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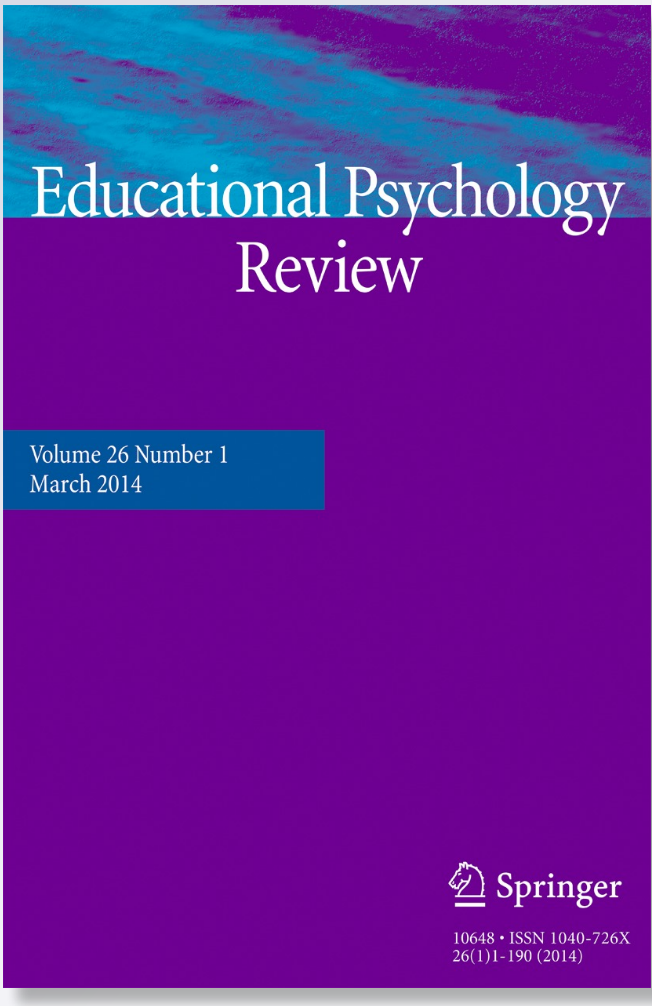
Educational Psychology Review

ISSN 1040-726X
Volume 26
Number 1

Educ Psychol Rev (2014) 26:9-25
DOI 10.1007/s10648-014-9249-3



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REVIEW ARTICLE

Concreteness Fading in Mathematics and Science Instruction: a Systematic Review

Emily R. Fyfe · Nicole M. McNeil · Ji Y. Son · Robert L. Goldstone

Published online: 14 January 2014

Abstract A longstanding debate concerns the use of concrete versus abstract instructional materials, particularly in domains such as mathematics and science. Although decades of research have focused on the advantages and disadvantages of concrete and abstract materials considered independently, we argue for an approach that moves beyond this dichotomy and combines their advantages. Specifically, we recommend beginning with concrete materials and then explicitly and gradually fading to the more abstract. Theoretical benefits of this "concreteness fading" technique for mathematics and science instruction include (1) helping learners interpret ambiguous or opaque abstract symbols in terms of well-understood concrete objects, (2) providing embodied perceptual and physical experiences that can ground abstract thinking, (3) enabling learners to build up a store of memorable images that can be used when abstract symbols lose meaning, and (4) guiding learners to strip away extraneous concrete properties and distill the generic, generalizable properties. In these ways, concreteness fading provides advantages that go beyond the sum of the benefits of concrete and abstract materials.

Keywords Concrete manipulatives · Abstract symbols · Learning and instruction

A longstanding controversy concerns the use of concrete versus abstract materials during mathematics and science instruction. Concrete materials connect with learners' prior knowledge, are grounded in perceptual and/or motor experiences, and have identifiable correspondences between their form and referents. In contrast, abstract materials eliminate extraneous perceptual properties, represent structure efficiently, and are more arbitrarily linked to their

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referents. Although the concreteness of materials varies on a continuum, researchers often refer only to the extremes, pitting concrete and abstract materials against each other.

Here, we argue that concreteness fading is a promising instructional technique that moves beyond the concrete versus abstract debate and exploits the advantages and minimizes the disadvantages of both. Concreteness fading refers specifically to a three-step progression by which the concrete, physical instantiation of a concept becomes increasingly abstract over time. This fading technique offers a set of unique advantages that surpass the benefits of concrete or abstract materials considered in isolation. In this review, we focus on the theoretical benefits of this concreteness fading technique, discuss supporting evidence in mathematics and science instruction, and lay out unanswered questions and directions for expansion.

Concrete Versus Abstract Materials

Concrete materials, which include physical, virtual, and pictorial objects, are widely used in Western classrooms (Bryan et al. 2007), and this practice has support in both psychology and education (e.g., Bruner 1966; Piaget 1970). There are at least four potential benefits to using concrete materials. First, they provide a practical context that can activate real-world knowledge during learning (Schliemann and Carraher 2002). Second, they can induce physical or imagined action, which has been shown to enhance memory and understanding (Glenberg et al. 2004). Third, they enable learners to construct their own knowledge of abstract concepts (Brown et al. 2009). Fourth, they recruit brain regions associated with perceptual processing, and it is estimated that 25–40 % of the human cortex is dedicated to visual information processing (Evans-Martin 2005). Despite these benefits, there are several reasons to caution against the use of concrete materials during learning. Specifically, they often contain extraneous perceptual details, which can distract the learner from relevant information (e.g., Belenky and Schalk 2014; Kaminski et al. 2008), draw attention to themselves rather than their referents (e.g., Uttal et al. 1997), and constrain transfer of knowledge to novel problems (e.g., Goldstone and Sakamoto 2003; Sloutsky et al. 2005).

A number of researchers recommend avoiding concrete materials in favor of abstract materials, which eliminate extraneous perceptual details. Abstract materials offer increased portability and generalizability to multiple contexts (Kaminski et al. 2009; Son et al. 2008). They also focus learners' attention on structure and representational aspects, rather than on surface features (Kaminski et al. 2009; Uttal et al. 2009). However, abstract materials are not without shortcomings. For example, solving problems in abstract form often leads to inefficient solution strategies (Koedinger and Nathan 2004), inflexible application of learned procedures (McNeil and Alibali 2005), and illogical errors (Carraher and Schliemann 1985; Stigler et al. 2010). In general, abstract materials run the risk of leading learners to manipulate meaningless symbols without conceptual understanding (Nathan 2012).

Concreteness Fading

Given that both concrete and abstract materials have advantages and disadvantages, we propose a solution that combines their advantages and mitigates their disadvantages. Specifically, we argue for an approach that begins with concrete materials and gradually and explicitly fades toward more abstract ones. This concreteness fading technique exploits the continuum from concreteness to abstractness and allows learners to initially benefit from the grounded, concrete context while still encouraging them to generalize beyond it.

Concreteness fading was originally recommended by Bruner (1966). He proposed that new concepts and procedures should be presented in three progressive forms: (1) an *enactive* form, which is a physical, concrete model of the concept; (2) an *iconic* form, which is a graphic or pictorial model; and finally (3) a *symbolic* form, which is an abstract model of the concept. For example, in mathematics, the quantity "two" could first be represented by two physical apples, next by a picture of two dots representing those apples, and finally by the Arabic numeral 2. The idea is to start with a concrete, recognizable form and gradually strip away irrelevant details to end with the most economic, abstract form. We use the term concreteness fading to refer to the three-step progression by which the physical instantiation of a concept becomes increasingly abstract over time. Since Bruner's time, several researchers have adopted similar approaches to the concrete versus abstract debate, advocating the use of concrete materials that are eventually decontextualized or faded to more abstract materials (Goldstone and Son 2005; Gravemeijer 2002; Lehrer and Schauble 2002; Lesh 1979).

Concreteness fading offers a unique solution to linking multiple instantiations of a concept that distinguishes it from alternative approaches to the concrete versus abstract debate. For example, it is widely assumed that providing multiple external representations will benefit learners relative to providing a single representation in isolation (e.g., Ainsworth 1999; Gick and Holyoak 1983). Although some studies have found support for multiple representations, others have found limited benefits (see Ainsworth 2006). For example, when multiple representations are presented simultaneously, novices often experience cognitive overload due to the burden on limited cognitive resources (e.g., Chandler and Sweller 1992). Further, learners may struggle to understand how the representations are related to one another and fail to extract key concepts (e.g., Ainsworth et al. 2002). Thus, learners need support in relating or integrating the different representations (e.g., Berthold and Renkl 2009; Schwonke et al. 2009). Concreteness fading involves the presentation of multiple examples but overcomes the limitations of simply presenting them simultaneously by varying the concreteness in a specific progression. That is, the examples are clearly aligned on the dimension of concreteness, and there is a specified directionality in that the concrete examples are the referents for the iconic and abstract examples.

A related approach to presenting multiple representations is to ask learners to explicitly compare them (e.g., Kotovsky and Gentner 1996). Comparison involves the simultaneous presentation of two examples and the mapping of elements across examples to highlight similarities or differences. For example, in one study, preschoolers were tested on their ability to detect patterns (i.e., ABA) by asking them to match one pattern (e.g., small square, large square, small square) with one of two other patterns (e.g., diamond, circle, diamond versus circle, diamond, diamond; Son et al. 2011). Children who were trained to align and compare the elements in each set were better able to detect and generalize the pattern than children who did not make this direct comparison. Although concreteness fading may support spontaneous comparison, the focus is not on mapping similarities and differences, but on linking the examples as mutual referents. Further, concreteness fading does not require that two representations be held in mind simultaneously; it only requires the presentation of one example at a time. Concreteness fading is a unique solution to linking concrete and abstract examples that provides advantages over a general presentation and comparison of multiple representations.

Theoretical Benefits of Concreteness Fading

On the surface, the benefits of concrete and abstract materials seem to be in opposition. It appears we have to choose grounded, contextualized knowledge or portable, abstract knowledge. However, we argue that both goals can be achieved simultaneously. Specifically, we suggest that concreteness fading allows concepts to be both grounded in easily understood concrete contexts and also generalized in a manner that promotes transfer. In this way, concreteness fading offers the advantages of concrete and abstract materials considered separately, the advantages of presenting concrete and abstract materials together, and finally, the advantages of presenting them together in a specific gradual sequence from concrete to abstract. Here, we explicate the unique theoretical benefits of this progressive fading sequence for both learning and transfer. See Fig. 1 for a schematic theoretical model of concreteness fading and its potential benefits.

Benefits of concreteness fading emerge because it starts with a concrete format and only later explicitly links to the more difficult abstract symbols. Concrete materials are advantageous initially because they allow the concept to be grounded in familiar, meaningful scenarios (Baranes et al. 1989; Carraher et al. 1985). The presentation of this initial concrete stage gives rise to at least three specific advantages.

First, it helps learners interpret ambiguous abstract representations in terms of well-understood concrete objects (Goldstone and Son 2005; Son et al. 2012). Abstract symbols are devoid of context and often difficult to interpret. This can lead to the manipulation of meaningless symbols without understanding. However, people interpret ambiguous objects in terms of unambiguous familiar objects (e.g., Medin et al. 1993). For example, when an ambiguous man-rat drawing is preceded by a drawing of a man, it is spontaneously interpreted as a man. But when it is preceded by a drawing of a rat, it is interpreted as a rat (Leeper 1935). Thus, if the concrete materials precede the abstract materials, the learner can successfully interpret the ambiguous abstract materials in terms of the already understood concrete context. This process may underlie children's improved performance on symbolic equations (e.g., 5+2=4+__) when they are preceded by equations constructed from concrete manipulatives

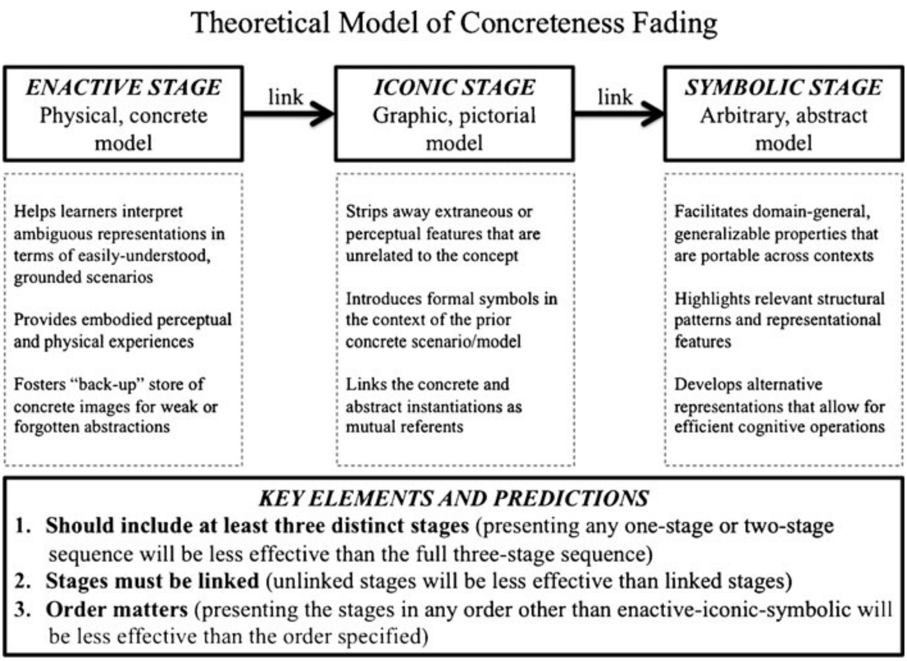

Fig. 1 Theoretical model of concreteness fading

(Sherman and Bisanz 2009). Children can interpret arbitrary symbols, such as the equal sign, in terms of familiar ideas, such as balancing or sharing. If teachers present materials in this fading sequence, learners can link the concrete and abstract materials as mutual referents, thus mitigating the disadvantage of ambiguity in the abstract materials.

Second, the initial concrete stage takes advantage of embodied cognition by giving learners experience with perceptual and physical processes that are constrained to give rise to proper inferences. According to an embodied theory of cognition, high-level cognitive processes (e.g., language, mathematics) stem from action and perception (Barsalou 2003; Lakoff and Nunez 2000). Thus, comprehension of abstract symbols requires mapping those symbols onto bodily experiences or representations of those experiences. For example, having children act out a sentence with toys helps them connect words (i.e., abstract symbols) to particular objects and actions and enhances comprehension of the text (e.g., Glenberg et al. 2004). The concrete, or enactive, stage engages learners in physical and perceptual processes thereby providing the necessary bodily experience. Further, these embodied experiences are linked to the abstract symbols, thus facilitating the mapping between abstract concepts and perceptual processes.

Third, the concrete stage enables learners to acquire a store of images that can be used when abstract symbols are forgotten or disconnected from the underlying concept. According to Bruner (1966), once an understanding of the abstract concept has been achieved, learners do not give up their imagery, but rather rely on this stock of representations as a means of relating new problems to those already mastered. The stored images provide learners with an accessible, back-up representation that can be used when the abstract symbols are detached from their referent. For example, elementary school students who initially learned about the equal sign in the context of balancing a seesaw were presented with challenging open-ended sentences later in the school year (e.g., 5+6=__+2). Students who were in doubt about their solutions were encouraged to rely on their knowledge of a seesaw and check if their solution made sense (Mann 2004). Concreteness fading not only encourages teachers to focus on both concrete and abstract understanding but also provides learners with abstractions explicitly linked to a stock of images. Each of these images can act as a salient retrieval cue for connecting an otherwise reclusive abstraction.

The concrete, enactive stage provides numerous benefits for the learning of a concept. Equally important, however, are the two subsequent stages: the iconic and symbolic stages of Bruner's (1966) sequence. If given only concrete materials (or concrete and abstract materials in isolation), it is likely that the learner's knowledge would remain too tied to the concrete context and would not transfer to dissimilar situations. Indeed, Kaminski et al. (2008) found that students who learned a math concept only through an abstract example outperformed those who learned through a concrete example followed by an abstract example. However, the examples in this study were presented one after another *in isolation*, rather than in a fading progression. It is through the gradual and explicit fading that learners are able to strip the concept of extraneous, concrete properties and grasp the more portable, abstract properties (Bruner 1966).

Stripping away extraneous features facilitates transfer by highlighting structural information and reducing the salience of concrete information. Indeed, the problem of transferring knowledge to novel contexts has often been attributed to learners' inability to detect the underlying structure of the problem at hand (e.g., Ross 1987). Further, drawing attention to structure is one of the central tenets of several theories of learning, including the preparation for future learning account (e.g., Schwartz et al. 2011) and schema theory (e.g., Chi et al. 1981). By slowly decontextualizing the concrete materials, concreteness fading hones in on the pertinent, structural features and results in a faded representation that can be a useful stand-in for a variety of specific contexts. This mitigates the disadvantageous context specificity of concrete materials and allows learners to more easily extend learned material to new and superficially dissimilar spheres of application.

In sum, concreteness fading can benefit learning and transfer by starting with a concrete, recognizable format then gradually and explicitly removing context-specific elements to generate a more abstract representation. Resulting knowledge is not only grounded and meaningful but also abstract and portable. The fading process allows learners to more explicitly link the concrete materials and abstract symbols as mutual referents. Importantly, the result is a rich, grounded understanding of the underlying concept that is connected to conventional, abstract symbols.

Support for Concreteness Fading

Although empirical support for concreteness fading is increasing, most evidence in favor of this method is admittedly indirect and restricted to the domains of mathematics and science. Various features of concreteness fading have been found to be helpful in learning contexts. For example, concreteness fading involves presenting the same concept in three different instantiations and linking their common structures. Several studies have found benefits of presenting multiple instantiations of a concept, as it affords the possibility of extracting commonalities (e.g., Gick and Holyoak 1983). In addition, concreteness fading reduces representational support as domain knowledge increases, which may have analogous benefits to fading out instructional support more generally (e.g., Kalyuga 2007; Wecker and Fischer 2011). Learners with low domain knowledge tend to benefit from methods with heavy instructional support, whereas higher-knowledge learners benefit from methods with little to no instructional support (see Kalyuga 2007). Further, concreteness fading maximizes smooth transitions from concrete to abstract as opposed to abrupt shifts. Recent studies suggest smooth transitions adapted to learner's knowledge level are particularly beneficial (e.g., Renkl et al. 2002).

More relevant indirect evidence comes from a variety of studies demonstrating benefits of a concrete-to-abstract sequence. For example, Koedinger and Anderson (1998) found high learning gains from a cognitive tutor that presented concrete, algebra word problems first, followed by an intermediate step, followed by the symbolic expression. In more recent research, worked examples accompanied by animations that faded from concrete to abstract were better at facilitating transfer performance than were worked examples that were not accompanied by any animations (Scheiter et al. 2010). However, it remains unclear whether the benefits were due to the fading or to the mere presence of animations. Similarly, physics simulation environments that contained concrete images that transitioned into more abstract images promoted better transfer than simulations that contained only concrete images (Jaakkola et al. 2009). Further, transitioning from physical experimentation with actual concrete models to a simulation environment promoted better conceptual understanding than physical experimentation alone (Zacharia 2007). Unfortunately, without an abstract or simulation-only condition, it is difficult to determine whether concrete models should have been included at all.

There has been some direct, experimental evidence in favor of concreteness fading, but more is needed. Goldstone and Son (2005) had undergraduates learn a scientific principle (i.e., competitive specialization) via computer simulations that varied in perceptual richness. The concrete elements appeared as colorful ants foraging for fruit, whereas the abstract elements appeared as black dots and green shapes. Students exhibited higher transfer when the elements in the display switched from concrete to abstract than when the elements remained concrete, remained abstract, or switched from abstract to concrete. However, the simulations only used two steps in the fading progression as opposed to the three steps recommended by Bruner (1966). Additionally, the concrete and abstract elements were both fairly concrete as they both represented ants and food—the abstract elements were simply stripped of perceptual detail. Also, the abstract elements were not arbitrarily linked to their referents, as most abstract symbols are.

Braithwaite and Goldstone (2013) also examined a two-step fading progression, but in the mathematical domain of combinatorics. The researchers used outcome listing (i.e., listing all the possible outcomes) and numerical calculation (i.e., using a formula to calculate the possible outcomes) as examples of concrete and abstract materials, respectively. Students were assigned to one of four conditions: concrete (outcome listing only), abstract (numerical calculation only), concrete first (listing followed by calculation), or abstract first (calculation followed by listing). Concrete first led to higher transfer than concrete or abstract first and to similar transfer as abstract. These results suggest that when both concrete and abstract materials are employed, concrete materials should precede the abstract. However, the combination was not necessarily more effective than using abstract materials alone. Again, only two-step fading was used as opposed to the three-step fading. Also, in this study, the concrete and abstract materials were both fairly abstract (i.e., graphic notation written on paper). Perhaps including the initial, enactive stage would have enhanced the effectiveness of this form of concreteness fading.

There is some direct empirical support for the full three-step concreteness fading progression. For example, McNeil and Fyfe (2012) had undergraduates learn modular arithmetic in one of three conditions: *concrete*, in which the concept was presented using meaningful images; *abstract*, in which the concept was presented using arbitrary, abstract symbols; or *concreteness fading*, in which the concept was presented using meaningful images that were faded into abstract symbols. The concreteness fading progression included an intermediate, "faded" stage that retained the identifiable correspondence between the form and referent, but was stripped of all extraneous perceptual detail. This allowed the concrete and abstract elements to be explicitly linked as mutual referents. Students completed a transfer test immediately, 1 week later, and 3 weeks later. Importantly, students in the concreteness fading condition exhibited the best transfer performance at all three time points.

Two additional studies provide direct support for concreteness fading but are currently unpublished. In the first experiment, children with low knowledge of math equivalence received instruction on math equivalence problems (e.g., 3+4=3+__) in one of four conditions (Fyfe and McNeil 2009). In the *concrete* condition, problems were presented using physical, concrete objects (e.g., toy bears on a balance scale). In the *abstract* condition, problems were presented as symbolic equations on paper. In the *concreteness fading* condition, problems were presented in three progressive formats: first with the physical objects, next with fading worksheets that used pictures to represent the objects, and finally with symbolic equations (see Fig. 2). In the *concreteness introduction* condition, problems were presented in the reverse progression such that it began with abstract materials and the concrete materials were gradually introduced. Following instruction, children solved five transfer problems. Children in the concreteness fading condition solved more transfer problems correctly than children in the other conditions.

A follow-up experiment tested the generalizability of the results by extending the target population to children who had high prior knowledge in the domain (Fyfe and McNeil 2013). In this study, high-knowledge children were taught an advanced procedure for solving equivalence problems that none of the children had used on a screening measure. Specifically, they were taught to "cancel" equivalent addends on opposite sides of the equal sign (e.g., canceling the 3s in 3+4=3+__). Children were assigned to the same four conditions described above. Following instruction, children solved transfer problems, including one

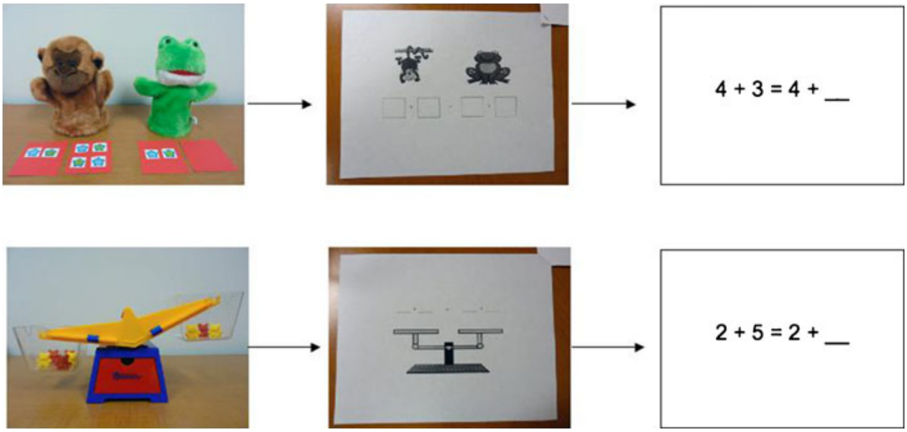

Fig. 2 Progression of materials used during instruction in the concreteness fading condition in Fyfe and McNeil (2009)

"challenge" problem (258+29+173=29+__+258) designed to elicit the cancel strategy. Children in all conditions had similar overall transfer performance, but the challenge problem revealed meaningful differences. Children in the concreteness fading condition were more likely than children in the other conditions to use the cancel strategy and to solve the challenge problem correctly.

Importantly, in both of these experiments, children in the concreteness *introduction* condition did not perform as well as children in the concreteness *fading* condition, ruling out a number of alternative explanations for why concreteness fading is effective. For example, children in both the concreteness fading and concreteness introduction conditions solved problems in three formats (i.e., concrete, worksheet, abstract), manipulated physical objects, and were encouraged to map concrete and abstract examples. Thus, these results suggest it was the specific progression from concrete to abstract that promoted children's mathematical knowledge.

Although evidence in favor of concreteness fading continues to increase, it is worth acknowledging situations in which a concrete-to-abstract sequence did not optimize learning outcomes. As mentioned previously, both Kaminski et al. (2008) and Braithwaite and Goldstone (2013) found that the concrete–abstract combination was not more effective than using abstract materials alone. In another study (Tapola et al. 2013), concrete materials alone faired better than the combination: A simulation on electricity in which the elements remained concrete promoted higher learning than one in which the elements switched from concrete to abstract. Finally, a recent experiment found benefits for a reverse abstract-to-concrete sequence. Middle school students who studied electrical circuits using abstract diagrams that transitioned to concrete illustrations exhibited higher transfer than those who studied concrete illustrations that transitioned to abstract diagrams (Johnson et al. 2014). There are several reasons why concreteness fading may not have optimized outcomes in these situations. First, concreteness fading was not ideally implemented. In all of these studies, only two stages were employed as opposed to the three stages recommended by Bruner (1966), the elements did not span the full concreteness continuum, and the stages were rarely linked in an explicit manner. Second, learner characteristics may have played a role. Concreteness fading assumes that learners can easily comprehend the concrete materials, are unfamiliar with the abstract materials, and have a certain level of "readiness" to learn. If learners lack sufficient prior knowledge to understand the concrete materials, then a fading progression may be too premature. On the other hand, if learners already have a sophisticated understanding of the abstract material, then they may benefit from working directly with the symbolic representations.

Although these examples suggest that there may be boundary conditions to the efficacy of concreteness fading, the number of studies supporting the technique continues to grow. As shown here, there is a variety of support in favor of concreteness fading, ranging from relevant existing literatures to direct, experimental evidence. Clearly, research suggests that concreteness fading can be an effective instructional technique, though the number of studies that instantiate the complete three-step transition is admittedly small. More research is needed to confirm the effectiveness of concreteness fading across tasks, settings, and populations.

Applications of Concreteness Fading

Despite the need for more direct support, concreteness fading has already been incorporated into several mathematics curricula and programs. For example, *MathVIDS* is a supplementary resource for teachers with struggling mathematics learners (Allsopp et al. 2006). One of the core recommended strategies is to teach through CRA, a concrete–representational–abstract sequence. With CRA, each math concept is first modeled with concrete materials, then with pictures that represent the concrete materials, and finally with abstract numbers and symbols. The website provides example materials for each stage of the CRA sequence as well as how to use them effectively. The CRA sequence is primarily studied and recommended by special education researchers (e.g., Butler et al. 2003; Peterson et al. 1988), though some suggest CRA may be a beneficial approach for all learners (Berkas and Pattison 2007).

At least four widely used curricula also use a form of concreteness fading in their lessons. *Singapore Math* is a teaching method and curriculum based on the national math practices of Singapore (Thomas and Thomas 2011). The overarching goal is to facilitate mastery of each basic concept before introducing new material. This is accomplished by covering fewer topics in greater depth and by employing a three-step process that progresses from concrete to pictorial to abstract (Wang-Iverson et al. 2010). As with concreteness fading, this sequence starts with physical, hands-on activity; moves to pictorial diagrams or models; and eventually transitions to conventional abstract symbols. *Singapore Math* continues to grow in popularity and is used by an increasing number of schools in the USA. However, despite reports of high math achievement in Singapore, there is little evidence on the effectiveness of the *Singapore Math* curricula as it is used in the USA (WWC 2009).

Mathematics in Context is a comprehensive middle school curriculum (Romberg and Shafer 2004) that supports mathematics knowledge via progressive formalization (Freudenthal 1991). The goal is to use students' prior, informal representations to support the development of more formal mathematics. In practice, this means the mathematical model is first instantiated in a context-specific, concrete situation and only later generalized over situations as a more abstract model. This instructional approach is based heavily on Gravemeijer's (2002) "bottom-up" progression, in which he recommends starting from situational mathematics and progressing to more formal mathematics. Longitudinal evidence suggests that *Mathematics in Context* leads to achievement gains (Romberg et al. 2005); however, its overall effectiveness remains unclear due to a lack of studies meeting sufficient evidence standards (WWC 2008).

Similarly, *Everyday Mathematics* is a research-based and field-tested elementary school curriculum that emphasizes the use of concrete, real-life examples as introductions to key concepts (Carroll and Issacs 2003). In this way, formal concepts and procedures are not presented in isolation, but are linked to informal, concrete situations. The goal is to help learners think carefully about mathematics by relating the abstract symbols to actions and referents that are familiar to young students. The curriculum also emphasizes facility with multiple representations, especially the ability to translate among representations, which is also a component of concreteness fading. *Everyday Mathematics* was found to have potentially positive effects on math achievement for elementary school students (WWC 2010).

Finally, *Building Blocks* is a supplemental mathematics curricula designed to develop preschool children's early math knowledge through the use of concrete manipulatives and print material (Clements and Sarama 2007). It embeds math learning in daily activities with the goal of relating children's informal math knowledge to more formal mathematics concepts. The general goal is to build upon children's early, situational experiences with mathematics to help them develop a strong foundation for more abstract mathematical thinking and reasoning. Although the use of explicit and gradual fading is not necessarily a formal component of the *Building Blocks* curricula, it is naturally built in as concepts are initially grounded in concrete experience and eventually connected to more conventional mathematics symbols and numbers. Importantly, *Building Blocks* has been evaluated by the What Works Clearinghouse and was found to have a positive effect on mathematics knowledge (WWC 2007).

Although most of these approaches are intended for teachers in classrooms, concreteness fading is also an important design component in technology-driven, individualized instruction. Tablet-based or web-based interventions, though newer and not as fully evaluated, also progress from concrete to abstract. Initial results of a randomized field trial show that one of these interactive programs, *Spatiotemporal* (*ST*) *Math*, a game-based intervention for touch tablets, shows signs of early promise (Rutherford et al. 2010). In *ST Math*, elementary students interact with simple, dynamic representations of visuospatial puzzles. These exercises increase in difficulty and can only be solved when a pattern or concept has been extrapolated. Only after students understand the concept visually are they introduced to corresponding symbolic representations and procedures. Examples of this sequence from pure visual to more symbolic can be viewed at the MIND Research website, www.mindresearch.net/demo/interactive.

A final example of interactive instruction that implements the three-stage fading sequence is the DragonBox system for teaching algebra, illustrated in Fig. 3. Learners begin playing a game with "monsters" that follow certain rules (i.e., rules of algebra): (a) day and night monsters annihilate each other in a vortex, (b) monsters above and below a line can be converted into a special monster that can be eliminated when positioned next to any other

Enactive Iconic Symbolic

Fig. 3 The enactive, iconic, and symbolic stages of concreteness fading as implemented by the DragonBox tutoring system for teaching algebra monster, and (c) a monster can be added to one side of a screen as long as it is also added to the other side. The goal of the game is to isolate a special "star box" on one side of the screen. Passing through an iconic stage, in which monsters are replaced with simple dice patterns, learners finally reach a stage in which algebraic symbols are used. The students learn that the "rules" they had learned before correspond to algebraic axioms (e.g., $A+[-A]=0$, the additive inverse property). Learners implicitly internalize the constraints of algebraic transformations during the concrete, enactive stage, and these constraints are progressively made more explicit and formalized.

Questions for Future Research

In general, concreteness fading has been incorporated into a variety of programs and curricula, all of which have shown potential for success. This is promising for the practical application of concreteness fading in real-world learning contexts, though admittedly more empirical support is needed. Once the efficacy of concreteness fading is established, there are several potential directions for future research. These include, but are not limited to, (1) explicating the underlying mechanisms, (2) optimizing the outcomes, and (3) specifying key moderators. Here, we briefly outline and discuss several issues related to these future directions.

What underlying mechanisms explain the benefits of concreteness fading? There are a number of possibilities, not all of which are mutually exclusive. One potential mechanism by which concreteness fading causes better learning outcomes is by supporting a persistent interpretation of the materials. The fading process may allow learners to maintain their already understood, concrete representation of a concept even as the materials become more abstract. Abstract symbols are arbitrarily linked to their referents and often difficult to interpret. However, people spontaneously interpret ambiguous elements in terms of things with which they are familiar (e.g., Leeper 1935). With concreteness fading, a familiar concrete scenario precedes the abstract scenario. One possibility is that learners maintain this concrete interpretation of the abstract symbols, even after the concrete image is no longer present (Bruner 1966). That is, the abstract symbols carry meaning because of the cognitive availability of the concrete context.

Another potential mechanism is by encouraging structural recognition and alignment (e.g., Kaminski et al. 2009). The fading method strips away extraneous, superficial details thereby reducing opportunities for these details to compete with, or be confused for, the deeper relational structure. Previous research has shown that when situations are presented only in a perceptually rich manner, the structure is not likely to be noticed (e.g., Gentner and Medina 1998; Markman and Gentner 1993). For example, when perceptually rich objects are used in a relational task, children often notice the concreteness of the objects (e.g., both sets contain a toy car), rather than the structural relation (e.g., both sets have objects that increase in size). Concreteness fading can overcome these limitations by narrowing in on the relational structure, rather than surface elements. Also, by highlighting the structure in multiple examples that vary in concreteness, the fading progression likely increases the learner's ability to recognize that structure in diverse contexts. Indeed, several studies have shown that providing at least two examples of a concept, rather than only one, greatly improves learners' ability to identify a novel instance of the concept (e.g., Christie and Gentner 2010; Graham et al. 2010; Kurtz et al. 2013).

Concreteness fading may also improve learning by grounding a concept in an intuitive manner (e.g., Baranes et al. 1989). This allows learners to gain confidence in easily graspable ideas before that grounding is systematically removed. This mechanism may also explain the success of a related instructional technique, which we refer to as prototype fading. A common approach is to present a simple, prototypical example of a concept and introduce increasingly distant examples. For instance, a math teacher might begin a unit on multiplication by first describing repeated addition, a prototypical example that works well for multiplying positive integers. Eventually, students' concept of multiplication extends further and repeated addition falls by the wayside as the basis for understanding multiplication (Devlin 2011). Similarly, a physics teacher might begin a unit on waves by first describing water waves, then sound waves, and finally light waves (Hofstadter and Sander 2013). Analogous to fading out concrete aspects of learning materials, prototype fading works by grounding a concept in an intuitive manner and then systematically removing aspects of that grounding to promote an understanding that is transportable to previously unimagined areas. The most striking cases of prototype fading are powered by the development of symbol systems that are built to express general patterns. One of the best ways to avoid being limited by concrete, superficial appearances is to translate situations into formal symbol systems that are unfettered by the constraints of enactive or iconic representations. Science as a whole historically progresses by the development of such formal understandings (Quine 1977), as algebraic notation, DNA, and the periodic table all attest.

Although the cognitive sciences offer several viable mechanisms, errorless learning might also provide potential mechanisms underlying the success of concreteness fading. Errorless learning was originally investigated in animal learning; pigeons were able to learn difficult discriminations by first presenting them with easily discriminable stimuli superimposed on the difficult stimuli (Terrace 1963). Although pigeons would first respond based on the easily discriminable stimuli, gradually those would be faded out so that only the difficult stimuli remained. Learning the more difficult information is incidental, but still powerful. Errorless learning may also explain why native Japanese speakers learn the distinction between "R" and "L" sounds most effectively when they are first trained on an exaggerated /r/–/l/ discrimination that accentuates the differences between the two sounds and then slowly progressed to a more typical /r/–/l/ discrimination (McClelland et al. 2002). The exaggeration helps learners focus on the relevant discriminating aspects, and then the gradual reduction of the exaggeration gives learners experience with naturalistic speech. Concreteness fading may be driven in a similar manner. Learners with no prior intention of learning abstractions may find themselves acquiring this information simply by responding to easily grasped concrete materials, which perfectly co-occur with relevant abstract characteristics. With concreteness fading, these concrete scaffolds eventually disappear, allowing learners to lean more on abstract information.

In addition to explicating the mechanisms by which concreteness fading works, future research should also work to optimize the technique. One way to optimize the technique is to test whether all types of concreteness are beneficial during the enactive stage. So far we have been positing a single concrete-to-abstract continuum, but it is likely more productive to distinguish several such continua. For example, imagine teaching the distance formula by describing a car ride from Chicago to New York. The teacher can vary perceptual concreteness (e.g., colorful car model versus line drawing of a car), narrative concreteness (e.g., provide a contextualized story versus provide only the relevant variables), familiarity concreteness (e.g., story involving the student versus story involving random person), or representational concreteness (e.g., sketch of the trip versus line graph of distance traveled). These different types of concreteness are unlikely to be cognitively equivalent (Son and Goldstone 2009), and they may interact to impact learning outcomes. For example, preschoolers exhibited better counting when they used perceptually rich, but unfamiliar objects than when they used objects that were perceptually rich and familiar, or perceptually bland (Petersen and McNeil 2013). Future work should continue investigating different types of abstractness and whether they can be combined to facilitate learning and transfer.

A related issue for optimizing concreteness fading is to manipulate components of the sequence. This would allow researchers to determine if it is possible, for example, to bypass the enactive stage. As currently conceptualized, concreteness fading ought to begin with actual physical objects, which provide perceptual and physical experiences. At least one study suggests that the enactive stage is necessary. Butler et al. (2003) compared a full CRA sequence to an RA sequence for teaching students with math disabilities about fractions. The CRA group used concrete manipulatives for the first few lessons, while the RA group used representational drawings. Students in the CRA group exhibited higher posttest achievement than students in the RA group. However, it is not clear if physical manipulation is always feasible or even necessary. For example, although McNeil and Fyfe (2012) employed a three-step fading progression, all three representations were graphic in nature and did not afford physical manipulation. Indeed, perceptually grounded computer simulations of scientific phenomena are often as effective, if not more, than enactive, physical instantiations of the phenomena (de Jong et al. 2013). One key difference between these studies is the knowledge level of the participants. Indeed, Bruner (1966) suggested that the enactive stage may not be necessary for learners with high prior knowledge.

This point highlights that individual differences play a large role in the learning process, often determining whether an instructional technique will be effective or not (Cronbach and Snow 1977). Thus, it will be important to determine which factors serve as potential moderators of concreteness fading. A large body of research indicates that learners' prior domain knowledge may be particularly important (see Kalyuga 2007). A relevant study by Goldstone and Sakamoto (2003) suggests that learners with low knowledge are more adversely affected by distracting concrete materials relative to learners with high knowledge. However, an experiment by Homer and Plass (2009) offers different conclusions. They found that concrete visualizations in a science tutorial were more effective for students with low knowledge than for students with high knowledge. These studies suggest that interactions between concreteness and individual differences may be complicated. Future work is needed to tease apart the distinctions between these studies and to consider the relationship between concreteness fading and knowledge levels.

Another potential moderator to consider is the level of direct instructional guidance. Although many influential learning theorists advocate discovery learning (e.g., Bruner 1961; Piaget 1973), recent evidence suggests that including some explicit instructional elements may be more beneficial than pure discovery alone (e.g., Alfieri et al. 2011; Mayer 2004). However, the optimal amount and type of instructional guidance remains unclear and likely depends on the task, learner, and instructional technique. One relevant study suggests the level of direct instruction influences learning from concrete materials. Kaminski and Sloutsky (2009) examined kindergarteners' ability to recognize common proportions across different instantiations. When explicit training and examples were provided, children who learned with either concrete or abstract materials successfully transferred their knowledge. However, when no explicit instruction was provided, only children who learned with abstract materials exhibited transfer. Thus, one possibility is that concreteness fading is more effective when there is a high level of direct instructional guidance. Another possibility is that concreteness fading is equivalent to direct instruction and may circumvent the need for more explicit training and instruction.

Conclusion

In sum, concreteness fading represents a promising instructional technique that moves beyond the concrete versus abstract debate and exploits advantages of multiple examples across the concreteness continuum. It refers specifically to the three-step progression by which the physical instantiation of a concept becomes increasingly abstract over time. This fading technique offers unique advantages that surpass the benefits of concrete or abstract materials considered in isolation. Additionally, it has some support in the research literature and is widely used in existing mathematics curricula, though more direct experimental evidence is needed.

Given the widespread use and endorsement of concrete materials by researchers and teachers alike, it is pertinent that we find optimal ways to use these materials to facilitate both learning and transfer. This will require considering the types of concrete materials to use, when to use them, and most importantly how to connect them to conventional, abstract symbols. Indeed, as Brown et al. (2009) note, "linking nonsymbolic conceptual understanding to more abstract, symbolic representations may be one of the most significant challenges teachers face today" (p. 162). We propose concreteness fading as a solution to that challenge, as it offers the best of both concrete and abstract instruction.

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