Representation of the utility functional by two fuzzy integrals

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Abstract

In the cumulative prospect theory (CPT) the preference (utility functional) is represented by the difference of two Choquet integrals. We investigate representations of the corresponding functional by two fuzzy integrals, specially the difference representations of the asymmetric Choquet integral with respect to a signed fuzzy measure with bounded chain variation. We discuss non-monotone, real valued functional \( L \), which is revised monotone and its asymmetric Choquet integral-based representation.

Keywords: cumulative prospect theory, signed fuzzy measures, chain variation, Choquet integral

1 Introduction

One of the most important integral based on a fuzzy measure \( m \) is the Choquet integral which is often used in economics, pattern recognition and decision analysis as nonlinear aggregation tool [4, 7, 19]. Two crucial properties of the Choquet integral, defined for non-negative measurable functions, are monotonicity and comonotonic additivity, see [2, 4, 13]. A general monotone (non-decreasing) set function, vanishing at the empty set, is called by various names, such as capacity, cooperative game, non-additive measure, fuzzy measure. In this paper we call them fuzzy measures. A fuzzy measure is non-negative set function, obviously the requirement \( m(\emptyset) = 0 \) ensures its non-negativity. A generalized fuzzy measure, a signed fuzzy measure, introduced by Liu in [9], is revised monotone set function, vanishing at the empty set, and it can take also negative values, see [13]. Murofushi et al. in [10] have used term non-negative fuzzy measure to denote a real-valued set function satisfying \( m(\emptyset) = 0 \), and some authors used term a signed fuzzy measure to denote such set function. In this paper we deal with a signed fuzzy measure in the sense of definition given in [9].

There exist two extensions of Choquet integral to the class of all measurable functions, the symmetric Choquet integral, introduced by Šipoš and the asymmetric Choquet integral, see [2, 4, 10, 13]. The second one is defined with respect to a real-valued set function \( m \), not necessary monotone. For the main field of applications of Choquet integral, decision under uncertainty, an universal set \( X \) is a state of nature and functions from \( X \) to \( \mathbb{R} \) are prospects. The preference relation \( \preceq \) is defined on the set of prospects and we say that the utility functional \( L \) represents a preference relation if and only if \( L(f) \leq L(g) \) for all pairs of prospects \( f, g \) such that \( f \preceq g \). Schmeidler [17, 18] has shown that preference can be represented by Choquet integral \( C_m \) (\( m \) is a fuzzy measure) model, so called Choquet expected utility model (cumulative utility). Choquet expected utility model is not an appropriate tool when the gain and loss must be considered at the same time. In the field of decision theory the cumulative prospect theory (CPT), introduced by Tversky and Kahneman [20], see [3], combines cumulative utility and a generalization of expected utility, so called sign dependent expected utility, related to bipolar scale, see [16]. CPT holds if there
exist two fuzzy measures, \( m^+ \) and \( m^- \), which ensure that the utility functional \( L \), model for preference representation, can be represented by the difference of two Choquet integrals, i.e.,

\[
L(f) = C_m^+(f^+) - C_m^-(f^-),
\]

where \( m^+ \) and \( m^- \) are two fuzzy measures, \( f^+ = f \lor 0 \) is the gain part of prospect \( f \), and \( f^- = (-f) \lor 0 \) is its loss part. Narukawa et al. proved in [11, 12] that comonotone-additive and monotone functional can be represented as a difference of two Choquet integrals and gave the conditions for which it can be represented by one Choquet integral.

Motivated by (1) the aim of this paper is to present some different representations of asymmetric Choquet integral w.r.t. a signed fuzzy measures. The paper is organized as follows. In the next section the short overview of basic notions and definitions is given. In Section 3 we introduce a chain variation of set functions and the space \( BV \), the family of set functions, vanishing at the empty set, with bounded chain variation. In this section we consider a difference representation of Choquet integral w.r.t. a signed fuzzy measure \( m \) with bounded chain variation. In Section 4 an interpreter and a frame for representation of the signed fuzzy measures is defined. We shall prove that for every signed fuzzy measure \( m \in BV \) there exists a representation of \( m \). Applying this result, we present another difference representation of Choquet integral w.r.t. \( m \). In Section 5 we introduce a revised monotone functional and discuss the conditions for its Choquet integral-based representation.

### 2 Preliminaries

Let \( X \) be an universal set. Let \( \mathcal{A} \) be a \( \sigma \)-algebra of subsets of \( X \). \((X, \mathcal{A}) \) is called a measurable space, see [13]. A set function \( \mu : \mathcal{A} \rightarrow [-\infty, \infty] \) with \( \mu(\emptyset) = 0 \) is called a signed measure, if for each sequence \( E_1, E_2, \ldots \) of mutually disjoint sets from \( \mathcal{A} \) the series \( \sum_{i=1}^{\infty} \mu(E_i) \) is defined and the equality

\[
\mu \left( \bigcup_{i=1}^{\infty} E_i \right) = \sum_{i=1}^{\infty} \mu(E_i)
\]

holds, and \( \mu \) assumes at most one of the values \( \infty \) and \( -\infty \). A fuzzy measure \( m \) is a non-negative real-valued set function defined on a \( \sigma \)-algebra \( \mathcal{A} \) with the following properties:

- (FM1) \( m(\emptyset) = 0 \).
- (FM2) \( E \subset F \rightarrow m(E) \leq m(F) \), for all \( E, F \in \mathcal{A} \).

A set function \( m : \mathcal{A} \rightarrow [-\infty, \infty] \) is called a signed fuzzy measure if \( m \) satisfies

- (SFM1) \( m(\emptyset) = 0 \).
- (SFM2) If \( E, F \in \mathcal{A}, E \cap F = \emptyset \), then
  \( a) m(E) \geq 0, m(F) \geq 0, m(E) \lor m(F) > 0 \Rightarrow m(E \cup F) \geq m(E) \lor m(F) \);  
  \( b) m(E) \leq 0, m(F) \leq 0, m(E) \land m(F) < 0 \Rightarrow m(E \cup F) \leq m(E) \land m(F) \);  
  \( c) m(E) > 0, m(F) < 0 \Rightarrow m(F) \leq m(E \cup F) \leq m(E) \).

The conjugate set function of real-valued set function \( m, m : \mathcal{A} \rightarrow \mathbb{R} \) is defined by \( \check{m}(E) = m(X) - m(\bar{E}) \), where \( \bar{E} \) denotes the complement set of \( E \), \( \bar{E} = X \setminus E \). Obviously, if \( m \) is a fuzzy measure, \( \check{m} \) is a fuzzy measure, too.

**Example 1** Let \( X \) be a set of 2n elements. Let \( A, B \subset X \) such that \( X = A \cup B \), \( A \cap B = \emptyset \) and \( \text{card}(A) = \text{card}(B) = n \). We define the set function \( m : \mathcal{P}(X) \rightarrow \mathbb{R} \) by:

\[
m(E) = \begin{cases} 
0, & E = X \\
\text{card}(X), & E = A \\
-\text{card}(X), & E = B \\
\text{card}(A \cap E) - \text{card}(B \cap E), & \text{else}.
\end{cases}
\]

\( m \) is a signed fuzzy measure.

We discuss the condition (SFM2) of revised monotonicity. Same as in the modified version of the example a workshop, given by Murofushi et al. in [10], let us consider the set \( X \) as the set of all workers in a workshop, and sets \( A \) and \( B \) are the sets of good and bad workers in sense of their efficiency, i.e., inefficiency. If we suppose that workers from group \( A \) work two times better if they work all together (with nobody else), and workers from \( B \) two times worse, and in the other cases "anybody is effective in the proportion to its quantitative membership to the 'good' group \( A \) or 'bad' group \( B \)". The signed fuzzy measure \( m \) in the above example is used to denote
the efficiency of the worker. The interpretation of revised monotonicity is in the assumptions that for disjoint groups \( E \) of 'good' and \( F \) of 'bad' workers, if they work together, then their productivity is not greater to productivity of \( E \) and not less to productivity of \( F \), for groups \( E \) and \( F \) of 'good'('bad') workers the simultaneous productivity is not less (not greater) to theirs individual productivity.

Let \( \mathcal{M} \) be the class of all non-negative measurable functions \( f \) on \( X \) and let \( \overline{\mathcal{M}} \) denotes the class of all measurable functions on \( X \). We introduce the Choquet integral with respect to a fuzzy measure \( m : \mathcal{A} \rightarrow [0, \infty] \) (a signed fuzzy measure \( m : \mathcal{A} \rightarrow \mathbb{R} \)) of a measurable function \( f : X \rightarrow [0, \infty] \) \((f : X \rightarrow [-\infty, \infty])\).

**Definition 1** ([2, 10, 13]) Let \((X, \mathcal{A})\) be a measurable space.

(i) The Choquet integral w.r.t. a fuzzy measure \( m : \mathcal{A} \rightarrow [0, \infty] \) is functional \( C_m : \mathcal{M} \rightarrow [0, \infty] \) defined by

\[
C_m(f) = \int_{0}^{\infty} m(\{x \mid f(x) \geq t\}) dt
\]

(ii) The asymmetric Choquet integral w.r.t. a set function \( m : \mathcal{A} \rightarrow \mathbb{R} \), is functional \( C_m : \overline{\mathcal{M}} \rightarrow [-\infty, \infty] \) defined by

\[
C_m(f) = \int_{-\infty}^{0} (m(\{x \mid f(x) \leq t\}) - m(X)) dt + \int_{0}^{\infty} m(\{x \mid f(x) \geq t\}) dt
\]

if both of the above Lebesgue integrals exist. When the expression \( \infty - \infty \) is occurred, the integral is not defined.

Let \( X \) be a finite set \( X = \{x_1, \ldots, x_n\} \). The Choquet integral of a function \( f \in \mathcal{M} \) w.r.t. \( m : \mathcal{A} \rightarrow [0, \infty] \) can be expressed as

\[
C_m(f) = \sum_{i=1}^{n} (f_{\alpha(i)} - f_{\alpha(i-1)}) m(E_{\alpha(i)}),
\]

where \( f \) admits a comonotone-additive representation \( f = \sum_{i=1}^{n} f_{\alpha(i)} 1_{E_{\alpha(i)}} \) and \( \alpha = (\alpha(1), \alpha(2), \ldots, \alpha(n)) \) is a permutation of index set \( \{1, 2, \ldots, n\} \) such that

\[
0 \leq f_{\alpha(1)} \leq \cdots \leq f_{\alpha(n)}.
\]

\( f_{\alpha(0)} = 0 \), sets \( E_{\alpha(i)} \) are given by

\[
E_{\alpha(i)} = \{x_{\alpha(i)}, \ldots, x_{\alpha(n)}\}
\]

and \( 1_E \) is characteristic function of a crisp subset \( E \) of \( X \). The asymmetric Choquet integral can be expressed in the terms of the Choquet integrals of non-negative functions \( f^+ \) and \( f^- \), the positive and negative parts of the function \( f \), i.e.

\[
C_m(f) = C_m(f^+) - C_m(f^-),
\]

where \( f^+ = f \vee 0 \) and \( f^- = (-f) \wedge 0 \), and \( \tilde{m} \) is the conjugate set function of \( m \).

### 3 Signed fuzzy measures with bounded chain variation

**Definition 2** The chain variation of a real-valued set function \( m \), \( m(\emptyset) = 0 \), for each \( E \in \mathcal{A} \), is defined by

\[
|m|(E) = \sup \{ \sum_{i=1}^{n} |m(E_i) - m(E_{i-1})| \mid \emptyset = E_0 \subset E_1 \subset \cdots \subset E_n = E, E_i \in \mathcal{A}, i = 1, \ldots, n \}.
\]

In the previous definition, the supremum is taken over all finite chains between \( \emptyset \) and \( E \).

The chain variation \( |m| \) of set function \( m \) is positive, monotone set function, vanishing at the empty set, and the inequality \( |m(E)| \leq |m|(E) \) is satisfied for each \( E \in \mathcal{A} \). Consequently, if \( m \) is a fuzzy measure, then \( |m|(E) = m(E) \), for all \( E \in \mathcal{A} \).

**Definition 3** A real-valued set function \( m \), \( m(\emptyset) = 0 \), is of bounded chain variation if

\[
|m|(X) < \infty.
\]

The family of all set functions of bounded chain variation, vanishing at the empty set, is denoted by \( BV \). The functional \( \|m\| = |m|(X) \) is a norm on a Banach space \( (BV, \| \|) \), see [1, 13]. Another important characterization of the space \( BV \) is given by the following theorem, see [1, 13].

**Theorem 1** A set function \( m : \mathcal{A} \rightarrow \mathbb{R}, m(\emptyset) = 0 \), belongs to \( BV \) if and only if it can be represented as difference of two fuzzy measures \( m_1 \) and \( m_2 \).
By means of Theorem 1, another representation of Choquet integral with respect to a signed fuzzy measure can be obtained [13] and this is illustrated in the following example.

**Example 2** Let \( X \) be a finite set, \( X = \{1, 2, 3, 4\} \), \( \mathcal{A} = \mathcal{P}(X) \) and let \( m \) be a signed fuzzy measure, \( m \in BV \), defined by

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<thead>
<tr>
<th>( m({1}) )</th>
<th>( m({2}) )</th>
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<tbody>
<tr>
<td>0.3</td>
<td>0.2</td>
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<tr>
<td>( m({4}) )</td>
<td>1</td>
<td>0.5</td>
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<td>( m({1,2}) )</td>
<td>0.1</td>
<td>-0.4</td>
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<tr>
<td>( m({1,3}) )</td>
<td>0.5</td>
<td>0.2</td>
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<tr>
<td>( m({2,3,4}) )</td>
<td>-0.4</td>
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For \( f \in \mathcal{M} \) defined by \( f(1) = -0.3 \), \( f(2) = 0.2 \), \( f(3) = -0.4 \) and \( f(4) = 0.6 \), we compute \( C_m(f) = -0.4 \cdot 0 + (-0.3 + 0.4) \cdot 0.2 + (0.2 + 0.3) \cdot (-0.4) + (0.6 - 0.2) \cdot (-0.4) = -0.34 \). On the other side, we obtain by the equality (2) the same result

\[
C_m(f^+) - C_m(f^-) = -0.24 - 0.1 = -0.34.
\]

However, \( m \in BV \), and therefore it can be represented as difference of two fuzzy measures \( m_1 \) and \( m_2 \), i.e., \( m = m_1 - m_2 \), (this representation is not unique). Let \( m_1 \) be defined by

<table>
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<tr>
<td>0.3</td>
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and \( m_2 \) be defined by

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<th>( m_2({3}) )</th>
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<tr>
<td>0</td>
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<td>0.4</td>
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<tr>
<td>0.5</td>
<td>0.3</td>
<td>0.6</td>
</tr>
<tr>
<td>1</td>
<td>0.5</td>
<td>0.8</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
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Then we have \( C_{m_1}(f) = -0.4 + (-0.3 + 0.4) \cdot 1 + (0.2 + 0.3) \cdot 0.2 + (0.6 - 0.2) \cdot 0 = -0.2 \) and \( C_{m_2}(f) = -0.4 + (-0.3 + 0.4) \cdot 0.8 + (0.2 + 0.3) \cdot 0.6 + (0.6 - 0.2) \cdot 0.4 = 0.14 \). Therefore we obtain

\[
C_m(f) = C_{m_1}(f) - C_{m_2}(f) = -0.34.
\]

By Definition 1 (ii) we have the next representation of \( C_m \), independently of the representation of \( m \) given by Theorem 1.

**Theorem 2** If \( m \) is a signed fuzzy measure, \( m \in BV \), then the asymmetric Choquet integral of \( f \in \mathcal{M} \) can be represented in the following manner

\[
C_m(f) = C_{m_1}(f) - C_{m_2}(f),
\]

where \( m_1 \) and \( m_2 \) are two fuzzy measures from Theorem 1 such that \( m = m_1 - m_2 \) and \( C_m(f) \) does not depend of the representation of \( m \).

## 4 Representation of signed fuzzy measures

In this section we shall consider a representation of a signed fuzzy measure \( m : \mathcal{A} \to [-\infty, \infty] \) which belongs to space \( BV \). We will link to it a signed measure \( \mu \) defined on a \( \sigma \)-algebra \( \mathcal{B} \) of subsets of a set \( Y \).

First, we will introduce an interpreter for measurable sets and a frame for representation, see [8, 13].

**Definition 4** A mapping \( H : \mathcal{A} \to \mathcal{B} \) is called an interpreter if \( H \) satisfies

(i) \( H(\emptyset) = \emptyset \) and \( H(X) = Y \).

(ii) \( H(E) \subset H(F) \), for all \( E \subset F \). A triple \( (Y, \mathcal{B}, H) \) is called a frame of \( (X, \mathcal{A}) \), if \( H \) is an interpreter from \( \mathcal{A} \) to \( \mathcal{B} \).

**Definition 5** Let \( m \) be a signed fuzzy measure defined on \( \mathcal{A} \). A quadruple \( (Y, \mathcal{B}, \mu, H) \) is called a representation of \( m \) (or \( (X, \mathcal{A}, m) \)) if \( H \) is an interpreter from \( \mathcal{A} \) to \( \mathcal{B} \), \( \mu \) is a signed measure on \( (Y, \mathcal{B}) \), and \( m = \mu \circ H \).

**Theorem 3** Every signed fuzzy measure \( m \in BV \), has its representation.

**Proof.** Let \( m \) be a signed fuzzy measure, and \( m \in BV \). There exist two fuzzy measures, \( m_1 \) and \( m_2 \), such that \( m(E) = m_1(E) - m_2(E) \) for all \( E \in \mathcal{A} \).
Let $Y$ be the open interval $(-m_2(X), m_1(X))$, and $\mathcal{B}$ the class of all Borel subsets of $Y$. We define the mapping $H : \mathcal{A} \to \mathcal{B}$ by $H(E) = (-m_2(E), m_1(E))$, for all $E \in \mathcal{A}$. $H(\emptyset) = \emptyset$ and $H(X) = (-m_2(X), m_1(X)) = Y$. For $E \subset F$ we have $m_1(E) \leq m_1(F)$ and $m_2(E) \leq m_2(F)$, and therefore

\[ H(E) = (-m_2(E), m_1(E)) \]
\[ \subset (-m_2(F), m_1(F)) = H(F). \]

Therefore $H$ is an interpreter from $\mathcal{A}$ to $\mathcal{B}$.

Let $\mu$ be a signed measure defined by

\[ \mu((a, b)) = \lambda((a, b) \cap Y^+) - \lambda((a, b) \cap Y^-), \]

for $(a, b) \in \mathcal{B}$, where $\lambda$ is a Lebesgue measure and $Y^+ = (0, m_1(X))$, $Y^- = Y \setminus Y^+$. Hence for every $E \in \mathcal{A}$ we have

\[ m(E) = m_1(E) - m_2(E) \]
\[ = \lambda((0, m_1(E)) - \lambda((-m_2(E), 0)) \]
\[ = \mu(H(E)) \]
\[ = \mu \circ H(E). \]

Therefore $(Y, \mathcal{B}, \mu, H)$ is a representation of $m$. □

**Remark 1**

(i) As it is mentioned before, the representation with two fuzzy measures $m_1$ and $m_2$ in Theorem 1 is not unique, hence the representation of $m$ given above is not unique, too.

(ii) If $m$ is a signed fuzzy measure, $m \in BV$, and $\tilde{m}$ is the conjugate set function of $m$, then a quadruple $(Y, \mathcal{B}, \mu, \tilde{H})$ is a representation of $\tilde{m}$, where the interpreter $\tilde{H}$ is defined by $\tilde{H}(E) = (-\tilde{m}_2(E), \tilde{m}_1(E))$, for all $E \in \mathcal{A}$ and $(Y, \mathcal{B}, \mu)$ is exactly the same as in the proof of Theorem 3.

We can apply Theorem 3 to obtain a representation of asymmetric Choquet integral of measurable function $f$ with respect to a signed fuzzy measure $m$. Murofushi et al. in [10] have proved similar representation theorem of the Choquet integral with respect to a non-additive set function $m$ of bounded chain variation. Although the revised monotonicity of a signed fuzzy measure $m$ is more restrictive condition, it is inessential in the proof of theorem.

**Theorem 4**

If $m$ is a signed fuzzy measure, $m \in BV$ and $f \in M$, then there exist two functions $I_1^f : Y \to [0, \infty]$ and $I_2^f : Y \to [0, \infty]$ such that the asymmetric Choquet integral of $f \in M$ can be represented by

\[ C_m(f) = \int I_1^f \ d\mu - \int I_2^f \ d\mu, \quad (4) \]

where, $f^+ = f \lor 0$, $f^- = (-f) \lor 0$ and the integrals on the right-hand side are the Lebesgue integrals. $C_m(f)$ does not depend of the representation of $m$ by means of Theorem 3.

**Proof.** Let $m$ be a signed fuzzy measure and let $\tilde{m}$ be its conjugate set function, with representations $(Y, \mathcal{B}, \mu, H)$ and $(\tilde{Y}, \mathcal{B}, \mu, \tilde{H})$, respectively. If we define for a non-negative measurable function $f \in M$, two functions $I_1^f$ and $I_2^f$ on $Y$ by

\[ I_1^f(y) = \sup \{ t \ | \ y \in H(\{ x \ | \ f(x) \geq t \}) \} \quad \text{and} \]
\[ I_2^f(y) = \sup \{ t \ | \ y \in \tilde{H}(\{ x \ | \ f(x) \geq t \}) \} \]

for all $y \in Y$, then (4) immediately follows by the equality (2).

Let $\tilde{m}_1$ and $\tilde{m}_2$ be two fuzzy measures such that $m = \tilde{m}_1 - \tilde{m}_2$, and $(\tilde{Y}, \mathcal{B}, \mu, \tilde{H})$ and $(\tilde{Y}, \mathcal{B}, \mu, \tilde{H})$ are the representations of $m$ and $\tilde{m}$, where $\tilde{H}$ and $\tilde{H}$ are the interpreters defined by: $\tilde{H}(E) = (-\tilde{m}_2(E), \tilde{m}_1(E))$ and $\tilde{H}(E) = (-\tilde{m}_2(E), \tilde{m}_1(E))$, for all $E \in \mathcal{A}$. We have

\[ m = \mu \circ H = \mu \circ \tilde{H} \]
\[ \text{and} \]
\[ \tilde{m} = \mu \circ \tilde{H} = \mu \circ \tilde{H} \]
\[ \text{and therefore for every} f \in M \]
\[ C_m(f^+) = \int I_1^f \ d\mu - \int I_2^f \ d\mu \quad \text{and} \]
\[ C_m(f^-) = \int I_1^f \ d\mu - \int I_2^f \ d\mu, \]
\[ \text{where} I_1^f \text{ and } I_2^f \text{ are defined on } \tilde{Y} \text{ by} \]
\[ \tilde{I}_1^f(y) = \sup \{ t \ | \ y \in \tilde{H}(\{ x \ | \ f(x) \geq t \}) \} \quad \text{and} \]
\[ \tilde{I}_2^f(y) = \sup \{ t \ | \ y \in \tilde{H}(\{ x \ | \ f(x) \geq t \}) \} \]
\[ \text{for all } y \in \tilde{Y}. \text{ Hence (4) unambiguously represents } C_m(f). \quad □ \]
Example 3 Let $m$ and $f$ be defined same as in the Example 2.

Let $(Y, \mathcal{B}, \mu, H)$ and $(Y, \mathcal{B}, \tilde{\mu}, \tilde{H})$ be the representations of $m$ and $	ilde{m}$ related to $m_1$ and $m_2$ given in Example 2. Therefore $Y = (-1, 1)$ and we have

$$I_{f_1}^1(y) = \begin{cases} 0, & y \in (-1, -0.6] \cup [0.2, 1) \\ 0.2, & y \in [0, 0.2) \cup (-0.6, -0.4] \\ 0.6, & y \in (-0.4, 0), \end{cases}$$

and

$$I_{f_2}^2(y) = \begin{cases} 0, & y \in (-1, -0.4] \cup [0.8, 1) \\ 0.3, & y \in [0, 0.8) \cup (-0.4, -0.2] \\ 0.4, & y \in (-0.2, 0). \end{cases}$$

We have

$$\int I_{f_1}^1 \, d\mu - \int I_{f_2}^2 \, d\mu = -0.24 - 0.1 = C_m(f).$$

5 Revised monotone functional

We have seen in the above examples that the values of the asymmetric Choquet integral w.r.t. a signed fuzzy measure, $C_m(f)$, can be negative, although the function $f$ is non-negative. Non-monotonicity of the $C_m$ requires a modification of the monotonicity property. A real valued functional $L$, $L: \overline{M} \to \mathbb{R}$, defined on the class of measurable functions $f: X \to \mathbb{R}$, can be viewed as an extension of the signed fuzzy measure $m$, so it is reasonable to require that $L(1_E) = m(E)$, for all $E \in \mathcal{A}$ ($1_E$ denotes characteristic function of the crisp $E$). In order to examine the properties of a real valued functional $L$, under which it can be represented by the asymmetric Choquet integral w.r.t. a signed fuzzy measure, it is useful to consider the concept of comonotone functions.

Recall that two measurable functions $f$ and $g$ on $X$ are called comonotone [4] if they are measurable with respect to the same chain $C$ in $\mathcal{A}$. The functional $L$ is comonotone additive iff

$$L(f + g) = L(f) + L(g)$$

for all comonotone functions $f, g \in \overline{M}$. We say that functional $L$ is positive homogeneous iff

$$L(af) = aL(f)$$

for all $f \in \overline{M}$ and $a \geq 0$.

We introduce a revised monotone functional $L$ defined on $\overline{M}$.

Definition 6 Let $L: \overline{M} \to \mathbb{R}$ be a functional on $\overline{M}$.

(i) $L$ is revised monotone iff

\[ a) \ L(f) \geq 0, \ L(g) \geq 0, \ L(f) \vee L(g) > 0 \ \Rightarrow \ L(f + g) \geq L(f) \vee L(g) \]

\[ b) \ L(f) \geq 0, \ L(g) \geq 0, \ L(f) \wedge L(g) < 0 \ \Rightarrow \ L(f + g) \leq L(f) \wedge L(g) \]

\[ c) \ L(f) > 0, \ L(g) < 0, \ \Rightarrow \ L(g) \leq L(f + g) \leq L(f) \]

for all functions $f, g \in \overline{M}$.

(ii) $L$ is comonotone revised monotone iff conditions a), b) and c) are satisfied for all comonotone functions $f, g \in \overline{M}$.

Note that for a non-negative functional $L$ acting on measurable non-negative functions on $X$, the revised monotonicity ensures the monotonicity.

Directly by definitions of the comonotone additive and the revised monotone functional $L$ we have the next proposition.

Proposition 1 The asymmetric Choquet integral w.r.t. a signed fuzzy measure $m$, $C_m: \overline{M} \to \mathbb{R}$ is the comonotone revised monotone functional.

Remark 2 Note that any additive functional $L: \overline{M} \to \mathbb{R}$ is a revised monotone functional. The Lebesgue integral with respect to a signed measure $\mu$ is a revised monotone functional. Proposition 1 is a consequence of this fact and Theorem 4.

For a finite set $X = \{x_1, x_2, \ldots, x_n\}$, we have the next theorem.

Theorem 5 Let $L$ be a real valued, revised monotone, positive homogeneous and comonotone additive functional on $\overline{M}$. Then there exists a signed fuzzy measure $m_L$, such that $L$ is the asymmetric Choquet integral w.r.t. $m$, i.e.

$$L(f) = C_{m_L}(f).$$

Proof. Let $m$ be a set function $m$ defined by

$$m_L(E) = L(1_E), \ \text{for} \ E \subseteq X.$$


Observe that for comonotone functions \(1_X\) and \(-1_E\), we have
\[
m_L(E) = L(1_E) = L(1_X + (-1_E)) = L(1_X) + L(-1_E),
\]
hence
\[
L(-1_E) = -\bar{m}(E), \ E \subseteq X.
\]
By definition of \(m_L\) and revised monotonicity of functional \(L\) we have:
1) \(m_L(\emptyset) = L(1_\emptyset) = L(0) = 0\)
2) a) for \(E, F \in \mathcal{A}\), \(E \cap F = \emptyset\), and
\(m_L(E) \geq 0, m_L(F) \geq 0, m_L(E) \vee m_L(F) > 0\) we have
\[
m(E \cup F) = L(1_{E \cup F}) = L(1_E + 1_F) \geq L(1_E) \vee L(1_F) = m_L(E) \vee m_L(F).
\]

Analogously, we obtain that \(m_L\) satisfies conditions (SFM2) b) and c), hence \(m_L\) is the revised monotone signed fuzzy measure. Now, we consider \(f \in \overline{M}\) and its comonotone additive representation \(f = f^+ + (-f^-)\), where
\[
f^+ = \sum_{i=1}^{n} (a_i - a_{i-1})1_{E_i},
\]
\[
-f^- = \sum_{i=1}^{n} (b_i - b_{i+1})(-1_{E_i}),
\]
\(a_i = f^+_{a(i)} = a_0 = 0, b_i = f^-_{a(n+1-i)}, b_{n+1} = 1,\) \(a_i\)’s are in non-decreasing, \(b_i\)’s are in non-increasing order, \(\alpha\) is a permutation, such that
\(-\infty < f_{a(1)} \leq \cdots \leq f_{a(n)} < \infty, E_i = E_{a(i)}, F_i = E_1 \setminus E_{a(n+1-i)}, E_{a(i)} = \{x_{a(i)}, \ldots, x_{a(n)}\} \) and \(E_{a(n+1)} = \emptyset\).

For every \(i\) and \(j\) the functions \(1_{E_i}\) and \(1_{E_j}\) are comonotone, and by comonotone additivity and positive homogeneity of the functional \(L\), we have
\[
L(f^+) = \sum_{i=1}^{n} (a_i - a_{i-1})L(1_{E_i})
\]
and
\[
L(-f^-) = \sum_{i=1}^{n} (b_i - b_{i+1})L(-1_{E_i})
\]
\[
= -\sum_{i=1}^{n} (b_i - b_{i+1})(-L(1_{E_i}))
\]
\[
= -\sum_{i=1}^{n} (b_i - b_{i+1})\bar{m}_L(E_i)
\]
\[
= -C_{m_L}(f^-).
\]
Therefore by the comonotonicity of functions \(f^+\) and \(-f^-\) we obtain that
\[
L(f) = L(f^+ + (-f^-))
\]
\[
= L(f^+) + L(-f^-)
\]
\[
= C_{m_L}(f^+) - C_{m_L}(f^-)
\]
\[
= C_{m_L}(f).
\]

\(\Box\)

**Remark 3** In the paper [15] we have consider the analogous situation for the Sugeno integral. An extension of the Sugeno integral in the spirit of the symmetric extension of Choquet integral proposed by M. Grabisch in [5, 6] is useful as a framework for cumulative prospect theory in an ordinal context. In the paper [15] there was considered a representation by two Sugeno integrals of the functional \(L\) defined on the class of functions \(f: X \rightarrow [-1, 1]\) on a finite set \(X\). In the case of infinitely countable set \(X\) there was obtained as a consequence of results on general fuzzy rank and sign dependent functionals that the symmetric Sugeno integral is comonotone-additive functional on the class of functions with finite support.

**Acknowledgement** The work has been supported by the project MNTRs "Mathematical models of nonlinearity, uncertainty and decision" and the project "Mathematical Models for Decision Making under Uncertain Conditions and Their Applications" supported by Vojvodina Provincial Secretariat for Science and Technological Development.

**References**


