \]

\[ \square \]

**Remark.** The upper bound is tight. For example, the 2-cycle \( C_2 \) attains the bound.

**Corollary 3.6** Let \( G \) be a connected plane graph. If \( G \) has an odd cycle, then \( \mu(D(G)) \leq n(G) \). In particular, if \( G \) has loops, then \( \mu(D(G)) \leq n(G) \).

**Proof.** If \( G \) has an odd cycle, it must have a face whose boundary contains an odd number of edges. Let \( e_1, e_2, \ldots, e_{2k}, e_{2k+1} (k \geq 0) \) be the edges in the boundary of the face. Note that the graph \( G/e_1/e_2/\cdots/e_{2k} \) obtained from \( G \) by contracting \( e_1, e_2, \ldots, e_{2k} \) successively is connected. \( G/e_1/e_2/\cdots/e_{2k} \) must have spanning trees which implies that \( G \) must have a spanning tree including the edges \( e_1, e_2, \ldots, e_{2k} \). Let \( T \) be such a spanning tree. It is not difficult to see that \( \mu(D(T + e_{2k+1})) = 1 \). To obtain \( G \), we add exactly \( n(G) - 1 \) edges to \( T + e_{2k+1} \), and so by Lemma 3.3, the corollary holds. \[ \square \]

**Remark 1.** The converse of Corollary 3.6 is not true. For example, the graph \( I_k \) consisting of two vertices connected by \( k \) parallel edges has no odd cycles. However \( \mu(D(I_k)) = 1 \) when \( k \) is odd and \( \mu(D(I_k)) = 2 \) when \( k \) is even, while \( n(I_k) = k - 1 \).

**Remark 2.** Corollary 3.6 can be further generalized. We call cycles \( C_1, C_2, \ldots, C_k \) of \( G \) independent if any two of them share at most one common vertex, and the graph, which consists of \( C_1, C_2, \ldots, C_k \) as vertices and the cycle pairs as edges if and only if they share one common vertex, is acyclic. Let \( o(G) \) be the maximum number of independent odd cycles of \( G \). It is not difficult to prove that \( \mu(D(G)) \leq n(G) - o(G) + 1 \).

**4 An algorithmic criterion for extremal graphs**

In this section, we shall design an algorithm to judge whether a graph is extremal or not.
A path is called a chain if each of its internal vertices has degree 2, and is a 
maximal chain if, in addition, the endvertices have degree not equal to 2. We
allow the two endvertices of a maximal chain to be the same. As an exception,
we also take the n-cycle graph \( C_n \) to be a maximal chain. We call a maximal
chain an odd (resp. even) maximal chain if it has odd (resp. even) number of
edges. By deleting a maximal chain, we mean deleting all edges and internal
vertices of the chain, and by contracting a maximal chain, we mean deleting
the chain firstly and then identifying its two endvertices.

An edge is \( e \) of \( G \) is called a bridge if \( k(G - e) > k(G) \).

**Lemma 4.1** If \( e \) is a bridge of a plane graph \( G \), then \( \mu(D(G/e)) = \mu(D(G)) \).

**Proof.** We may view \( G \) as the union of \( G_1 \), the bridge \( e \) and \( G_2 \), with \( e \)
connecting \( G_1 \) and \( G_2 \). Thus \( G/e \) comprises \( G_1 \) and \( G_2 \) with only one common
vertex. By Lemma 3.1, we know \( \mu(D(G/e)) = \mu(D(G_1)) + \mu(D(G_2)) - 1 \) and
\( \mu(D(G)) = \mu(D(G_1)) + \mu(D(G_2)) + \mu(e) - 2 = \mu(D(G_1)) + \mu(D(G_2)) - 1 \),
thus the lemma holds. \( \square \)

Note that bridges of graphs correspond to the nugatory crossings of link dia-
grams, and in this sense, Lemma 4.1 holds clearly.

**Lemma 4.2** Let \( G \) be a connected plane graph, \( e_1 \) and \( e_2 \) be a pair of parallel
edges of \( G \). If \( G - e_1 - e_2 \) is connected, then \( \mu(D(G)) \leq n(G) - 1 \).

**Proof.** By Lemma 2.3, we have \( \mu(D(G)) = \mu(D(G - e_1 - e_2)) \). Since \( G - e_1 - e_2 \)
is connected, we have \( n(G) = n(G - e_1 - e_2) + 2 \). By Theorem 3.4, \( \mu(D(G)) =
\mu(D(G - e_1 - e_2)) \leq n(G - e_1 - e_2) + 1 = n(G) - 1 \). \( \square \)

The following lemma is a critical one in the paper.

**Lemma 4.3** Let \( G \) be a plane graph with \( \delta(G) \geq 3 \), where \( \delta(G) \) is the mini-
mum degree of the graph \( G \). Then \( \mu(D(G)) \leq n(G) \).

**Proof.** Without loss of generality, we can suppose that \( G \) is connected. Now
we prove the lemma by induction on \( q(G) \), the number of edges of \( G \). Clearly
\( q(G) \geq 2 \). If \( q(G) = 2 \), the graph \( G \) must be a vertex with two loops incident
with it. Hence, we have \( \mu(D(G)) = 1 \) and \( n(G) = 2 \), and the lemma holds.
Now we suppose that the lemma holds for every connected plane graph \( H \)
with \( \delta(H) \geq 3 \) and \( q(H) < k(k \geq 3) \), and let \( G \) be a graph with \( \delta(G) \geq 3 \) and
\( q(G) = k \).
(1) $G$ has a loop $e$.

In this case, we have $\mu(D(G)) = \mu(D(G-e))$ by Lemma 3.1, and clearly, $G-e$ is a connected plane graph. Thus $\mu(D(G-e)) \leq n(G-e) + 1$ by Theorem 3.4. Hence, $\mu(D(G)) = \mu(D(G-e)) \leq n(G-e) + 1 = n(G)$.

(2) $G$ has a bridge $e$.

In this case, we have $\mu(D(G)) = \mu(D(G/e))$ by Lemma 4.1. Note that $G/e$ is a connected plane graph with $\delta(G/e) \geq 3$. Applying induction hypothesis, we have $\mu(D(G/e)) \leq n(G/e)$. Hence, $\mu(D(G)) = \mu(D(G/e)) \leq n(G/e) = n(G)$.

In the following, we suppose that $G$ has neither loops nor bridges. Choose an edge $e$ of $G$. Suppose that the two endvertices of $e$ are $u$ and $v$. Clearly, $d_G(u) \geq 3$ and $d_G(v) \geq 3$. Let $G' = G-e$. Clearly $G'$ is connected, since $e$ is not a bridge. There are three cases.

(1) $d_G(u) > 3$ and $d_G(v) > 3$.

In this case, it is easy to see that $\delta(G') \geq 3$. By induction hypothesis, we have $\mu(D(G')) \leq n(G')$. By Lemma 3.3, we have $\mu(D(G)) \leq \mu(D(G'))+1$. Hence, $\mu(D(G)) \leq \mu(D(G'))+1 \leq n(G') + 1 = n(G)$.

(2) $d_G(u) = 3$ and $d_G(v) > 3$.

Suppose that the other two vertices incident with $u$ are $u_1$ and $u_2$. There are two cases.

(a) $u_1, u_2, v$ are all different. See Fig. 4.

Let $G'' = G'/(u, u_1)/(u, u_2)$. Note that $G''$ must be connected and $\delta(G'') \geq 3$, since $d_G(v) > 3$. By induction hypothesis, we have $\mu(D(G'')) \leq n(G'')$. By Lemma 2.3, $\mu(D(G')) = \mu(D(G''))$, and clearly, $n(G') = n(G'').$ Hence, $\mu(D(G)) \leq \mu(D(G'))+1 = \mu(D(G''))+1 \leq n(G'') + 1 = n(G') + 1 = n(G)$.

(b) Otherwise, there are three different cases as shown in Fig. 5.

Note that $G$ always has a parallel pair and the graph obtained by deleting the parallel pair is still connected, since we have supposed that $G$ has no bridges. By Lemma 4.2, we have $\mu(D(G)) \leq n(G) - 1$.

Note that in this case we need not use the condition $d_G(v) > 3$.

(3) $d_G(u) = 3$ and $d_G(v) = 3$.

Suppose that the other two vertices incident with $u$ are $u_1$ and $u_2$ and the other two vertices incident with $v$ are $v_1$ and $v_2$. In this case, we can suppose that $u_1, u_2, u, v$ are all different and $v_1, v_2, u, v$ are also all different. Otherwise we can deal with it as case (2b). There are two cases.
Fig. 5.
(a) $u_1, u_2, v_1, v_2$ are all different. See Fig. 6.

Let $G'' = G'/(u, u_1)/(u, u_2)/(v, v_1)/(v, v_2)$. Clearly, $G''$ is connected

\[ G'' = G'/(u, u_1)/(u, u_2)/(v, v_1)/(v, v_2). \]

and $\mu(D(G')) = \mu(D(G''))$ and $n(G') = n(G'')$. By induction hypothesis, we have $\mu(D(G'')) \leq n(G'')$. Hence, $\mu(D(G)) \leq \mu(D(G')) + 1 = \mu(D(G'')) + 1 \leq n(G'') + 1 = n(G') + 1 = n(G)$.

(b) Otherwise.

$G$ must contain a triangle, and by Corollary 3.6, we have $\mu(D(G)) \leq n(G)$.

This completes the proof of Lemma 4.3.

\[ \square \]

**Remark.** The result of Lemma 4.3 can be further generalized to $\mu(D(G)) \leq n(G) - \delta(G) + 3$ when $G$ is a connected plane graph with $p(G) \geq 2$. When $\delta(G) = 1, 2, 3$, by Theorem 3.4 and Lemma 4.3, it holds; when $\delta(G) \geq 4$, we have $q(G) \geq \frac{\delta(G)p(G)}{2} \geq 2p(G) + \delta(G) - 4$ when $p(G) \geq 2$. By Theorem 3.4, we have $\mu(D(G)) \leq p(G) \leq q(G) - p(G) - \delta(G) + 4 = n(G) - \delta(G) + 3$.

**Lemma 4.4** Let $G$ be a connected plane graph.

1. Let $e$ be a bridge of $G$. Then $G/e$ is extremal if and only if $G$ is extremal.
2. Let $v$ be a vertex of degree 2 with exactly one adjacent vertex. Then $G - v$ is extremal if and only if $G$ is extremal.
3. Let $v$ be a vertex of degree 2 with two different adjacent vertices $x$ and $y$. Then $G/(v, x)/(v, y)$ is extremal if and only if $G$ is extremal.
4. $G$ is extremal if and only if each block of $G$ is extremal.
5. Suppose $e$ is not a bridge of $G$. If $G$ is extremal, then $G - e$ is extremal.

**Proof.**

1. If $e$ is a bridge, $G/e$ is still a connected plane graph and $n(G) = n(G/e)$. By Lemma 4.1, $\mu(D(G/e)) = \mu(D(G))$. Hence, $G/e$ is extremal if and only if $G$ is extremal.
2. If $v$ is a vertex of degree 2 with exactly one adjacent vertex, then $G - v$ is connected plane graph and $n(G - v) = n(G) - 1$. By Lemma 2.3, it is not difficult to see that $\mu(D(G - v)) = \mu(D(G)) - 1$. Hence, $G - v$ is extremal if and only if $G$ is extremal.
(3) If \( v \) is a vertex of degree 2 and the two vertices \( x \) and \( y \) adjacent with \( v \) are different, then \( G/(v, x)/(v, y) \) is a connected plane graph and \( n(G/(v, x)/(v, y)) = n(G) \). By Lemma 2.3, \( \mu(D(G/(v, x)/(v, y))) = \mu(D(G)) \). Hence, \( G/(v, x)/(v, y) \) is extremal if and only if \( G \) is extremal.

(4) Let \( B_1, B_2, \ldots, B_k \) be all the blocks of \( G \). Then \( G \) is extremal if and only if

\[
\sum_{i=1}^{k} \mu(D(B_i)) - (k - 1) = n(G) + 1 \quad (\text{By Lemma 3.1.})
\]

\[
\sum_{i=1}^{k} \mu(D(B_i)) - (k - 1) = \sum_{i=1}^{k} n(B_i) + 1
\]

\[
\mu(D(B_i)) = n(B_i) + 1 \quad \text{for every } i \quad (\text{By Theorem 3.4.})
\]

\[
\sum_{i=1}^{k} \mu(D(B_i)) = \sum_{i=1}^{k} (n(B_i) + 1)
\]

\[
\mu(D(B_i)) = n(B_i) + 1 \quad \text{for every } i
\]

(5) If \( e \) is not a bridge, \( G - e \) is a connected plane graph and \( n(G) = n(G - e) + 1 \). By Lemma 3.3, \( \mu(D(G - e)) \geq \mu(D(G)) - 1 = n(G) = n(G - e) + 1 \). By Theorem 3.4, we have \( \mu(D(G - e)) = n(G - e) + 1 \), hence \( G - e \) is extremal.

By Lemma 4.4 (2) and (3), we obtain:

**Corollary 4.5** Deleting an even maximal chain with two same endvertices and contracting an even maximal chain with two different endvertices both preserve the extremity of the graph.

By Lemma 4.4 (5) and (1), we obtain:

**Corollary 4.6** Let \( G \) be extremal, and \( C \) be a maximal chain of \( G \). If \( G - C \) is connected, then \( G - C \) is also extremal.

Let \( u, v \in V(G) \). Denote by \( G/\{u, v\} \) the graph obtained from \( G \) by identifying the two vertices \( u \) and \( v \). Clearly if \( u = v \) then \( G/\{u, v\} = G \). With this notation, Lemma 4.4 (2) and (3) can be combined to state: if \( v \) is a vertex of degree 2 of \( G \) adjacent with \( x \) and \( y \) (not necessarily distinct), then \( G \) is extremal if and only if \( (G - v)/\{x, y\} \) is extremal.

Now we design an algorithm to determine whether a graph is extremal.

**Algorithm.**

Input: A connected plane graph \( G \).

Output: \( H \) and \( c \). \( G \) is extremal if and only if \( H \) is trivial. When \( G \) is extremal, \( \mu(D(G)) = c + 1 \).

Initially, we set \( c = 0 \).

Step 1. Contract all bridges of \( G \) to obtain a connected bridgeless plane graph \( H \).

Step 2. If \( H \) is trivial or \( H \) has a loop or \( \delta(H) \geq 3 \), stop.

Step 3. Let \( x \) be a vertex with degree 2 of \( H \) and the two vertices adjacent with \( x \) be \( u \) and \( v \).
If $u = v$, let $H = H - x$. Set $c = c + 1$.  
If $u \neq v$, let $H = H/(x, u)/(x, v)$. 
Return to Step 2.

**Theorem 4.7** Let $G$ be a connected plane graph. Then $G$ is extremal if and only if the above algorithm outputs a trivial graph. Furthermore, if $G$ is extremal, then $\mu(D(G)) = c + 1$.

**Proof.** By Lemma 4.4 (1), it suffices to show that the graph $H$, obtained by contracting all bridges of $G$, is extremal. Note that Step 3 of the algorithm reduces the number of vertices of $H$ by one or two. The algorithm must stop with a trivial graph, a graph with loops or a graph $H$ with $\delta(H) \geq 3$.

Let $H$ be a bridgeless plane graph, $x$ be a vertex of degree 2 of $H$, and the two vertices adjacent with $x$ are $u$ and $v$. If $u = v$, by Lemma 4.4 (2), then $H$ is extremal if and only if $H - x$ is extremal, and the number of components of the corresponding link diagram will decrease by one; if $u \neq v$, by Lemma 4.4 (3), then $H$ is extremal if and only if $H/(x, u)/(x, v)$ is extremal, and the number of components of the corresponding link diagram will not change. After step 3, the graph obtained is still a bridgeless connected plane graph.

We have shown that $H$ is extremal if and only if the outputs of the algorithm is extremal, and by Corollary 3.6 and Lemma 4.3, if and only if the algorithm outputs a trivial graph.

5 Construction of extremal graphs

In this section, we shall study the construction of extremal graphs and prove that all extremal graphs can be obtained from $K_1$ by applying two graph operations repeatedly.

**Lemma 5.1** Let $G$ be a non-trivial extremal graph. Then $G$ has at least two vertices with degrees less than 3.

**Proof.** We shall prove the lemma by induction on $p(G)$, the number of vertices of $G$. Clearly, $p(G) \geq 2$. If $p(G) = 2$, $G$ must be $I_1$ or $I_2$, and so the lemma holds. Suppose the lemma holds for all non-trivial extremal graphs with vertex numbers less than $k$ ($k \geq 3$).

Let $G$ be an extremal graph with $p(G) = k$. By Lemma 4.3, we have $\delta(G) \leq 2$. Suppose $G$ has more than one block. By Lemma 4.4 (4), each block of $G$ must
be extremal, and thus, by induction hypothesis, each block has at least two vertices with degree less than 3. Choose two blocks corresponding to two leaves of block-cutvertex tree of $G$, from which we can obtain two vertices with degree less than 3 of $G$.

Now we suppose that $G$ has only one block, which implies $\delta(G) = 2$. Let $v$ be a vertex of degree 2 of $G$. Note that $G$ has no loops since $G$ is extremal. Let the two vertices adjacent with $v$ be $x$ and $y$ which must be different since $G$ has only one block. By Lemma 4.4 (3), $G/(v, x)/(v, y)$ is extremal. Note that $G/(v, x)/(v, y)$ can not be trivial since $G$ has only one block. Hence, by induction hypothesis, $G/(v, x)/(v, y)$ has at least two vertices with degree less than 3. Since $G/(v, x)/(v, y)$ has a vertex $u$ different from $x = y$ with degree less than 3, $u$ and $v$ will be the two vertices of $G$ with degree less than 3.

Lemma 5.2 Let $G$ be the plane graph obtained from two connected plane graphs $G_1$ and $G_2$ by adding two chains $C_1$ and $C_2$ connecting $G_1$ and $G_2$ as shown in Fig. 7. If $G$ is extremal, then $G_1$ and $G_2$ are both extremal. Conversely, if $C_1$ and $C_2$ are odd maximal chains, the maximal chains, except $C_1$ and $C_2$, in the boundaries of the face $F_1$ or the face $F_2$ of $G$ are all even, and $G_1$ and $G_2$ are both extremal, then $G$ is extremal.

![Fig. 7.](image)

**Proof.** It is not difficult to see that $n(G) = n(G_1) + n(G_2) + 1$. Suppose $G$ is extremal. Let $e_1$ be an edge of the chain $C_1$. By Lemmas 3.3, 4.1 and 3.1, we have $\mu(D(G)) \leq \mu(D(G-e_1)) + 1 = \mu(D(G-C_1)) + 1 = \mu(D(G_1)) + \mu(D(G_2))$. If $G_1$ is not extremal, by Theorem 3.4, we have $\mu(D(G)) \leq \mu(D(G_1)) + \mu(D(G_2)) \leq n(G_1) + n(G_2) + 1 = n(G)$, which means $G$ is not extremal, a contradiction. Hence, $G_1$ is extremal. Similarly, $G_2$ is also extremal.

Conversely, since the maximal chains, except $C_1$ and $C_2$, in the boundary of the two faces $F_1$ or $F_2$ of $G$ are all even, it is not difficult to see that $\mu(D(G)) = \mu(D(G_1)) + \mu(D(G_2))$. Hence, $\mu(D(G)) = n(G_1) + 1 + n(G_2) + 1 = n(G) + 1$, and so $G$ is extremal.

**Remark.** The condition that the maximal chains, except $C_1$ and $C_2$, in the boundaries of one of the two faces $F_1$ and $F_2$ of $G$ are all even in Lemma
5.2 is necessary. See the example shown in Fig. 8. The graph has nullity 3 and the corresponding link diagram has 2 components, thus it is not extremal. However, $G_1$ and $G_2$ are both 2-cycles, which are extremal.

![Fig. 8. A counterexample.](image)

**Lemma 5.3** Let $G$ be the plane graph obtained from two connected plane graphs $G_1$ and $G_2$ by adding two edges $(x_1, x_2)$ and $(y_1, y_2)$, where $x_i, y_i \in V(G_i)$ and lie on a same face of $G_i$ for $i = 1, 2$ as shown in Fig. 9. Let the plane graph $G_1 / \{x_1, y_1\}$ be extremal. Then $G$ is extremal if and only if the plane graph $G_2 / \{x_2, y_2\}$ is extremal.

![Fig. 9.](image)

**Proof.** Let $G'_i = G_i / \{x_i, y_i\}$ for $i = 1, 2$. We first prove the necessity by induction on $p(G'_i)$. If $G'_i$ has only one vertex, $G_1$ must also have only one vertex $x_1 = y_1$. Since $G$ is extremal, by Lemma 4.4, $G'_2 = (G - x_1) / \{x_2, y_2\}$ is extremal, and so the lemma holds. Now let $G'_i$ have $k$ vertices ($k \geq 2$). Then, by Lemma 5.1, $G'_1$ must have at least two vertices with degree less than 3, which implies that $G'_1$ must have one vertex $v$ different from $x_1 = y_1$ with degree less than 3.

Suppose $v$ has degree one and the edge incident with $v$ is $e$. It is clear that $G'_1 / e$ and $G / e$ are both extremal. By induction hypothesis, $G'_2$ is extremal. Suppose $v$ has degree 2 and the two vertices adjacent with $v$ are $u_1$ and $u_2$ in $G'_1$. By Lemma 4.4, we have $(G'_1 - v) / \{u_1, u_2\}$ and $(G - v) / \{u_1, u_2\}$ are both extremal. By induction hypothesis, $G'_2$ is extremal.

Now we prove the sufficiency. Let $C_i$ be an even chain with endvertices $x_i$ and $y_i$ for $i = 1, 2$ such that $C_1$ and $C_2$ both lie in the unbounded face of $G$ and they are disjoint. Since $G'_i$ is extremal, by Corollary 4.5, we have $G_i + C_i$ is
extremal. By Lemma 5.2, \( G + C_1 + C_2 \) is extremal, then by Corollary 4.6, we deduce that \( G \) is extremal.

Remark. \( G_i / \{x_i, y_i\} \) being extremal implies that \( G_i \) must also be extremal; that follows by Corollary 4.6, since \( G_i / \{x_i, y_i\} \) is extremal if and only if \( G_i + C_i \) is extremal, where \( C_i \) is an even chain.

**Theorem 5.4** The set \( \Phi \) of all extremal graphs can be obtained as follows.

1. The trivial graph \( K_1 \in \Phi \).
2. Let \( G \) be the plane graph obtained from two connected plane graphs \( G_1 \) and \( G_2 \) by adding an edge joining a vertex from \( G_1 \) to a vertex from \( G_2 \). If \( G_1, G_2 \in \Phi \), then \( G \in \Phi \).
3. For \( i = 1, 2 \), let \( G_i \) be a connected plane graph with \( x_i, y_i \in V(G_i) \) such that \( G_i / \{x_i, y_i\} \in \Phi \). Let \( G \) be the plane graph obtained from \( G_1 \) and \( G_2 \) by adding two new edges \( (x_1, x_2) \) and \( (y_1, y_2) \) as shown in Fig. 9. Then \( G \in \Phi \).

**Proof.**
We first prove that if \( G \in \Phi \), then \( G \) is extremal. Obviously, \( K_1 \) is extremal.

By Lemmas 4.1, 3.1 and 5.3, we know that the graph \( G \) produced by either of the two graph operations in Theorem 5.4 is also extremal. Now we prove that if \( G \) is extremal, then \( G \in \Phi \). We shall prove it by induction on \( p(G) \), the number of vertices of \( G \). If \( p(G) = 1 \), \( G \) must be \( K_1 \), \( K_1 \in \Phi \). Now we suppose that all extremal graphs with vertex number less than \( k \) belong to \( \Phi \), and let \( G \) be an extremal graph with \( p(G) = k \). Since \( G \) is an extremal graph, by Lemma 4.3, \( G \) has one vertex \( v \) with degree less than 3. There are two cases.

1. \( d_G(v) = 1 \).
   In this case, since \( G \) is extremal, it is clear that \( G_2 = G - v \) is extremal. By induction hypothesis, \( G_2 \in \Phi \). Since \( G \) can be viewed as \( K_1 \) and \( G_2 \) connected by an edge, we have \( G \in \Phi \).
2. \( d_G(v) = 2 \).
   Suppose \( N_G(v) = \{x_2, y_2\} \). There are also two cases.
   (a) If \( x_2 = y_2 \), then by Lemma 4.4 (2), \( G_2 = G - v \) is extremal. By induction hypothesis, \( G_2 \in \Phi \). \( G \) can be viewed as the graph obtained from \( K_1 \) and \( G_2 \) by adding two parallel edges, hence \( G \in \Phi \).
   (b) If \( x_2 \neq y_2 \), let \( G_2 = G - v \). By Lemma 4.4 (3), we have \( G_2 / \{x_2, y_2\} = G / (x_2, y_2) \in \Phi \). Since \( G \) can be viewed as the graph obtained from \( K_1 \) and \( G_2 \) by adding two edges joining \( K_1 \) and \( G_2 \), hence \( G \in \Phi \).
This completes the proof of Theorem 5.4. □

**Remark.** From the proof of Theorem 5.4, the construction of all extremal graphs can be further simplified as follows: to obtain all extremal graphs it suffices to choose the $G_1$ in (3) of Theorem 5.4 to be $K_1$.

6 The dual characterization of extremal graphs

It’s relatively simple to characterize extremal graphs using their dual graphs. In this section, we shall give the dual characterization of extremal graphs. We need the following lemmas.

**Lemma 6.1** Let $G$ be a connected plane graph and $e$ be an edge of $G$. Then
\[ \mu(D(G/e)) - 1 \leq \mu(D(G)) \leq \mu(D(G/e)) + 1. \]

**Proof.** There are two cases.

**Case 1.** If $e$ is a loop, $G/e = G - e$. By Lemma 3.1 $\mu(D(G)) = \mu(D(G/e))$, and so the lemma holds.

**Case 2.** If $e$ is not a loop, there are also two cases.

(1) If $\alpha_1$ and $\alpha_2$, the dashed curves as shown in Fig. 10, belong to different components of $D(G)$, then $\mu(D(G)) = \mu(D(G/e)) + 1$, and so the lemma holds.

(2) If $\alpha_1$ and $\alpha_2$ belong to the same component of $D(G)$. There are two cases again.

(a) Along the component, if the order of the four endpoints of the two short arcs $\alpha_1$ and $\alpha_2$ is $A, B, C, D$, then $\mu(D(G)) = \mu(D(G/e)) - 1$; and

(b) if the order of the four endpoints of the two short arcs $\alpha_1$ and $\alpha_2$ is $A, B, D, C$, then $\mu(D(G)) = \mu(D(G/e))$.

The lemma holds for both subcases. This completes the proof of Lemma 6.1. □
Lemma 6.2 Let $G$ be a simple plane graph with $q(G) \geq 1$. Then $\mu(D(G)) < p(G)$.

Proof. We prove the lemma by induction on $q(G)$. If $q(G) = 1$, then $G$ must be the union of $P_2$ and some isolated vertices. It is not difficult to see that $\mu(D(G)) = p(G) - 1$, and so the lemma holds. We suppose that the lemma holds for each simple plane graph $G$ with $1 \leq q(G) < k$ ($k \geq 2$). Now let $G$ be a simple plane graph with $q(G) = k$.

Suppose $e \in E(G)$. It is clear that $G/e$ has no loops, however $G/e$ may have parallel edges with multiplicities 2. Let $G/e$ be the graph obtained from $G/e$ by deleting all such parallel pairs, if there are any. Then, by Lemma 2.3, $\mu(D(G/e)) = \mu(D(G/e))$. Now $G/e$ is a simple plane graph. There are two cases.

1. $q(G/e) = 0$.

In this case, $G$ is a union of the generalized theta graph $\theta(1,2,\ldots,2)$ and some isolated vertices, where $n \geq 1$. It is not difficult to obtain that $\mu(D(\theta(1,2,\ldots,2))) = n$. Note that $p(\theta(1,2,\ldots,2)) = n + 2$, and so the lemma holds.

2. $q(G/e) \geq 1$.

In this case, by induction hypothesis, we have $\mu(D(G/e)) < p(G/e)$, hence, by Lemma 6.1, $\mu(D(G)) \leq \mu(D(G/e)) + 1 = \mu(D(G/e)) + 1 < p(G/e) + 1 = p(G)$, thus the lemma also holds.

This completes the proof of Lemma 6.2.

Theorem 6.3 Let $G$ be a connected plane graph with dual graph $G^*$. Then $G$ is extremal if and only if the plane graph $H^*$, obtained by deleting all loops of $G^*$, is trivial or there is an even non-negative number of edges between each pair of vertices of $H^*$.

Proof. Note that $G$ is extremal if and only if $\mu(D(G^*)) = p(G^*)$, and if and only if $\mu(D(H^*)) = p(H^*)$.

If $H^*$ is trivial, we have $\mu(D(H^*)) = p(H^*) = 1$. If each pair of vertices of $H^*$ has even number of edges, we delete parallel pairs repeatedly until we get a graph with $p(H^*)$ isolated vertices. By Lemma 2.3, such deletions do not change the number of components of the corresponding link diagram. Note that the link diagram corresponding to the graph consisting of $p(H^*)$ isolated vertices has $p(H^*)$ components, thus we have $\mu(D(H^*)) = p(H^*)$. Conversely, if $H^*$ is not trivial and there exists a pair of vertices of $H^*$, there is an odd number of edges between them. We delete the parallel pairs of $H^*$ repeatedly,
finally we obtain a simple plane graph $H'$ with $q(H') \geq 1$. By Lemmas 2.3 and 6.2, $\mu(D(H^*)) = \mu(D(H')) < p(H') = p(H^*)$. This completes the proof of Theorem 6.3.

7 The structural criterion of bridgeless extremal graphs

In this section, we further characterize the structure of bridgeless extremal graphs. A simple criterion on the structure of graphs is given to judge whether a bridgeless connected plane graph is extremal or not. Some simple necessary conditions for a bridgeless connected plane graph to be extremal are derived.

Let $G$ be a connected plane graph. We define an associate graph $as(G)$ of $G$ as follows: the vertices of $as(G)$ correspond to the faces of $G$ including the unbounded face; to every odd maximal chain $C$ of $G$, there is an edge $e(C)$ which connects the two vertices (not necessarily distinct) of $as(G)$ corresponding to the two faces divided by the odd maximal chain $C$.

Lemma 7.1 Let $G$ be a connected plane graph with associate graph $as(G)$. Then $\mu(D(G)) = \mu(D(as(G)))$.

Proof. Let $G^*$ be the dual of $G$, we have $\mu(D(G)) = \mu(D(G^*))$. Note that $as(G)$ can be obtained from $G^*$ by deleting all parallel pairs of $G^*$ which, by Lemma 2.3, does not change the number of components of the corresponding link diagrams. Hence, the lemma holds.

Theorem 7.2 Let $H$ be a bridgeless connected plane graph. Then $H$ is extremal if and only if any two adjacent faces of $H$ have exactly even number of odd maximal chains lying in their common boundary.

Proof.
1. The sufficiency.
Let $as(H)$ be the associate graph of the bridgeless plane graph $H$. Under the condition of the theorem, if there exist edges between two vertices of $as(H)$, the number of the edges must be even. By deleting parallel edges of $as(H)$ repeatedly, we obtain a graph consisting of $p(as(H))$ isolated vertices. By Lemma 2.3, $\mu(D(as(H))) = p(as(H))$, which is the number of faces of $H$ and hence, is equal to $n(H) + 1$. By Lemma 7.1, $\mu(D(H)) = \mu(D(as(H))) = n(H) + 1$. Hence, $H$ is extremal.

2. The necessity.
Since $H$ is bridgeless, $as(H)$ is loopless. If $H$ does not satisfy the condition of the theorem, then there exist two faces $F_1$ and $F_2$ such that there is an
odd number of odd maximal chains in their common boundary. Deleting the parallel pairs of $as(H)$ repeatedly, finally we obtain a simple plane graph $as(H)$ with $\mu(D(as(H))) = \mu(D(as(H)))$. Note that $q(as(H)) \geq 1$, and so by Lemma 6.2, we have $\mu(D(as(H))) < p(as(H)) = p(as(H)) = n(H) + 1$, a contradiction.

An example with two odd maximal chains and four even maximal chains together with the corresponding link diagram, is shown in Fig. 11.

![Fig. 11. An example with odd maximal chains.](image)

By the suppression of a vertex of degree 2 in a graph $G$, we mean deleting the vertex and joining the two vertices previously adjacent with it by a new edge. The reduction $R(G)$ of a graph $G$ is the graph obtained by suppressing the vertices of degree 2 of $G$ successively until suppression is no longer possible. Obviously there is a one-to-one correspondence between the maximal chains of $G$ and the edges of $R$. We mention that in [13], the author lists all reductions with nullity less than 6, where the reduction is called the homeomorphically irreducible star graph.

Let $R(H)$ be the reduction of $H$. Note that two edges of $R(H)$ which correspond to odd maximal chains of $H$ constitute a 2-edge cut of $R(H)$ if and only if the two odd maximal chains lie in the common boundary of two faces of $H$. Hence, the condition of the theorem is equivalent to the following statement: Each edge of $R(H)$ which corresponds to an odd maximal chain of $H$ can be paired with exactly odd number of edges of $R(H)$ which correspond to odd maximal chains of $H$ to form 2-edge cuts of $R(H)$. Hence, we have the following corollary.

**Corollary 7.3** Let $H$ be a bridgeless plane graph with reduction $R(H)$. If $R(H)$ has no 2-edge cuts, then $H$ is extremal if and only if $H$ has no odd maximal chains.

An example together with the corresponding link diagram is shown in Fig. 12. The graph is $K_4$ with each edge divided into two edges. Note that each face of such graphs, including the unbounded face, corresponds to one component of the corresponding link diagram. We mention here that the figure appeared in [14] and such a phenomenon has been observed by its author.

**Corollary 7.4** Let $H$ be a bridgeless connected plane graph. If $H$ has odd
number of odd maximal chains, then $H$ cannot be extremal.

Proof. By Theorem 7.2, it is obvious.

Remark 1. Corollaries 3.6 and 7.4 are independent, in the sense that there exist non-extremal examples which satisfy the condition of one of the two and do not satisfy the condition of the other. For example, $K_4$ has odd cycles and has even number of odd maximal chains; $I_3$ has odd number of odd maximal chains and has no odd cycles.

Remark 2. The converse of Corollary 7.4 is not true, i.e. there exist non-extremal graphs with even number of odd maximal chains. For example, $K_4$ has 6 odd maximal chains and it is not extremal, since $\mu(D(K_4)) = n(K_4) = 3$.

Lemma 7.5 Let $H$ be a plane graph, and $H'$ be the plane graph obtained from $H$ by deleting an odd maximal chain of $H$. Let $m(H)$ (resp. $m(H')$) be the number of odd maximal chains of $H$ (resp. $H'$). Then $m(H') = m(H) - 1$, $m(H) - 3$ or $m(H) - 5$.

Proof. Suppose that $C$ is an odd maximal chain of $H$ with endvertices $u$ and $v$ and $H' = H - C$. Let $H_1$ be the plane graph obtained from $H$ by deleting $C$, then adding another odd chain $C''$ with the same endvertices $v$. It is not difficult to see that $m(H_1) = m(H)$ or $m(H_1) = m(H) - 2$. Note that $H'$ can be obtained from $H_1$ by deleting $C''$. Similarly, we have $m(H') = m(H_1) - 1$ or $m(H') = m(H_1) - 3$. Hence we have $m(H') = m(H) - 1$, $m(H) - 3$ or $m(H) - 5$, and so the lemma holds.

By Corollary 7.4 and Lemma 7.5, we obtain the following necessary condition for a bridgeless connected plane graph to be extremal.

Corollary 7.6 Let $H$ be a bridgeless connected plane graph. If $H$ is extremal, then $H - e$ has bridges for any $e \in E(H)$. 

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Proof. If $H$ is bridgeless and extremal, by Lemma 4.4 (5), $H - e$ is also extremal. Thus, if $H - e$ is bridgeless, by Corollary 7.4, $m(H)$ and $m(H - e)$ must be both even. Hence, $m(H) - m(H')$ is even. However, if $H - e$ is bridgeless, then the edge $e$ itself is an odd maximal chain of $H$. Thus, by Lemma 7.5, the difference $m(H) - m(H')$ should be odd, a contradiction. □

Remark 1. Let $H$ be a bridgeless connected plane graph, and $H + e$ be the plane graph obtained from $H$ by adding a new edge $e$ connecting two vertices (not necessarily distinct) of $H$. By Corollary 7.6, $H + e$ is not extremal.

Remark 2. The converse of Corollary 7.6 is not true. We take for example, the connected plane graph with odd number of odd maximal chains with length greater than or equal to 3.

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