Periodic solution for state-dependent impulsive shunting inhibitory CNNs with time-varying delays

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In this paper, we consider existence and global exponential stability of periodic solution for state-dependent impulsive shunting inhibitory cellular neural networks with time-varying delays. By means of $B$-equivalence method, we reduce these state-dependent impulsive neural networks system to an equivalent fix time impulsive neural networks system. Further, by using Mawhin’s continuation theorem of coincide degree theory and employing a suitable Lyapunov function some new sufficient conditions for existence and global exponential stability of periodic solution are obtained. Previous results are improved and extended. Finally, we give an illustrative example with numerical simulations to demonstrate the effectiveness of our theoretical results.

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1. Introduction

A model of artificial neural networks named as shunting inhibitory cellular neural networks is introduced by Bouzerdoum and Pinter in Bouzerdoum and Pinter (1993) and has many potential applications in the areas such as pattern recognition, signal and image processing, optimization problems, biology, vision, speech and parallel computations. All of these applications tediously depend on the dynamical characteristics such as stability and periodicity of the designed network. In practical applications, because of the finite speed of the switching and transmission of signals, the time delays are indispensable in the networks. For these reasons, stability and periodicity of shunting inhibitory cellular neural networks with constant or time-varying delays were extensively studied and sufficient conditions for global exponential stability are constructed in Cao (1999), Cao, Chen, and Huang (2005), Chen, Cao, and Huang (2004), Hien, Loan, and Tuan (2008), Huang and Cao (2003), Li, Liu, and Zhu (2005); Li, Meng, and Zhou (2008), Liu (2009), Liu and Huang (2007a, 2007b), Meng and Li (2008), Ou (2009), Wang and Lin (2009), Wu and Fu (2009), Zhang, Yang, Long, and He (2010), Zhong and Liu (2007) and references therein. On the other hand, the instantaneous perturbations and abrupt changes in the voltages at certain instant, which are created by circuit elements, are exemplary of impulsive phenomena that can affect the evolutionary process of the neural networks. Therefore, stability and periodicity of impulsive shunting inhibitory cellular neural networks with delay (see, for example Akhmet & Yılmaz, 2014, Gui & Ge, 2006a, 2006b, 2007, Li & Xing, 2007, Lin & Jun, 2009, Sun, Wang, & Gao, 2009, Wang, Li, & Xu, 2010, Xia, Cao, & Huang, 2007, Yang, 2009, Yang & Cao, 2007, Yang, Zhang, Wu, Chen, & Yang, 2010 and Zhang & Gui, 2009) which are neither purely continuous nor discrete have been widely considered.

The main necessity of the present paper is to find sufficient conditions which guarantee the existence and global exponential stability of periodic solution for neural networks with discontinuities. Besides, in the present paper, different from the most existing studies, we introduce a more general class of shunting inhibitory cellular neural networks including time-varying delays and related to the state-dependent impulsive phenomena. The aim of defining this new class is that the moments of impulses are arbitrary in $\mathbb{R}_+$, that is, solutions with different initial data have different impulse time. As it is mentioned in Liu, Li, and Liao (2011), in real world problems, the impulses of many systems do not occur at fixed time, like for example, population control systems, saving rates control systems, some circuit control systems and so on. These types of systems are called state-dependent impulsive differential systems or impulsive systems with variable-time impulses. For more detailed discussion of real world applications of state-dependent impulsive systems please see Refs. Akhmet (2010) and Yang (2001).
Therefore, considering the system with non-fix moments of impulses is more general than the fixed time impulses. This results in more theoretical and technical challenges, since simple transformations are not allowed. To the best of our knowledge, these types of impulsive neural networks were considered in Li et al. (2011), Sayh & Yilmaz (2014) and Yilmaz (2014) and problems related to stability, almost periodicity and robustness of bidirectional associative memory neural networks were analyzed. One should underline that there are no results on existence and global exponential stability of periodic solution of shunting inhibitory cellular neural networks having variable coefficients with state-dependent impulse and time-varying delays in the literature. Therefore, our results are generalization of the studies (Akhet & Yilmaz, 2014; Gui & Ge, 2006a, 2006b, 2007; Li & Xing, 2007; Lin & Jun, 2009; Sun et al., 2009; Wang et al., 2010; Xia et al., 2007; Yang, 2009; Yang & Cao, 2007; Yang et al., 2010; Zhang & Gui, 2009) with fix-time impulses to the state-dependent impulse time \( t = \theta_k + r_k(x) \).

To solve the problem we used the technique of the reduction of the analyzed system to a system with fixed moments of impulses by using the \( B \)-equivalence method, which was studied widely in Akalin and Akhmet (2005), Akhmet (2005, 2010) and Akhmet and Perestyuk (1990) for ordinary differential equations and applied delay differential equation in Liu and Wang (2006). Then, we find some sufficient constraints by using Mawhin’s continuation theorem of coincidence degree theory and employing an appropriate Lyapunov function to guarantee the existence and global exponential stability of periodic solution of the considered networks.

2. Model description and preliminaries

Let \( \mathbb{Z}_+ \), \( \mathbb{R}_+ \) and \( \mathbb{R} \) be the sets of positive integers, nonnegative real numbers and real numbers, respectively. Consider the following variable-time impulsive neural networks with time-varying delay:

\[
\begin{aligned}
        x'_i(t) &= -a_i(t)x_i(t) - \sum_{j \in N(i), \theta_k \leq t} b_{ij}(t)f_j(x_j(t))x_i(t) \\
        &- \sum_{j \in E_{N(i), \theta_k \leq t}} c_{ij}(t)(\delta_j(x_j(t)) - \rho_{ij}(t))) \\
        &+ x_i(t) + L_i(t), \quad t \geq 0 \\
        \Delta x_i(t) &= e_{ij}x_j(t) + I_i(x(t))
\end{aligned}
\]  

(2.1)

where \( k \in \mathbb{Z}_+ \), \( x \in \mathbb{R}^{m \times n} \), \( t \in \mathbb{R}_+ \) and \( c_{ij} \) denote the cell at the \((i,j)\) position of the lattice, the \( r \)-neighborhood \( N_r(i) \) of \( c_{ij} \) is

\[
N_r(i,j) = \{ e_{ij} : \max(|h-i|, |l-j|) \leq r, \ 1 \leq h \leq m, 1 \leq l \leq n \},
\]

\( i = 1, 2, \ldots, m, j = 1, 2, \ldots, n \). Also, \( e_{ij} \) is a bounded sequences such that \((1 + e_{ij}) \neq 0, i = 1, 2, \ldots, m, j = 1, 2, \ldots, n, k \in \mathbb{Z}_+ \), \( r \) is a real valued continuous functions defined on \( \mathbb{R}^{m \times n} \), \( k \in \mathbb{Z}_+ \). Moreover, the sequence \( r \) satisfies the condition \( \lim_{t \to \infty} r(t) = 0 \).

In the system (2.1), \( x_i(t) \) denotes the membrane potential of the cell \( c_{ij} \) at time \( t \); the function \( a_i(t) > 0 \) denotes the passive decay rate of the membrane potential of the cell \( c_{ij} \) at time \( t \); the continuous bounded nonnegative functions \( f_j(x_j(t)) \) denotes the measures of activation to its incoming potentials of the cell \( c_{ij} \) at time \( t \); the continuous function \( \rho_{ij}(t) \) corresponds to the transmission delay along the axon of the \((h, l)\) cell from the \((i,j)\) cell and satisfies \( 0 \leq \rho_{ij}(t) \leq \rho \) (\( \rho \) is a constant); \( \rho = \max_{1 \leq i \leq m, 1 \leq j \leq n} \rho_{ij} \); the continuous bounded nonnegative functions \( \delta_j(x_j(t)) \) denotes the measures of activation to its incoming potentials of the cell \( c_{ij} \) at time \( t \); \( i \) denotes the bounded external bias on the \((i,j)\) cell at time \( t \); \( b_{ij}(t) \) and \( c_{ij}(t) \) correspond to the positive bounded synaptic connection weight of the cell \( c_{ij} \) on the cell \( c_{ij} \) at time \( t \). It will be assumed that \( a_i, b_{ij}, c_{ij}, I_{ij}, \delta_j : \mathbb{R} \to \mathbb{R}, \ i = 1, 2, \ldots, m, j = 1, 2, \ldots, n, k \in \mathbb{Z}_+ \) are continuous functions.

Now, we will present the following class of maps and norm:

\[
C[X, Y] = \{ \phi : X \to Y | \phi(\cdot) \text{ is a continuous mapping from the topological space } X \text{ to the topological space } Y \}
\]

Set \( X(t) = (x_1(t), \ldots, x_n(t), \ldots, x_m(t), \ldots, x_{mn}(t)) \), for \( \forall x \in \mathbb{R}^{m \times n} \) its norm is given by \( \|x\| = \max_{1 \leq i \leq m} \left\{ |x_i| \right\} \).

The system (2.1) is supplemented with initial data given by

\[
x(s) = \varphi(s)
\]  

(2.2)

where \( s \in [-\rho, 0], \varphi \in C([-\rho, 0], \mathbb{R}^{m \times n}) \).

Using the constant \( \rho, C([-\rho, 0], \mathbb{R}^{m \times n}) \) is a Banach space with the norm \( \| \cdot \| \) given by \( \| \varphi \| = \sup_{-\rho \leq s \leq 0} \| \varphi(s) \| \). Also, we assume \( \| \varphi(s) \| \leq \bar{h}, \bar{h} \in \mathbb{R}_+ \).

In the present study, we do not necessitate smoothness and monotonicity of the activation functions \( f_i(\cdot) \) and \( g_i(\cdot) \), \( i = 1, 2, \ldots, m, j = 1, 2, \ldots, n \).

From now on the following assumptions will be needed throughout the paper:

(A1) Each \( a_i(\cdot) \) is positive, continuous and bounded, that is, there exist \( \bar{a}_i \) and \( \underline{a}_i \) such that

\[
0 < \underline{a}_i \leq a_i(\cdot) \leq \bar{a}_i
\]

additionally, we denote \( \underline{a} = \min \{ \underline{a}_i \}, \bar{a} = \max \{ \bar{a}_i \} \) where \( i = 1, 2, \ldots, m, j = 1, 2, \ldots, n \).

(A2) There exists a Lipschitz constant \( \ell > 0 \) such that

\[
|f_j(x) - f_j(y)| + |g_j(x) - g_j(y)| + |I_{ij}(x) - I_{ij}(y)| \leq \ell \|x - y\|, \ \forall x, y \in \mathbb{R}
\]

(A3) There exists a positive number \( \theta \in \mathbb{R} \) such that \( \theta_{k+1} - \theta_k \geq \theta \) holds for all \( k \in \mathbb{Z}_+ \) and the surfaces of discontinuity \( \Gamma_k : t = \theta_k + r_k(x), k \in \mathbb{Z}_+ \) satisfy the following conditions:

\[
0 < \theta_k + r_k(x) < \theta_{k+1} + r_{k+1}(x), \quad |\theta_k| \to +\infty \text{ as } |k| \to +\infty,
\]

\[
\tau_k(S + E_k)x + I_k(x) \leq \tau_k(x), \quad x \in \mathbb{R}^{m \times n},
\]

where \( S \) is an \( (m \times n) \times (m \times n) \) identity matrix and

\[
E_k = \text{diag}(e_{11k}, \ldots, e_{mnk}) = \left( \begin{array}{ccc} e_{11k} & 0 & \cdots & 0 \\ 0 & e_{12k} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e_{mnk} \end{array} \right)
\]

and \( \mathbf{l}_k = \left( \begin{array}{c} I_{11k} \\ I_{12k} \\ \vdots \\ I_{mnk} \end{array} \right) \).

(A4) \( \ell (r_1 + r_2) < 1 \), where

\[
r_1 = \max_{1 \leq i \leq m, 1 \leq j \leq n} \left\| \left( \sum_{b_{ij}(t) \in \mathbb{N}} b_{ij}(t)p_1^{+} \right) + \sum_{c_{ij}(t) \in \mathbb{N}} c_{ij}(t)p_2^{+} \right\| + +\infty
\]

\[
r_2 = \max_{1 \leq i \leq m, 1 \leq j \leq n} \left\| (I_{ij}(t)) \right\|, \ p_1 = \max_{1 \leq i \leq m, 1 \leq j \leq n} \left\{ f_j(x_j(0)) \right\},
\]

\[
p_2 = \max_{1 \leq i \leq m, 1 \leq j \leq n} \left\{ g_j(x_j(t) - \rho_{ij}(t)) \right\}
\]

with \( \|x(t)\| \leq \bar{h} \).
Using Akhmet (2010), Bainov and Stamova (1999), Ballinger and Liu (1999), Benchohra, Henderson, Ntouyas, and Ouahab (2004), Hale and Lunel (1993), Liu and Ballinger (2002, 2004), Liu and Wang (2006) and Yang and Xu (2006) it can be shown that if conditions (A1)–(A2) hold, then the system (2.1) with the initial data (2.2) has a unique solution on an interval $I$ in $\mathbb{R}_+$. By virtue of Theorem 5.3.1 in Akhmet (2010) and papers (Liu & Ballinger, 2002; Liu & Wang 2006) with assumptions (A3)–(A4), every solution $x(t)$ of (2.1) with the initial data (2.2) intersects each surface of discontinuity $I_k$, $t = \theta_k + \tau_k(x)$, $k \in \mathbb{Z}_+$, at most once, that is, there is no beating. Furthermore, by using Theorem 3.5 in Liu and Ballinger (2002) with (A1)–(A4), continuation of solutions of delay-differential equation

$$
\begin{align*}
\dot{x}(t) &= -a_q(t)x(t) - \sum_{h \in \mathcal{N}_N(i,j)} b^h_q(t)f_h(x_h(t))x_q(t) \\
&\quad - \sum_{h \in \mathcal{N}_N(i,j)} c^h_q(t)g_h(x_q(t) - \rho_h(t))) \\
&\times x_h(t) + L_q(t), \quad t \geq 0
\end{align*}
$$

(2.3)

and the condition $\theta_k \to +\infty$ as $k \to \infty$, one can prove that every solution $x(t) = x(t, \theta_k, \psi)$, $(\theta_k, \psi) \in \mathbb{R}_+ \times \mathbb{R}^{m \times n}$, of (2.1) with (2.2) is continuous on $\mathbb{R}_+$.

2.1. The reduction to the system with fixed-moments of impulses

In this part of the paper, in order to reduce to the system with fixed-moments of impulses and to investigate the global asymptotic properties of solutions of Eqs. (2.1), we give the techniques of $B$-topology and $B$-equivalence method which were introduced and developed in Akhmet (2010) for the systems of differential equations with state-dependent impulses and generalized to the delay differential equations in Liu and Wang (2006). We refer the reader to the book Akhmet (2010) and the paper Liu and Wang (2006), for a more detailed discussion.

For a fixed $k \in \mathbb{Z}_+$, let $x^k(t)$ be a solution of the system of delay-differential equations (2.3) which passes the point $(\theta_k, x)$. Denote by $t = \xi_k$ the time when the solution of (2.3) intersects the surface of discontinuity $I_k$, $t = \theta_k + \tau_k(x(\xi_k))$, $k \in \mathbb{Z}_+$. Suppose that $x^k(t)$ is also a solution of (2.3) passing through $(\xi_k, (S + \xi_k^k) x^k(\xi_k) + L_k(x^k(\xi_k)))$. Next, we define a mapping $U_k(x) : \mathbb{R}^{m \times n} \to \mathbb{R}^{m \times n}$ such that for each $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, $U_k(x(\theta_k))$ is given by

$$
U_k(x(\theta_k)) = \int_{\frac{\theta_k}{\xi_k}} \left( -a_i(s)x^k_i(s) - \sum_{h \in \mathcal{N}_N(i,j)} b^h_i(s)f_h(x_h^k(s))x^k_i(s) \\
- \sum_{h \in \mathcal{N}_N(i,j)} c^h_i(s)g_h(x^k_i(s) - \rho_h(s)))x^k_i(s) \\
\times f_h(x_h^k(s))x^k_i(s) - \sum_{h \in \mathcal{N}_N(i,j)} c^h_i(s)g_h \\
\times (x^k_h(s) - \rho_h(s)))x^k_i(s) + L_k(s) \right) ds,
$$

where

$$
x^k_i(\xi_k) = x_i(\theta_k) + \int_{\frac{\theta_k}{\xi_k}} \left( -a_i(s)x^k_i(s) - \sum_{h \in \mathcal{N}_N(i,j)} b^h_i(s)f_h(x_h^k(s))x^k_i(s) \\
\times f_h(x_h^k(s))x^k_i(s) - \sum_{h \in \mathcal{N}_N(i,j)} c^h_i(s)g_h \\
\times (x^k_h(s) - \rho_h(s)))x^k_i(s) + L_k(s) \right) ds.
$$

Then, construct a system of impulsive delay-differential equations with fixed moments, which has the form

$$
\begin{align*}
y'_{ij}(t) &= -a_{ij}(t)y_{ij}(t) - \sum_{b^h_{ij} \in \mathcal{N}_N(i,j)} b^h_{ij}(t)f_h(y_h(t))y_{ij}(t) \\
&\quad - \sum_{b^h_{ij} \in \mathcal{N}_N(i,j)} c^h_{ij}(t)g_h(y_h(t) - \rho_h(t))) \\
&\times y_{ij}(t) + L_{ij}(t), \quad t \geq 0
\end{align*}
$$

(2.4)

$$
\Delta y_{ij}(\xi_k) = e_{ij}y_{ij}(\xi_k)
$$

where $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n \in \mathbb{Z}_+$.

We denote $\mathcal{P}(C([\xi_k, \xi_k+\epsilon]; \mathbb{R}^{m \times n})), J \subset \mathbb{R}_+, \epsilon > 0$, the space of all piecewise continuous functions $\psi : J \to \mathbb{R}^{m \times n}$ with points of discontinuity of the first kind $\theta_k$, $k \in \mathbb{Z}_+$ which continuous from the right.

Let $x(t)$ be a solution of Eq. (2.1) on $\mathcal{U}$ ($\mathcal{U}$ can be an interval or a real half-line) with initial condition (2.2).

**Definition 2.1.** A solution $y(t) \in \mathcal{P}(C([\xi_k, \xi_k+\epsilon]; \mathbb{R}^{m \times n}))$ of (2.4) is said to be in the $\varepsilon$-neighborhood of a solution $x(t) \in \mathcal{P}(C([\xi_k, \xi_k+\epsilon]; \mathbb{R}^{m \times n}))$ if:

(i) the measure of the symmetrical difference between the domains of existence of these solutions does not exceed $\varepsilon$;

(ii) discontinuity points of $y(t)$ are in $\varepsilon$-neighborhoods of discontinuity points of $x(t)$;

(iii) for all $t \in \mathcal{U}$ outside of $\varepsilon$-neighborhoods of discontinuity points of $x(t)$ the inequality $\|x(t) - y(t)\| < \varepsilon$ holds.

The topology defined by $\varepsilon$-neighborhoods of piecewise continuous solutions will be called the $B$-topology. One can easily verify that it is Haudsford topology. Topologies and metrics for spaces of discontinuous functions were introduced and developed in Akhmet (2005) and Akhmet and Perestyuk (1990).

For any $\alpha, \beta \in \mathbb{R}_+$ we define the oriented interval $[\alpha, \beta]$ as

$$
[\alpha, \beta] = \begin{cases}
[\alpha, \beta], & \text{if } \alpha \leq \beta \\
[\beta, \alpha], & \text{otherwise}
\end{cases}.
$$

(2.5)

**Definition 2.2.** Systems (2.1) and (2.4) are said to be $B$-equivalent, if for any solution $x(t)$ of (2.1) defined on an interval $\mathcal{U}$ with the discontinuity points $\xi_k$, $k \in \mathbb{Z}_+$, there exists a solution $y(t)$ of system (2.4) satisfying

$$
x(t) = y(t), \quad t \in \mathbb{R}_+ \setminus \bigcup_{k \in \mathbb{Z}_+} [\xi_k, \theta_k].
$$

(2.6)

In particular,

$$
x(\theta_k) = y(\theta_k), \quad x(\xi_k) = y(\xi_k)
$$

(2.7)

$$
x(\theta_k) = y(\theta_k), \quad x(\xi_k) = y(\xi_k)
$$

(2.8)

Conversely, for each solution $y(t)$ of Eq. (2.4), there exists a solution $x(t)$ of system (2.1), which satisfies the conditions (2.6)–(2.8).

The proof of following lemma is quite similar to that of Theorem 5.8.1 in Akhmet (2010) and Lemma 3.1 in Liu and Wang (2006), so we omit it here.

**Lemma 2.1.** Assume that conditions (A1)–(A4) are satisfied by (2.1), then there are mapping $U_k(y) : \mathbb{R}^{m \times n} \to \mathbb{R}^{m \times n}$, $k \in \mathbb{Z}_+$, such that corresponding to each solution $x(t)$ of Eq. (2.1), there is a solution $y(t)$ of the system (2.4) satisfying $x(t) = y(t)$ if $t \in \mathbb{R}_+ \setminus \bigcup_{k \in \mathbb{Z}_+} [\xi_k, \theta_k]$. Moreover, the functions $U_k(x)$ satisfy the inequality

$$
\|U_k(x) - U_k(y)\| \leq \hat{k}(\ell)|x - y|,
$$

$$
\hat{k}(\ell) = k(\ell, \hat{h})
$$

is a bounded function, uniformly with respect to $k \in \mathbb{Z}_+$, for all $x, y \in \mathbb{R}^{m \times n}$, such that $|x| \leq \hat{h}$ and $|y| \leq \hat{h}$. 

\[\]
3. Existence and global exponential stability of periodic solution

In this section, we will show existence and exponential stability of periodic solution of system (2.4). Then using $\mathcal{B}$-equivalence technique, we will show existence and exponential stability of periodic solution of the system (2.1).

From now on, we need the following assumptions, lemmas and definitions to prove our main results.

**(A5)** System (2.1) is $(w, q)$-periodic, that is, there exist constants $w \in \mathbb{R}^+$, $q \in \mathbb{Z}_+$ such that $a_i(t + w) = a_i(t)$, $b_{ij}^\theta(t + w) = b_{ij}^\theta(t)$, $a_{ij}^\eta(t + w) = a_{ij}^\eta(t)$, $l_i^g(t + w) = l_i^g(t)$.

Definition 3.1. Assume (A5) holds. Then the system (2.4) is also $(w, q)$-periodic, that is, $U_y$ also satisfies $U_{y_{i+kq}} = U_y$.

Definition 3.2. Assume $y_i(t_0, \varphi_i^0)$ is a $(w, q)$-periodic solution of system (2.4) with initial value $\varphi_i^0 \in C$. If there exist positive constants $N, \eta$ such that for any solution $y_i(t, t_0, \varphi_i^0)$ with initial value $\varphi_i^0 \in C$

\[\|y_i(t, t_0, \varphi_i^0) - y_i(t, t_0, \varphi_i^0)\| \leq N \|\varphi_i^0 - \varphi_i^0\| e^{-\eta(t-t_0)},\]

for all $t \geq t_0$, then $y(t, t_0, \varphi_i^0)$ is called to be globally exponentially stable.

Now, consider the following non-impulsive delay differential system,

\[\frac{dz_i(t)}{dt} = -a_i(t)z_i(t) - \sum_{b_{ij}^\theta \in N_{(i,j)}} b_{ij}^\theta(t) f_{ij}y_i(t) - \sum_{c_{ij}^\eta \in N_{(i,j)}} c_{ij}^\eta(t) g_{ij}y_i(t) - \sum_{0 < q_i \leq s(t)} l_i^g(t) y_i(t) + \sum_{0 < q_i \leq s(t)} (1 - \varphi_i(t)) l_i^g(t) ,\]

with initial condition $z_i(s) = \varphi_i(s), s \in [-\rho, 0], i = 1, 2, \ldots, m, j = 1, 2, \ldots, n$.

The proof of the following lemma is quite similar to that of Theorem 1 in Yan and Zhao (1998) and we will omit it here.

**Lemma 3.2.** Assume (A7) holds, then

(i) if $z = (z_1, \ldots, z_{1n}, \ldots, z_{mn})^T$ is a solution of (3.9), then

\[y = \left(\prod_{0 < q_i < t} (1 - \varphi_i(t)) z_{11}, \ldots, \prod_{0 < q_i < t} (1 - \varphi_i(t)) z_{1n}, \ldots, \prod_{0 < q_i < t} (1 - \varphi_i(t)) z_{mn}\right)\]

is a solution of (2.4)

(ii) if $y = (y_1, \ldots, y_{1n}, \ldots, y_{mn})^T$ is a solution of (2.4), then

\[z = \left(\prod_{0 < q_i < t} (1 - \varphi_i(t))^{-1} y_1, \ldots, \prod_{0 < q_i < t} (1 - \varphi_i(t))^{-1} y_{1n}, \ldots, \prod_{0 < q_i < t} (1 - \varphi_i(t))^{-1} y_{mn}\right)\]

is a solution of (3.9).

To prove the existence of periodic solution of system (2.4), we will use the technique introduced in Gaines & Mawhin (1977) as follows:

Let $X, Z$ be normed vector spaces, $L : Dom L \subset X \rightarrow Z$ be a linear mapping and $N : X \rightarrow Z$ be a continuous mapping. The mapping $L$ will be called a Fredholm mapping of index zero if $\text{dim Ker } L = \text{codim Im } L < +\infty$ and $\text{Im } L$ is closed in $Z$. If $L$ is a Fredholm mapping of index zero there exist continuous projectors $P : X \rightarrow X$ and $Q : Z \rightarrow Z$ such that $\text{Im } P = \text{Ker } L$, $\text{Ker } Q = \text{Im } L = \text{Im } (I - Q)$. It follows that mapping $L|_{\text{Dom } L \cap \text{Ker } R} : (I - P)X \rightarrow \text{Im } L$ is invertible. We denote the inverse of that map by $K_P$. If $\Omega$ is an open bounded subset of $X$, the mapping $N$ will be called L-compact on $\bar{\Omega}$ if $QN(\bar{\Omega})$ is bounded and $K_P(I - Q)N : \bar{\Omega} \rightarrow X$ is compact. Since $\text{Im } Q$ is isomorphic to $\text{Ker } L$, there exists an isomorphism $J : \text{Im } Q \rightarrow \text{Ker } L$.

Now, for convenience of the reader, we state Mawhin’s continuation theorem (Gaines & Mawhin, 1977, p. 40) as follows:

**Lemma 3.3** (Gaines & Mawhin, 1977). Let $X$ and $Z$ be two normed spaces, $L : Dom L \subset X \rightarrow Z$ be a Fredholm operator with index zero, $\Omega \subset X$ is an open bounded set and $N : X \rightarrow Z$ is a continuous operator which is $L$-compact on $\Omega$. Assume

(i) for each $\tilde{x} \in (0, 1), x \in \partial \Omega \cap \text{Dom } L, Lx \neq \tilde{x} Nx$,

(ii) for each $x \in \partial \Omega \cap \text{Ker } L, QNx \neq 0$,

(iii) $\deg \{QN, \Omega \cap \text{Ker } L, 0\} \neq 0$.

Then, the equation $Lx = Nx$ has at least one solution in $\bar{\Omega} \cap \text{Dom } L$.

Now, we will give the following lemma with its proof to obtain our main results.

**Lemma 3.4.** Assume that the conditions (A1)–(A2) and (A5)–(A7) hold, then the system (2.4) has at least one $(w, q)$-periodic solution.
**Proof.** By employing Lemma 3.2, we only need to show that the system (3.9) has at least one $w$-periodic solution. Introduce the following notation

$$H_y(t) = -a_y(t)z_y(t) - \sum_{b^h \in N_l(i,j)} b^{h}_y(t)f_{ij}$$

$$+ \left( \prod_{0 < c_{kl} < t} (1 - v_{hlk})z_{hlk}(t) \right) z_{hlk}(t) - \sum_{c^{h_l}_y \in N_l(i,j)} c^{h}_{y}(t)g_{ij}$$

$$+ \left( \prod_{0 < c_{kl} < t - \rho_{hlk}(t)} (1 - v_{hlk})z_{hlk}(t - \rho_{hlk}(t)) \right) z_{hlk}(t)$$

In order to employ Lemma 3.3, we define $Z = \mathbb{Y} = \{z(t) \in C(\mathbb{R}, \mathbb{R}^{m \times n}) : z(t + w) = z(t)\}$ and the norm $\|z\| = \max_{[0,w]} \sum_{i,j} |z_{ij}(t)|, \quad \forall z \in Z$.

Clearly, $Z$ and $\mathbb{Y}$ are Banach spaces with the defined norm $\| \cdot \|$. Let,

$$L_z = \frac{dz(t)}{dt} \quad \text{and} \quad P_z = \frac{1}{w} \int_0^w z(t) dt, \quad z \in Z;$$

$$Q_y = \frac{1}{w} \int_0^w y(t) dt, \quad y \in \mathbb{Y}.$$

Also,

$$N : Z \to Z \quad \text{such that} \quad Nz(t) = H_y(t).$$

Then, it can be easily deduced that $Ker L = \mathbb{R}^{m \times n}, Im L = \{y | y \in \mathbb{Y} : \int_0^w y(s) ds = 0\}$ and $dim Ker L = m \times n = Codim Im L$. Thus, $Im L$ is closed in $\mathbb{Y}$ and $L$ is a Fredholm mapping of index zero. Also, $P, Q$ are continuous projectors such that $Im P = Ker L, \quad Ker Q = Im (I - Q) = Im L$.

Therefore, the inverse $K_P : Im L \to Ker P \cap Dom L$ of $L_P$ is in the form

$$K_P(y) = \int_0^w y(s) ds - \frac{1}{w} \int_0^w \int_0^w y(s) ds dt.$$ 

Hence,

$$QNz = \left( \frac{1}{w} \int_0^w H_{11}(t) dt, \ldots, \frac{1}{w} \int_0^w H_{1n}(t) dt, \ldots, \frac{1}{w} \int_0^w H_{n1}(t) dt, \ldots, \frac{1}{w} \int_0^w H_{nn}(t) dt \right)^T.$$

and

$$K_P(I - Q)Nz = \left( \int_0^t H_{0h}(t) dt \right)_{mn \times 1} + \left( \frac{t}{w} - \frac{1}{2} \right) \int_0^t H_{0h}(t) dt \right)_{mn \times 1} - \frac{1}{w} \int_0^w \int_0^t H_{0h}(s) ds dt \right)_{mn \times 1}.$$

Clearly, $QN$ and $K_P(I - Q)N$ are continuous. It is not difficult to show that $K_P(I - Q)N(\Omega)$ and $QN(\Omega)$ are relatively compact for any open bounded set $\Omega \subset Z$ by employing Arzelà-Ascoli theorem. So, $N$ is $L$-compact on $\Omega$ for any open bounded set $\Omega \subset \mathbb{R}$.

Now, we will search an appropriate open, bounded subset $\Omega$, for the application of Lemma 3.3. The operator equation $Lz = \tilde{\lambda}Nz$ with $\tilde{\lambda} \in (0, 1)$ implies,

$$z_y'(t) = -\tilde{\lambda}a_y(t)z_y(t) + \tilde{\lambda} \left\{ - \sum_{b^h \in N_l(i,j)} b^{h}_{y}(t)f_{ij}$$

$$+ \left( \prod_{0 < c_{kl} < t} (1 - v_{hlk})z_{hlk}(t) \right) z_{hlk}(t) - \sum_{c^{h_l}_y \in N_l(i,j)} c^{h}_{y}(t)g_{ij}$$

$$+ \left( \prod_{0 < c_{kl} < t - \rho_{hlk}(t)} (1 - v_{hlk})z_{hlk}(t - \rho_{hlk}(t)) \right) z_{hlk}(t)$$

$$+ \prod_{0 < c_{kl} < t} (1 - v_{hlk})^{-1}L_y(t) \right\}. \quad (3.10)$$

Assume that $z(t) \in Z$ is a solution of (3.10) for some $\tilde{\lambda} \in (0, 1)$. Integrating $z_yz_y'$ over the interval $[0, w]$, we get

$$0 = \frac{1}{2} \left( z_y^2(w) - z_y^2(0) \right) = \int_0^w z_y(t)z_y'(t) dt$$

$$= \tilde{\lambda} \int_0^w \left\{ -a_y(t)z_y(t)z_y'(t) - \sum_{b^h \in N_l(i,j)} b^{h}_{y}(t)f_{ij}$$

$$+ \left( \prod_{0 < c_{kl} < t} (1 - v_{hlk})z_{hlk}(t) \right) z_{hlk}(t) - \sum_{c^{h_l}_y \in N_l(i,j)} c^{h}_{y}(t)g_{ij}$$

$$+ \left( \prod_{0 < c_{kl} < t - \rho_{hlk}(t)} (1 - v_{hlk})z_{hlk}(t - \rho_{hlk}(t)) \right) z_{hlk}(t)$$

$$+ \prod_{0 < c_{kl} < t} (1 - v_{hlk})^{-1}L_y(t) \right\} dt.$$

That implies,

$$\int_0^w a_y(t)z_y^2(t) dt$$

$$= \int_0^w \left\{ - \sum_{b^h \in N_l(i,j)} z_y^2(t) b^{h}_{y}(t) f_{ij} \left( \prod_{0 < c_{kl} < t} (1 - v_{hlk})z_{hlk}(t) \right)$$

$$- \sum_{c^{h_l}_y \in N_l(i,j)} z_y^2(t) c^{h}_{y}(t) g_{ij}$$

$$+ \left( \prod_{0 < c_{kl} < t - \rho_{hlk}(t)} (1 - v_{hlk})z_{hlk}(t - \rho_{hlk}(t)) \right) \right\} dt$$

$$+ \int_0^w \prod_{0 < c_{kl} < t} (1 - v_{hlk})^{-1}z_y(t)L_y(t) dt.$$ 

Then, using the assumption (A1) and boundedness of coefficients and activation functions, we obtain that

$$\left\{ a_y + \sum_{b^h \in N_l(i,j)} b^{h}_{y} + \sum_{c^{h_l}_y \in N_l(i,j)} c^{h}_{y} \right\} \int_0^w |z_y(t)|^2 dt$$

$$\leq \tilde{\lambda} \int_0^w \int_0^w (1 - v_{hlk})^{-2} dt \left( \int_0^w |z_y(t)|^2 dt \right)^{1/2}$$

$$= \tilde{\lambda} \int_0^w \int_0^w |z_y(t)|^2 dt \right)^{1/2}.$$
Thus,
\[
\left(\int_0^w |z_j(t)|^2 dt \right)^{1/2} \leq \bar{L}_j W_{ij} \left\{ a_j + \sum_{b \in E_{N(i,j)}} b_{ij}^0 + \sum_{c \in E_{N(i,j)}} c_{ij}^0 \right\} =: S_{ij}.
\]  

(3.11)

Let \( \beta \in [0, w] \) not \( \neq \) \( \theta_k \) such that \( |z_j(\beta)| = \inf_{z \in [0, w]} |z_j(t)|. \) Then, using (3.11), we get

\[
|z_j(\beta)| w = |z_j(\beta)| \int_0^w dt \leq \int_0^w |z_j(t)| dt \leq \left( \int_0^w 1^2 dt \right)^{1/2} \left( \int_0^w |z_j(t)|^2 dt \right)^{1/2} \leq \sqrt{w} S_{ij}.
\]

Hence,

\[
|z_j(\beta)| \leq \frac{S_{ij}}{\sqrt{w}}.
\]

(3.12)

Using (3.12) and \( z_j(t) = z_j(\beta) + \int_\beta^t z_j'(t) dt, \) we have

\[
|z_j(t)| \leq \frac{S_{ij}}{\sqrt{w}} + \int_\beta^t |z_j(t)| dt.
\]

(3.13)

Furthermore, the conditions (A1), (A7) and Eq. (3.10) result in

\[
\int_\beta^t |z_j(t)| dt \leq \left\{ a_j + \sum_{b \in E_{N(i,j)}} b_{ij}^0 R_{ij}^1 + \sum_{c \in E_{N(i,j)}} c_{ij}^0 R_{ij}^1 \right\}
\times \int_0^w |z_j(t)| dt + \bar{L}_j \int_0^w \prod_{0 < c < t} (1 - v_{ahk})^{-1} dt
\]

\[
\leq \left\{ \bar{a}_j + \sum_{b \in E_{N(i,j)}} b_{ij}^0 R_{ij}^1 + \sum_{c \in E_{N(i,j)}} c_{ij}^0 R_{ij}^1 \right\}
\times \sqrt{w} \left( \int_0^w |z_j(t)|^2 dt \right)^{1/2} + \bar{L}_j \sqrt{w} \left( \int_0^w \prod_{0 < c < t} (1 - v_{ahk})^{-2} dt \right)^{1/2}
\]

\[
\leq \left\{ \bar{a}_j + \sum_{b \in E_{N(i,j)}} b_{ij}^0 R_{ij}^1 + \sum_{c \in E_{N(i,j)}} c_{ij}^0 R_{ij}^1 \right\}
\times \sqrt{w} S_{ij} + \bar{L}_j \sqrt{w} W_{ij} := D_{ij}.
\]

Hence, we obtain,

\[
|z_j(t)| \leq \frac{S_{ij}}{\sqrt{w}} + D_{ij} =: Q_j,
\]

(3.14)

Denote,

\[ K = mn \max_{1 \leq i \leq m, 1 \leq j \leq n} \{ Q_j \} \]

then \( K \) is independent of \( \lambda. \) Now, define \( \Omega = \{ z \in \mathbb{Z} : \| z \| = K + 1 \}. \)

It is obvious that \( \Omega \) satisfies the condition (i) in Lemma 3.3. When, \( z \in \partial \Omega \cap \text{Ker} L, \) where \( z = (z_1, \ldots, z_m), \) is a constant vector in \( \mathbb{R}^{m \times w} \) with the property \( \| z \| = K + 1. \) Furthermore, take, \( J : \text{Im} Q \rightarrow \text{Ker} L, R \rightarrow R \) and let \( K \) be greater such that

\[ z^T Q N z < 0. \] Thus, for any \( z \in \partial \Omega \cap \text{Ker} L, Q N z \neq 0. \) Additionally, let \( \Phi (\tilde{\gamma}; z) = -\tilde{\gamma} z + (1 - \gamma) \text{QN} z, \) then for any \( z \in \partial \Omega \cap \text{Ker} L, \)

\[ z^T \Phi (\tilde{\gamma}; z) < 0, \]

we obtain,

\[ \deg \{ QN, \Omega \cap \text{Ker} L \} \neq 0. \]

Hence, conditions (ii) and (iii) of Lemma 3.3 are also satisfied. Therefore, the system (3.9) has at least one \( w \)-periodic solution. Consequently, the system (2.4) has at least one \((w, q)\)-periodic solution. This completes the proof. \( \square \)

Now, we need to prove global exponential stability of periodic solution. Assume that \( y^*(t) = y_{11}(t), y_{12}(t), \ldots, y_{m1}(t), \ldots, y_{qm}(t) \) is a \((w, q)\)-periodic solution of the system (2.4) with initial data \( y_0'(s) = \psi_0(s), s \in [-\rho, 0] \) and \( y(t) = y_{11}(t), \ldots, y_{q1}(t), y_{q2}(t), \ldots, y_{qm}(t) \) is an arbitrary solution of the system (2.4) with initial data \( y(s) = \psi(s), s \in [-\rho, 0]. \) Let us define a new dependent variable \( u(t) = y(t) - y^*(t) \), then we obtain,

\[
\begin{align*}
\sum_{b \in E_{N(i,j)}} b_{ij}^0(t) y_{b1}(t) y_{b2}(t) & - \sum_{c \in E_{N(i,j)}} c_{ij}^0(t) y_{c1}(t) y_{c2}(t) \\
& + \sum_{b \in E_{N(i,j)}} b_{ij}^0(t) y_{b1}(t) y_{b2}(t) + \sum_{c \in E_{N(i,j)}} c_{ij}^0(t) y_{c1}(t) y_{c2}(t)
\end{align*}
\]

(3.15)

\[
\Delta u_{ij} = -v_{ijk} u_{ij} \]

where \( i = 1, 2, \ldots, m; j = 1, 2, \ldots, n. \) In order to prove exponential stability of periodic solution of the system (2.4), it is enough to prove exponential stability of zero solution of system (3.15).

We will need the following assumptions:

(A8) there exist positive constants \( p_{11}, \ldots, p_{1n}, \ldots, p_{m1}, \ldots, p_{mn} \)

such that

\[
\left( a_j - \sum_{b \in E_{N(i,j)}} b_{ij}^0 R_{ij}^1 - \sum_{c \in E_{N(i,j)}} c_{ij}^0 R_{ij}^1 \right) p_{ij} = 0.
\]

(3.16)

\[
\left( \sum_{b \in E_{N(i,j)}} b_{ij}^0 R_{ij}^1 + \sum_{c \in E_{N(i,j)}} c_{ij}^0 R_{ij}^1 \right) Q_j p_{ij} > 0
\]

where \( i = 1, 2, \ldots, m; j = 1, 2, \ldots, n. \)

(A9) There exists a constant \( \gamma < \lambda, \) such that

\[
\gamma_k \leq \lambda - \alpha_k \]

where \( \Delta \theta_k = \theta_k - \theta_{k-1} \) for \( k \in \mathbb{Z}_+, \gamma_k = \max \{ 1, \max_{1 \leq j \leq m} (\gamma_{ij} - a_j + \sum_{b \in E_{N(i,j)}} b_{ij}^0 R_{ij}^1 + \sum_{c \in E_{N(i,j)}} c_{ij}^0 R_{ij}^1) p_{ij} + \sum_{c \in E_{N(i,j)}} c_{ij}^0 Q_j e^{i(k\lambda - 1/v_{ijk})} \} \]

\[ \lambda = \min_{1 \leq j \leq m} \{ \lambda_j = \langle \lambda_j - a_j + \sum_{b \in E_{N(i,j)}} b_{ij}^0 R_{ij}^1 + \sum_{c \in E_{N(i,j)}} c_{ij}^0 R_{ij}^1 \rangle p_{ij} + \sum_{c \in E_{N(i,j)}} c_{ij}^0 Q_j e^{i(k\lambda - 1/v_{ijk})} \} \]

\[ |1 - v_{ijk}| = 1 - v_{ijk} \]

Lemma 3.5. Assume that the conditions (A1)–(A2) and (A5)–(A9) are valid. Then periodic solution of the system (2.4) is globally exponentially stable.

Proof. Define the following function:

\[
G_0(\lambda_{ij}) = \left( \lambda_j - a_j + \sum_{b \in E_{N(i,j)}} b_{ij}^0 R_{ij}^1 + \sum_{c \in E_{N(i,j)}} c_{ij}^0 R_{ij}^1 \right) p_{ij}
\]

\[
+ \left( \sum_{b \in E_{N(i,j)}} b_{ij}^0 Q_j e^{i(k\lambda - 1/v_{ijk})} \right) p_{ij}.
\]
Using the condition (A8), we have

\[
G_q(0) = \left( -a_{ij} + \sum_{b_{ij} \in E_N(i,j)} b_{ij}^0 R_{ij}^0 + \sum_{c_{ij} \in E_N(i,j)} c_{ij}^0 R_{ij}^0 \right) p_{ij}
+ \left( \sum_{b_{ij} \in E_N(i,j)} b_{ij}^0 + \sum_{c_{ij} \in E_N(i,j)} c_{ij}^0 \right) q_{ij} p_{hi} < 0.
\]

Additionally, \( \frac{d}{dt} G_q(\lambda_q) = p_{ij} + \sum_{e_{ij} \in E_N(i,j)} c_{ij}^0 Q_{ij} e^{\lambda_{ij} q_{ij} \rho_{ij} p_{hi}} > 0 \), and \( \lim_{\lambda_q \to \infty} G_q(\lambda_q) \to \infty \) which results in that \( G_q(\lambda_q) \) is a strictly monotone increasing function. Hence, it can be concluded that there exists \( \delta_q > 0 \) such that

\[
G_q(\delta_q) = 0 \quad \text{and} \quad G_q(\lambda_q) < 0 \quad \text{for all} \quad \lambda_q \in (0, \delta_q).
\]

Therefore, for each \( i = 1, 2, \ldots, m, j = 1, 2, \ldots, n \), we have \( G_q(\lambda) < 0 \), where \( \lambda = \min_{1 \leq i \leq m, 1 \leq j \leq n} \{ \lambda_q \} \). We can find a constant \( \eta > 1 \) such that \( \sup_{-\rho \leq s \leq 0} |q_{ij}(s) - \varphi_q(s)| < \eta p_{ij} = \sigma_q \) and thus for \( i = 1, 2, \ldots, m, j = 1, 2, \ldots, n \), we have

\[
\left( \lambda - a_{ij} + \sum_{b_{ij} \in E_N(i,j)} b_{ij}^0 R_{ij}^0 + \sum_{c_{ij} \in E_N(i,j)} c_{ij}^0 R_{ij}^0 \right) \sigma_q
+ \left( \sum_{b_{ij} \in E_N(i,j)} b_{ij}^0 + \sum_{c_{ij} \in E_N(i,j)} c_{ij}^0 \right) q_{ij} R_{ij}^0 \sigma_q
= \left( \lambda - a_{ij} + \sum_{b_{ij} \in E_N(i,j)} b_{ij}^0 R_{ij}^0 + \sum_{c_{ij} \in E_N(i,j)} c_{ij}^0 R_{ij}^0 \right) p_{ij}
+ \left( \sum_{b_{ij} \in E_N(i,j)} b_{ij}^0 + \sum_{c_{ij} \in E_N(i,j)} c_{ij}^0 \right) q_{ij} R_{ij}^0 \eta < 0. \quad (3.16)
\]

Let us define \( q_{ij}(t) = |u_{ij}(t)|, \quad i = 1, 2, \ldots, m, j = 1, 2, \ldots, n. \) Then, we have

\[
\frac{d^+ q_{ij}(t)}{dt} = \text{sign}(u_{ij}(t)) \left[ -a_{ij} |u_{ij}(t)| - \sum_{b_{ij} \in E_N(i,j)} b_{ij}^0 |f_{ij}(y_{hi}(t))| y_{hi}(t) \right.
\times (y_{hi}(t)) y_{ij}(t) + \sum_{b_{ij} \in E_N(i,j)} b_{ij}^0 |f_{ij}(y_{hi}(t))| y_{ij}(t)
- \sum_{c_{ij} \in E_N(i,j)} c_{ij}^0 |g_{ij}(y_{hi}(t))| y_{ij}(t)
\left. + \sum_{c_{ij} \in E_N(i,j)} c_{ij}^0 \left| g_{ij}(y_{hi}(t) - \rho_h(t)) \right| y_{ij}(t) \right]
\leq \left[ -a_{ij} |u_{ij}(t)| + \sum_{b_{ij} \in E_N(i,j)} b_{ij}^0 |f_{ij}(y_{hi}(t))| |u_{ij}(t)| \right.
+ \sum_{b_{ij} \in E_N(i,j)} b_{ij}^0 |f_{ij}(y_{hi}(t)) - f_{ij}(y_{hi}(t))| |y_{ij}(t)|
+ \sum_{c_{ij} \in E_N(i,j)} c_{ij}^0 |g_{ij}(y_{hi}(t) - \rho_h(t))| |u_{ij}(t)|
+ \sum_{c_{ij} \in E_N(i,j)} c_{ij}^0 \left| g_{ij}(y_{hi}(t) - \rho_h(t)) \right| y_{ij}(t)
- g_{ij}(y_{hi}(t) - \rho_h(t)) \right] |y_{ij}(t)|
\leq -a_{ij} q_{ij}(t) + \sum_{b_{ij} \in E_N(i,j)} b_{ij}^0 R_{ij}^0 q_{ij}(t)
+ \sum_{c_{ij} \in E_N(i,j)} c_{ij}^0 R_{ij}^0 q_{ij}(t)
\leq \left[ -a_{ij} + \sum_{b_{ij} \in E_N(i,j)} b_{ij}^0 R_{ij}^0 + \sum_{c_{ij} \in E_N(i,j)} c_{ij}^0 R_{ij}^0 \right] q_{ij}(t)
+ \sum_{b_{ij} \in E_N(i,j)} b_{ij}^0 q_{ij} R_{ij}^0 (t - \rho_h(t))
+ \sum_{c_{ij} \in E_N(i,j)} c_{ij}^0 q_{ij} R_{ij}^0 (t - \rho_h(t)) \quad (3.17)
\]

for \( t \geq t_0, t \neq \theta_k, k \in \mathbb{Z}_+ \). We will use mathematical induction to continue the proof. We claim for \( t \in [t_0, \theta_1) \) that

\[
V_q(t) = e^{e^t q_{ij}(t)} < \sigma_q, \quad i = 1, 2, \ldots, m, \quad j = 1, 2, \ldots, n. \quad (3.19)
\]

Assume that it is not true. Then, we can find some \( i, j, s_i \in [0, \theta_1) \) such that \( V_q(s_i) = \sigma_q \) and \( V_{hi}(t) < \sigma_{hi} \) for \( t \in [0, s_i), h = 1, 2, \ldots, m, l = 1, 2, \ldots, n \) which results in that \( \frac{d V_q(s_i)}{dt} \geq 0 \). Then using (3.18), we have,

\[
0 \leq \frac{d^+ V_q(s_i)}{dt} \leq \left( \lambda - a_{ij} + \sum_{b_{ij} \in E_N(i,j)} b_{ij}^0 R_{ij}^0 + \sum_{c_{ij} \in E_N(i,j)} c_{ij}^0 R_{ij}^0 \right) V_q(s_i)
+ \sum_{b_{ij} \in E_N(i,j)} b_{ij}^0 V_q R_{ij}^0 V_q(s_i)
\]
\[ + \sum_{\epsilon^{h_{N(i,j)}}} c_{ij}^{h} Q_{ij} e^{\lambda_{ji} t} V_{hi}(s_{1} - \rho_{hi}(s_{1})) \]

\[ \leq \left( \lambda - \alpha_{ij} + \sum_{b^{h_{N(i,j)}}} b_{ij}^{h} R_{ij} \right) + \sum_{c^{h_{N(i,j)}}} c_{ij}^{h} Q_{ij} e^{\lambda_{ji} t} \sigma_{ij} \]

\[ \left. + \sum_{b^{h_{N(i,j)}}} b_{ij}^{h} Q_{ij} e^{\lambda_{ji} t} \right) \sigma_{hi} \]

for all \( t \in [0, t_{0}), k \in \mathbb{Z}_{+} \). Also, we have \( \gamma < \lambda \). Hence, origin of the system (3.15) is globally exponentially stable. This completes the proof.

The last inequality (3.23) implies that the system (2.4) has a unique \((w, q)\)-periodic solution \( y^{*}(t) \) satisfying

\[ \| y(t) - y^{*}(t) \| \leq K \cdot e^{-\epsilon (t-t_{0})} \| \psi(s) - \phi^{*}(s) \|, \]

where \( y(t), \epsilon, \phi \) is an arbitrary solution of system (2.4), \( \epsilon = \lambda - \gamma \), \( K = e^{-\gamma t_{0}} \), with \( \| \psi(s) - \phi^{*}(s) \| \geq \max_{k \in \mathbb{Z}_{+}, t_{0} \leq t \leq t_{1}} \| \sigma_{ij} \| \).

Using the \( B \)-equivalence method which was widely explained in Akhmet (2010) and Liu and Wang (2006), the solution \( y(t) \) of (2.4) coincides with the solution \( x(t) \) of Eq. (2.1) at discontinuity points \( t \in (\theta_{k}, \theta_{k+1}) \). The continuous dependence, in the \( B \)-topology, of solutions of Eq. (2.1) on initial data and the right side implies the following result.

**Theorem 3.1.** Assume that conditions (A1)–(A9) are valid. Then (2.1) has a unique globally exponentially stable \((w, q)\)-periodic solution.
Taking absolute value of both sides for each $i = 1, 2, \ldots, m, j = 1, 2, \ldots, n$, and using the condition (A1) we obtain,
\[
|y_i(t) - y_j(t)| \leq |U_{ij}(y_i(\theta_k)) - U_{ij}(x_j(\theta_k))| \\
+ |e_{ij}(y_i(\theta_k) - e_{ij}(x_j^0(\theta_k))| + |e_{ij}(x_j^0(\theta_k))| \\
+ \left| \int_{\theta_k}^{t} \left( a_{ij}(y_i(s) - y_j(s)) \\
+ \sum_{b^h \in N_{(i,j)}} \tau_{ij}^h \left( R^2_j + \ell h \right) \right) ds \right|
\]

Then, adding all terms with using the conditions (A1)–(A2), $U_{ij}(0) = 0$, Lemma 2.1 and boundedness of $e_{ij}$, we get
\[
\|x(t) - y(t)\| \leq \hat{\ell}k(\ell) + \|e_{ij}\| \|y(\theta_k) - x^*(\theta_k)\| \\
+ \hat{\ell}k(\ell) + \|e_{ij}\| \|x^*(\theta_k)\| \\
+ C_1 \int_{\theta_k}^{t} \|x(s) - y(s)\| ds \\
+ C_2 \int_{\theta_k}^{t} \|x(s - \rho) - y(s - \rho)\| ds
\]

where
\[
C_1 = \max_{1 \leq i, j \leq m, 1 \leq s \leq n} \left\{ a_{ij} + \sum_{b^h \in N_{(i,j)}} \tau_{ij}^h \left( R^2_j + \ell h \right) \right\},
\]
\[
C_2 = \max_{1 \leq i, j \leq m, 1 \leq s \leq n} \sum_{c^h \in N_{(i,j)}} \tau_{ij}^h \left( R^2_j + \ell h \right).
\]

We note that for $t \in [\theta_k, \xi_k]$, we can easily show,
\[
\int_{\theta_k}^{t} \|x(s - \rho) - y(s - \rho)\| ds \leq \int_{\theta_k}^{t} \|x(s) - y(s)\| ds.
\]

It follows that,
\[
\|x(t) - y(t)\| \leq \hat{\ell}k(\ell) + \|e_{ij}\| \|y(\theta_k) - x^*(\theta_k)\| \\
+ \hat{\ell}k(\ell) + \|e_{ij}\| \|x^*(\theta_k)\| \\
+ (C_1 + C_2) \int_{\theta_k}^{t} \|x(s) - y(s)\| ds.
\]

By using the Gronwall–Bellman lemma [Akhmet, 2010], we find that
\[
\|x(t) - y(t)\| \leq \hat{\ell}k(\ell) + \|e_{ij}\| e^{(C_1+C_2)\ell \Theta} (\|y(\theta_k) - x^*(\theta_k)\| + \|x^*(\theta_k)\|)
\]

Then, using the inequality (3.24), we obtain,
\[
\|x(t) - y(t)\| \leq (\hat{\ell}k(\ell) + \|e_{ij}\|) e^{(C_1+C_2)\ell \Theta} (K \cdot e^{-\ell \Theta^2 t})
\]

Next, combining last inequality with (3.25), we get
\[
\|x(t) - x^*(t)\| \leq \hat{\ell}k(\ell) + \|e_{ij}\| e^{(C_1+C_2)\ell \Theta} (K \cdot e^{-\ell \Theta^2 t})
\]

Remark 3.1. In Theorem 3.1, we extend the results obtained in Akhmet and Yılmaz (2014), Gui and Ge (2006a, 2006b, 2007), Li and Xing (2007), Lin and Jun (2009), Sun et al. (2009), Wang et al. (2010), Xia et al. (2007), Yang (2009), Yang and Cao (2007), Yang et al. (2010) and Zhang and Gui (2009) to the state-dependent impulsive case. In other words, the results obtained in the previous articles are just the specific case of our results with $\tau(x) = 0$. Therefore, the results of this paper are completely new and more advanced version of the previously constructed results.

Remark 3.2. We remove the condition $0 < \sigma_{ij} < 2$ which was needed in the previous studies (Gui & Ge, 2006a, 2006b, 2007; Lin & Jun, 2009; Sun et al., 2009; Yang et al., 2010; Zhang & Gui, 2009). Therefore, we can let on the impulse magnitude as large as possible. This implies that, our results have more improved functionality of solving real-world problems.

4. An illustrative example

In this paper, theoretical results guarantee that simulation of reduced system with fixed moments of impulses is fully adequate to the original system because of the $B$-equivalence method. Therefore, we will give the following example to simulate the results of Theorem 3.1.

Consider the following neural networks system with fixed moments of impulses. In what follows, let $\theta_k = \pi r + (1 - 1/k)\pi/2$, $k \in \mathbb{Z}_+$ be the sequence of impulse action and network defined as
\[
y_i(t) = \frac{1}{2} (y_i + |y_i - 1|), \quad y_i(t) = \tan (\theta_k), \quad \ell = 1.
\]

Therefore, we will give the following example to simulate the results of (2.1).

Theorem 3.1. Therefore, the results of this paper are completely new and more advanced version of the previously constructed results.
phasizes that, our theoretical results allow the impulse magnitude and periodicsolution for the considered system. Also, one should emphasize that, our theoretical results allow the impulse magnitude as large as possible. This shows that, our results are more realistic for solving real-world problems. Finally, we give one example with numerical simulations to show the effectiveness and applicability of our results.

5. Conclusion

In this paper, it is the first time that the global exponential stability of periodic solution for state-dependent impulsive shunting inhibitory cellular neural networks with variable coefficients and time-varying delays is examined in literature. Although, systems with state-dependent impulses are commonly exist in both biological and artificial neural networks, it is very difficult to analyze such system. To solve problem, first we reduced the system to a fix time impulsive system by using $B$-equivalence method, then we used Mawhin’s continuation theorem of coincide degree theory and an appropriate Lyapunov function. At the end, we obtained easily verifiable sufficient conditions for exponential stability of periodic solution for the considered system. Also, one should emphasize that, our theoretical results allow the impulse magnitude as large as possible. This shows that, our results are more realistic for solving real-world problems. Finally, we give one example with numerical simulations to show the effectiveness and applicability of our results.

Clearly, system \((4.26)\) is \((2\pi, 2)\)-periodic and \((1 - v_{ij}) \neq 0\). Thus, conditions \((A1)\)–\((A7)\) hold. Then, using specified parameters above, we can calculate \([Q_{ij}] = \begin{bmatrix} 6.4349 & 6.4032 \\ 6.3826 & 6.3687 \end{bmatrix}\). This implies

\[
\begin{align*}
\left( a_{ij} - \sum_{b^h \in N(i,j)} b^h R^j_{ij} - \sum_{c^h \in N(i,j)} c^h R^j_{ij} \right) p_{ij} \\
- \left( \sum_{b^h \in N(i,j)} b^h \ell + \sum_{c^h \in N(i,j)} c^h \ell \right) Q_{ij} p_{ij} & \geq 3.75
\end{align*}
\]

where \(p_{ij} = 1\), so condition \((A8)\) holds. Additionally, the condition \((A9)\)

\[ y_k \leq e^{\gamma t} \Delta b_k \]

satisfied with \(\gamma = 1\) and \(\lambda = 1.1\), also, \((\lambda_{ij} - a_{ij} + \sum \alpha_{i \in N(i,j)} R^j_{ij} + \sum \beta_{i \in N(i,j)} R^j_{ij}) p_{ij} + (\sum \gamma_{i \in N(i,j)} R^j_{ij} + \sum \delta_{i \in N(i,j)} R^j_{ij}) Q_{ij} p_{ij} + (\sum \varepsilon_{i \in N(i,j)} R^j_{ij} + \sum \zeta_{i \in N(i,j)} R^j_{ij}) Q_{ij} e^{\gamma t} p_{ij} < -2.4372\) where \(y_k = 2.6\) and \(\lambda_{ij} = 1.1\). Hence, the conditions of Lemma 3.5 satisfied, so periodic solution of the specified network which is same as the periodic solution of the corresponding $B$-equivalence system is globally exponentially stable by Theorem 3.1. These results can be seen from the following numerical simulations Figs. 1–5.
Fig. 5. State trajectory $y_2(t)$ of the system (4.26) with initial data $y_2(t_0) = -0.25$ for $t \in [-1, 0]$. This trajectory is equivalent to the trajectory $x_2(t)$ of the original $B$-equivalent system with the same initial data.

References


