Positivity and contractivity in the dynamics of clusters’ splitting with derivative of fractional order

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Abstract: Classical models of clusters’ fission have failed to fully explain strange phenomena like the phenomenon of shattering (Ziff et al., 1987) and the sudden appearance of infinitely many particles in some systems with initial finite particles number. Furthermore, the bounded perturbation theorem presented in (Pazy, 1983) is not in general true in solution operators theory for models of fractional order \( \gamma \) (with \( 0 < \gamma \leq 1 \)). In this article, we introduce and study a model that can be understood as the fractional generalization of the clusters’ fission process. We make use of the theory of strongly continuous solution operators for fractional models (analogues of \( C_0 \)-semigroups for classical models) and the subordination principle for fractional evolution equations (Bazhlekov, 2000, Prüss, 1993) to analyze and show existence results for clusters’ splitting model with derivative of fractional order. In the process, we exploit some properties of Mittag-Leffler relaxation function (Berberan-Santos, 2005), the He’s homotopy perturbation (He, 1999) and Kato’s type perturbation (Banasiak, 2006) methods. The Cauchy problem for multiplication operator in the fractional dynamics is first considered, before we perturb it. Some additional concepts like Laplace transform, Hille-Yosida theorem and the dominated convergence theorem are use to finally show that there is a solution operator to the full fractional model that is positive and contractive.

Keywords: Fractional Cauchy problem, Contractive solutions, Fragmentation, Perturbation, Solution operators, Positivity

MSC: 26A33, 34A12, 35D10

1 Model’s motivation and introduction

The concepts of fractional derivatives and fractional integral started in 1695 when L’Hospital questioned about the meaning of the operator \( d^n y/dx^n \) if \( n = 1/2 \); that is “what if \( n \) is fractional?”. Leibniz then replied as \( d^{1/2} y/dx^{1/2} \) will be equal to \( x \sqrt{dy/dx} \). Despite its three centuries of age, fractional calculus remains lightly unpopular amongst science and engineering community. However, there is a growing interest in extending the normal calculus with integer orders to noninteger orders (real or complex order) [13, 36, 39, 44] because its applications have attracted a great range of attention in the past few years. Indeed, it has turned out recently that many phenomena in different fields, including sciences, engineering and technology can be described very successfully by the models using fractional order differential equations. As example, the concept of fractional Laplacian operator in the theory of Lévy flights [12] is a typical application of fractional derivatives, which leads to the theory of sub- and super-diffusion, well applicable in reaction-diffusion systems. In the field of mathematical epidemiology, especially the
the representation of the rate of change (accumulation or loss) in the system; that is, gain rate minus loss rate, at infinitesimal bounded space. However, does this system always replicate the real picture of the changing it describes? The answer is not totally positive according to the above arguments. Thus, differential equations with fractional derivative have become a useful tool for describing nonlinear phenomena of science and engineering models. Nonlinear phenomena occurring in the dynamics of fragmentation of clusters are well indicated to be analyzed here. Recall that the process of fragmentation of clusters occurs in many branches of natural sciences ranging from physics, through chemistry, engineering, biology, ecology and numerous domains of applied sciences, such as the depolymerization, the rock fractures and of breakage of droplets. Consequently, there are three essential motivations for considering the fractional clusters’ fragmentation model of this article:

(1): Classical models of clusters’ fission with normal derivative \( \frac{d}{dt} \) cannot fully explain strange phenomena like the sudden appearance of an infinite number of particles in some systems which contained, at the beginning, a finite number of particles and the phenomenon of shattering \([55]\). The latter is seen as an explosive or dishonest Markov process, see e.g. \([1, 38]\) and has been associated with an infinite cascade of breakup events creating a ‘dust’ of particles of zero size which, however, carry non-zero mass. Recall that fragmentation processes are difficult to analyse as they involve evolution of two intertwined quantities: the distribution of mass among the particles in the ensemble and the number of particles in it, that is why, though linear, they display non-linear features such as phase transition which, in this case, is called “shattering” and consists in the formation of a “dust” as explained above. Quantitatively, one can identify this process by disappearance of mass from the system even though it is conserved.

Due to the inability of getting exact solutions in fragmentation models, various authors have used several functional analytic approaches to investigate the dynamics of the system. These methods include semigroup theory \([6, 21, 43]\), perturbation theory \([2, 3, 20]\), approximation techniques \([19, 33]\), and probabilistic methods \([23]\). The efficiency of these methods is limited as these problems are reformulated in abstract spaces that are norm dependent and the overall behavior of the dynamics changes radically as different metrics are included in the system. In \([40]\), the authors provided explicit solutions to clusters’ fragmentation equations with general fragmentation rates, giving a general framework for understanding particles distributions in fragmentation processes as time evolves. Although the result is a breakthrough in the analysis of clusters’s fragmentation equations with arbitrary fragmentation rates, the phenomenon of shattering remains partially unexplained.

(2): As said above, \( \frac{d}{dt} \) is seen as the representation of the rate of change (accumulation or loss) in the system, considered at infinitesimal bounded space. But sometimes the infinitesimal space contains traps (of various sizes) where the variable under study is temporarily parked. Will the \( \frac{d}{dt} \) replicate the real picture of accumulation or loss then? Similarly these trap pictures could be islands or forbidden zones in the infinitesimal space where the variable (particle, mass, density, flux etc.) cannot reside; accordingly, the rate of accumulation or loss will be different than \( \frac{d}{dt} \). Hence, the fractional differentiation \( \frac{D^\gamma}{dt} \), with \( \gamma \in \mathbb{R} \) or \( \mathbb{C} \) may give the sub- or super-rate of accumulation or loss with index \( \gamma \) representing the heterogeneity distribution of the infinitesimal space (traps or islands)! However, substituting \( \frac{d}{dt} \) by the fractional derivative \( D^\gamma_t \) requires some considerations as we will show in the following sections.
Most work on fragmentation with normal derivative \( \frac{\partial}{\partial t} \) has been performed under assumption that the ensemble of particles is well mixed so that the particle distribution is uniform in space. However, recent approach using individual based models, [47], yield in a natural way to systems in which splitting particles are distributed in space according to some prescribed probability density, leading however, to models with explicit space dependence. We note that similar models were also considered earlier in [31] but with emphasis only on well-posedness.

These reasons are the sources of the increasing volition to try new approaches and extend classical models to models with fractional derivative (see [4, 8, 9, 16–18] and investigate them with various and different techniques in order to establish broader outlooks on the real phenomena they describe. For example, the authors [8] successfully generalized the advection-dispersion equation (to the fractional one) by using various techniques including the well-known action of Fourier transform on integer derivatives to rational order. With this in mind, and following the same approach as [2, 14, 15, 18, 42, 49], our model can be obtained by evaluating, on the one hand, the changing due to fragmentation, and on the other hand, the fractional variation that, we hope, (as explained above) will replicate the real picture of that changing. That is, for a cluster of size \( x \), the loss due to the fragmentation is caused by its fission and the gain due to the fragmentation is caused by the fission of a bigger cluster to form groups of size \( x \):

Hence, we propose the following fractional model of pure fragmentation process:

\[
D_t^\gamma (u(x,t)) = -a(x)u(x,t) + \int_0^\infty a(y)b(x|y)u(t,y)dy, \quad x \geq 0, \quad 0 < \gamma \leq 1, \quad t > 0, \tag{1}
\]

where \( D_t^\gamma \) is given as

\[
D_t^\gamma (u(x,t)) = \lim_{t \to 0} \frac{g_\gamma(t)u(x,t) - u(x,t)}{t}, \tag{2}
\]

with \( g_\gamma(t) \) the fractional time evolution, considered as universal attractor of semigroups of coarse grained macroscopic time evolutions [5, 53]. It is shown that [5, 49],

\[
D_t^\gamma (u(x,t)) = \frac{1}{\Gamma(-\gamma)} \int_0^\infty u(x,t-r) - u(x,t) \frac{r^{-\gamma}}{r^{\gamma+1}} dr, \quad 0 < \gamma \leq 1, \tag{3}
\]

which is the fractional derivative of \( u(x,t) \) in the sense of Marchaud, see [49]. Note that the first term in the right-hand-side of equation (1) symbolizes the loss due to fragmentation and the second term is the gain due to fragmentation.

Recall that fission models of type (1) with “normal” time derivative of order one \( \frac{\partial}{\partial t} \), have been comprehensively analyzed in numerous works (see the references below in the next section). Conservative and nonconservative regimes for fragmentation equations have been thoroughly investigated - see [20, 55], and, in particular, the breach of the mass conservation law (called shattering) has been attributed to a phase transition creating a dust of “zero-size” particles with nonzero mass, which are beyond the model’s resolution. Shattering can be interpreted from the probabilistic point of view as the explosion in the Markov process describing fragmentation [50], and from an analytic point of view as dishonesty of the semigroup associated with the model. Transport-type models With convection were investigated in [41] where the author showed that the convection part does not affect the breach of the conservation laws. In [20] the authors studied the non local fragmentation and showed that the process is conservative if at infinity daughter particles tend to go back into the system with a high known probability.

### 2 Model description and method

#### 2.1 Fragmentation differential equation

In this section, important concepts that will help analyze the fractional model are defined. For more details about analysis on fragmentation differential equations, we refer the reader to [6, 16, 20, 24, 34, 41, 50, 54] and the
The classic model for fragmentation process is given by the integrodifferential equation
\[ \frac{\partial}{\partial t} u(x, t) = -a(x)u(x, t) + \int_{x}^{\infty} a(y)b(x|y)u(t, y)dy, \quad x, t > 0, \quad (4) \]
we assume that it is subject to the initial condition
\[ u(x, 0) = f(x) \quad x > 0. \quad (5) \]
It describes the evolution of the mass density \( u(x, t) \) of particles having mass \( x \) at time \( t \); the particles of mass \( x \) undergo fission at a rate \( a(x) \). We assume that \( a(x) \) is non-negative and that
\[ a \in L_\infty, \text{loc}(0, \infty). \quad (6) \]
Further, \( b(x|y) \) describes the distribution of daughter particles masses \( x \) spawned by the fragmentation of a parent particle of mass \( y > x \).

In absence of any other mechanism, the mass of all daughter particles must be equal to the mass of the parent. This conservation law is mathematically expressed by
\[ \int_{0}^{y} xb(x|y)dx = y. \quad (7) \]
Similarly, the expected number of particles produced by a particle of mass \( y \) is given by
\[ n(y) = \int_{0}^{y} b(x|y)dx. \quad (8) \]
We note that \( n(y) \) may be infinite. The total mass of the ensemble at a time \( t \) is given by the first moment of \( u \); that is,
\[ M(t) = \int_{0}^{\infty} xu(x, t)dx. \]
From the physical point of view the total mass of fragmenting particles cannot increase, so the most appropriate Banach space to work in is
\[ X_1 := L_1(\mathbb{R}_+, xd) = \left\{ u: \int_{0}^{\infty} |u(x)|xd < +\infty \right\}. \quad (9) \]
By \( A \) we denote the pointwise multiplication \( \phi(x) \to -a(x)\phi(x) \) defined on a set of, say, measurable functions. Similarly, by \( B \) we denote the expression
\[ [B\phi](x) = \int_{x}^{\infty} a(y)b(x|y)\phi(y)dy, \quad (10) \]
defined first on all positive measurable functions for which the above integral is finite almost everywhere and then extended by linearity to a suitable linear subspace of measurable functions. This allows us to define the following operators: Let \( A \) and \( B \) be defined by
\[ Au = Au \quad \text{on} \quad D(A) = \{ u \in X_1; \quad Au \in X_1 \}, \quad (11) \]
and
\[ Bu = Bu \quad \text{restricted to} \quad D(A) \quad (12) \]
is a well-defined positive operator \([41]\). Then (4)-(5) can be written as an abstract Cauchy problem in \( X_1 \):
\[ \frac{\partial}{\partial t} u = Au + Bu, \quad t > 0 \quad (13) \]
\[ u(0) = f. \quad (14) \]
2.2 Brief overview of Kato-Voigt perturbation theory

Let \((A, D(A))\) be a generator of a \(C_0\)-semigroup on a Banach space \(X\) and \((B, D(B))\) be another operator in \(X\). The purpose of the perturbation theory is to find conditions that ensure that there is an extension \(K\) of \(A + B\) that generates a \(C_0\)-semigroup on \(X\) and characterize this extension.

One of the simplest and possibly the most often used perturbation result can be obtained using Kato-Voigt theorem in the sense that it allows to establish the existence of a smallest substochastic semigroup associated with a specific Cauchy problem. Let us define the terms stochastic and substochastic semigroups.

Definition 2.1. The strongly continuous semigroup of operators \((G(t))\) on the Banach space \(X\) is said to be (i) substochastic if \(S(t) \geq 0\) and \(\|G(t)\| \leq 1\) for all \(t \geq 0\), (ii) stochastic if, in addition, it satisfies \(\|G(t)\| = \|\psi\|\) for all non-negative \(\psi \in X\).

The following theorem holds.

Theorem 2.2. (Kato’s Theorem in \(L_1\) setting)

Let \(X = L_1(\Omega)\) and suppose that the operators \(A\) and \(B\) satisfy:
(1) \((A, D(A))\) generates a substochastic semigroup \((S_A(t))\);
(2) \(D(B) \supset D(A)\) and \(Bu \geq 0\) for \(u \in D(B)_+\);
(3) For all \(u \in D(A)_+\),
\[
\int_{\Omega} (Au + Bu)d\mu \leq 0.
\]

Then there exists a smallest substochastic semigroup, \((G_K(t))\), generated by an extension, \(K\), of \(A + B\). Moreover, \(K\) is characterized by
\[
(I - K)^{-1}\psi = \sum_{n=0}^{\infty} (I - A)^{-1}[B(I - A)^{-1}]^n \psi, \quad \forall \psi \in X.
\]

Proof. [6, Corollary 5.17].

2.3 Fractional fission evolution equation

We are interested in investigating the fractional version of (13)-(14) given by:
\[
D_t^\gamma u = Au + Bu, \quad 0 < \gamma \leq 1, \quad x, t > 0,
\]
\[
u(0) = f.
\]
with \(D_t^\gamma\) defined in (3). However, substituting \(\partial_t\) by \(D_t^\gamma\) in (13) is justified in the sense that the presence of \(\partial_t\) in (13) reflects a basic symmetry of the time translation invariance and the basic principle of locality. In fact, from the relation
\[
\frac{d}{dt}g(t) = \lim_{t \to 0} \frac{g(t) - g(t - t)}{t} = -\lim_{t \to 0} \frac{g(t)g(t) - g(t)}{t},
\]
we see that \(-\frac{d}{dt}\) is identified as the infinitesimal generator of time translation \(g(t)g(t) = g(t - t)\). Hence, this considers \(g(t)\) as the expression of the general time evolution, which is the same consideration done in the definition (2) and (3) of \(D_t^\gamma\). Thus, the derivative of fractional order \(D_t^\gamma\), \(0 < \gamma \leq 1\) was found to be, in general, infinitesimal generator of coarse grained macroscopic time evolution. It is shown [30] that all macroscopic time evolutions have fractional derivatives, with order less than unity, as their infinitesimal generators. Therefore, in Proposition 2.5 below, we provide a relation between the generator \(D_t^\gamma\), \(0 < \gamma \leq 1\) of the macroscopic time evolution and \(\frac{d}{dt}\).
Definition 2.3. Consider an operator $Q_{\gamma}$ applying in the fractional model

$$D_{t}^{\gamma} (u(x,t)) = Q_{\gamma} u(x,t) , \quad 0 < \gamma \leq 1, \; x, \; t > 0,$$

subject to the initial condition

$$u(x,0) = f(x) \quad x > 0$$

and defined in the Banach space $X_1$. A family $(G_{Q_{\gamma}}(t))_{t>0}$ of bounded operators on $X_1$ is called a solution operator of the fractional Cauchy problem (20)-(21) if

(i) $G_{Q_{\gamma}}(0) = I_{X_1}$;
(ii) $G_{Q_{\gamma}}(t)$ is strongly continuous for every $t \geq 0$;
(iii) $Q_{\gamma} G_{Q_{\gamma}}(t) f = G_{Q_{\gamma}}(t) Q_{\gamma} f$ for all $f \in D(Q_{\gamma})$;
(iv) $G_{Q_{\gamma}}(t)D(Q_{\gamma}) \subset D(Q_{\gamma})$;
(v) $G_{Q_{\gamma}}(t) f$ is a (classical) solution of the model (20)-(21) for all $f \in D(Q_{\gamma})$, $t \geq 0$.

It is well known [6, 19, 41] that an operator $\widetilde{Q} \in \mathcal{G}(M, \omega)$ means $\widetilde{Q}$ generates a $C_0$-semigroup $(G_{\widetilde{Q}}(t))_{t>0}$ so that there exists $M > 0$ and $\omega$ such that

$$\|G_{\widetilde{Q}}(t)\| \leq Me^{\omega t} \quad (22)$$

We are not trying to mix the two concepts (solution operator and $C_0$-semigroup), but, by analogy if the fractional Cauchy problem (20)-(21) has a solution operator $(G_{Q_{\gamma}}(t))_{t>0}$ verifying (22), then we say that $Q_{\gamma} \in \mathcal{G}(M, \omega)$. The solution operator $(G_{Q_{\gamma}}(t))_{t>0}$ is contractive if

$$\|G_{Q_{\gamma}}(t)\|_{X_1} \leq 1, \quad (23)$$

and we say $Q_{\gamma} \in \mathcal{G}(1, 0)$.

Note that if we have a contraction solution operator, we can use Definition 2.3 to identify the fractional Cauchy problem of which it is a solution. Usually, however, we are interested in the reverse question, that is, in finding the solution operator, that is contractive, for a given fractional model. The answer is given by the following theorem (seen as an analogue of Hille-Yosida theorem).

Theorem 2.4. An operator $A_{\gamma} \in \mathcal{G}(1, 0)$ for $0 < \gamma \leq 1$ if and only if

(a) $(0, \infty) \subset \rho(A_{\gamma})$.

(b) $\lambda^{\gamma-1} R(\lambda^{\gamma}, A_{\gamma}) f = \int_{0}^{\infty} e^{-\lambda r} G_{\gamma}(r) f dr,$

with $\lambda > 0$, $f \in X_1$ where $(G_{\gamma}(t))_{t>0}$ is a family of strongly continuous operators satisfying (23) and $\rho(A_{\gamma})$ is the resolvent set of the operator $A_{\gamma}$:

$$\rho(A_{\gamma}) = \{ \lambda \in \mathbb{R} ; \; \lambda I - A_{\gamma} : D(A_{\gamma}) \rightarrow X_1 \text{ is invertible and } (\lambda I - A_{\gamma})^{-1} \text{ bounded and linear} \}. \quad (25)$$

Proof. This theorem is a particular version of [46, Theorem 1.3] and the proof follows the same steps. \qed

In our analysis we will need some interesting properties of the Mittag-Leffler relaxation function $E_{\gamma}(-x)$ which arises in the description of complex relaxation processes and that corresponds to a relaxation function when $x$ is a positive real number like the time variable. We know that, see [22, 37],

$$E_{\gamma}[z] = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\gamma + 1)} \quad (26)$$
Using the Laplace transform $\mathcal{L}$ and its inverse, it can be proved, see [11, 26, 52], that
\[
E_{\frac{1}{2}}(\gamma) = \frac{2\pi}{\alpha} \int_0^\infty \frac{E_{2\beta}(\gamma^2)}{\gamma^2 + \gamma^2 \beta^2} \, d\gamma, \quad 0 < \gamma \leq 1.
\]
\[
= \sin(\gamma\pi) \frac{\pi}{\alpha} \int_0^\infty \frac{r^{\gamma-1}e^{-\frac{1}{\gamma}r} \, dr}{r^{\gamma^2 + 2\gamma^2(\gamma\pi)}} + 1, \quad 0 < \gamma \leq 1
\]
\[
= \int_0^\infty \sum_{n=0}^\infty \frac{(-r)^n}{n!} e^{-\gamma r} \, dr, \quad 0 < \gamma \leq 1
\]
\[
= \frac{1}{2\pi i} \int_\Gamma e^{(\nu-xr)^2} r^{\gamma-1}e^{-\gamma r} \, dr, \quad 0 < \gamma \leq 1.
\]
where $\Gamma$ is a contour domain which encircles the origin counterclockwise, going from $-\infty$ to $-\infty$.

To proceed, we consider the full operator in (17) as a perturbation of the following Cauchy problem representing the loss part of the fractional fission process:
\[
D_t^\gamma u(x, t) = -a(x)u(x, t) \quad (= Au(x, t)), \quad 0 < \gamma \leq 1,
\]
\[
u(x, 0) = f(x), \quad x, t > 0.
\]
Take
\[
f \in D(A),
\]
we can use the point of view of $C_0$-semigroup theory to say that the operator $A$ is the infinitesimal generator of a $C_0$-semigroup $(G_A(t))_{t \geq 0}$, from the original semigroup $(S(t))_{t \geq 0}$ (the multiplication semigroup) generated by the differentiation operator $\frac{d}{dt}$ and obtained via a simple integration as
\[
(S(t)u)(x, t) = e^{-a(x)t}u(x, t).
\]
The existence of $(S(t))_{t \geq 0}$ comes from assumption that the fission rate satisfies (6). Therefore the infinitesimal generators of $(G_A(t))_{t \geq 0}$ may be interpreted as the distributions, see [25] evaluated on the right time translation group, which leads to the following proposition:

**Proposition 2.5.** (a) $(S(t))_{t \geq 0}$ is an equibounded semigroup and the norms of all operators $S(t), \quad t \in \mathbb{R}$ are bounded above by 1;  
(b) $A = D_t^\gamma, \quad 0 < \gamma \leq 1$, let $u \in D(A)$. The infinitesimal generators $A$ of $C_0$-semigroups $(G_A(t))_{t \geq 0}$ are related to $\frac{d}{dt}$ by the representations
\[
\left(\frac{d}{dt}\right)^\gamma u(t) = -D_t^\gamma (u(t)) = \lim_{\varepsilon \to 0+} \frac{1}{\Gamma(\gamma)} \int_\varepsilon^\infty r^{-\gamma-1}[1 - e^{-a(t)r}]u(t)dr
\]
and
\[
\left\|S(t)u - u\right\| \leq Mt^\gamma \|Au\| \quad \text{for some } M > 0
\]
(c) For every $u \in D(A)$ we have $G_A(t)Au = AG_A(t)u$.

**Proof.** (a): By condition (6), the operator $A$ is bounded linear operator. Then, we know that [16, 20, 41] it is the infinitesimal generator of a positive semigroup of contractions (or substochastics semigroup defined in Definition 2.1) and the assertion follows.

To prove (b) we exploit the Marchaud type representation (3) of infinitesimal generators (see also [32, 49]). Let $\delta$ be the Dirac measure, then if we take $C^\infty_c(\mathbb{R})$ as the set of test functions on $\mathbb{R}$, we know that, see [48], the fractional derivative distribution $\delta^\gamma, \quad 0 < \gamma \leq 1$, of $\delta$ can be expressed by
\[
\langle \psi, \delta^\gamma \rangle = \frac{1}{\Gamma(\gamma)} \int_0^\infty r^{-\gamma-1}[\psi(t) - \psi(t + r)]dr
\]
and approximated by the family $\{F_{\varepsilon}\}_{\varepsilon>0}$ of finite Borel measure on $\mathbb{R}_+$ given as

$$
\langle \varphi, F_{\varepsilon}' \rangle = \frac{1}{\Gamma(-\gamma)} \int_{\varepsilon}^{\infty} r^{-\gamma-1} [\varphi(t) - \varphi(t + r)] dr, \quad \text{for all } \varphi \in C^1_0(\mathbb{R}).
$$

The family $\{F_{\varepsilon}\}_{\varepsilon>0}$ generates the bounded linear operators $F_{\varepsilon}$ defined by

$$
F_{\varepsilon} u(t) = \frac{1}{\Gamma(-\gamma)} \int_{\varepsilon}^{\infty} r^{-\gamma-1} [u(t) - S(r)u(t)] dr, \quad u \in X_1.
$$

Using Laplace transform $L$, we can show, see [51], that

$$
L \left( F_{\varepsilon} \right) (s) = \frac{s}{\Gamma(-\gamma)} \frac{1}{\Gamma(1 + \gamma)} \left[ \left( \frac{s}{\varepsilon} \right)^{-\gamma} - \left( \frac{s}{\varepsilon} + 1 \right)^{-\gamma} \right] (s), \quad s > 0,
$$

which yields, by uniqueness theorems for Laplace transformation and well known properties of convolution operator $*$,

$$
F_{\varepsilon}' = 0^* \left[ \frac{1}{\Gamma(-\gamma)\Gamma(1 + \gamma)} \left( \left( \frac{\varepsilon}{s} \right)^{-\gamma} + \left( \frac{\varepsilon}{s} + 1 \right)^{-\gamma} \right) \right]
$$

and then

$$
A \int_{0}^{\infty} \frac{1}{\Gamma(-\gamma)\Gamma(1 + \gamma)} \left( \left( \frac{\nu}{s} \right)^{-\gamma} + \left( \frac{\nu}{s} + 1 \right)^{-\gamma} \right) S(r)u(t) dr = \frac{1}{\Gamma(-\gamma)} \int_{\varepsilon}^{\infty} r^{-\gamma-1} [u(t) - S(r)u(t)] dr.
$$

Now taking $\varepsilon \to 0$, the assertion (b) follows by using the fact that $A$ is a closed operator [21] and

$$
\lim_{\varepsilon \to 0} \int_{0}^{\infty} \frac{1}{\Gamma(-\gamma)\Gamma(1 + \gamma)} \left( \left( \frac{\nu}{s} \right)^{-\gamma} + \left( \frac{\nu}{s} + 1 \right)^{-\gamma} \right) S(r)u(t) dr = u(t).
$$

To prove (33) and (c), we make use of [43, Theorem 6.13 p.74]. We just need to show that $0 \notin \rho(A)$. In fact since $a \in L_{\infty, \text{loc}}(0, \infty)$ then $0 \notin \text{sup}\{(0, \infty)\}$ hence, $a$ has a bounded inverse $1/a$ and then, $A$ as well. Thus $0 \notin \rho(A)$, which completes the proof.

3 Analysis of fractional fission problem

3.1 Homotopy perturbation for the loss part of fractional fission problem

The He homotopy method is more a computational device that allows us to approximate the solution of the fractional model in order to, for instance, proceed some simulations. We provide a simple understanding of the homotopy perturbation method related to the loss part of the fractional fission model. Then, consider the following fractional differential equation:

$$
D_t^\gamma u(x, t) = -a(x)u(x, t), \quad t > 0, \quad x \in \mathbb{R}, \quad 0 < \gamma \leq 1
$$

where $D_t^\gamma$ is given by (3). The fractional homotopy perturbation method, introduced by He [27, 28] states that a homotopy for equation (37) can be constructed in the following manner:

$$
(1 - p) \left( D_t^\gamma u(x, t) \right) + p \left[ D_t^\gamma u(x, t) + a(x)u(x, t) \right] = 0
$$

giving

$$
D_t^\gamma u(x, t) = p [-a(x)u(x, t)]
$$
with $p$ the embedding parameter taken in $[0, 1]$. In case $p = 0$, equation (38) is a fractional differential equation, $D^\gamma_t (u(x,t)) = 0$ which is easy to solve. When $p = 1$, equation (38) turns out to be the original one (37). The basic assumption is that the method admits solutions that can be written as a power series in $p$:

$$u(x, t) = u_0(x, t) + pu_1(x, t) + p^2u_2(x, t) + p^3u_3(x, t) + \cdots \quad (39)$$

It is obvious that the solutions of the original equation (37) can be approximated by tending $p \to 1$, that is

$$u(x, t) \approx u_0(x, t) + u_1(x, t) + u_2(x, t) + u_3(x, t) + \cdots \quad (40)$$

Substituting (39) into (38) and comparing coefficients of terms with identical powers of $p$, we obtain a system of equations given in the following forms:

$$p^0 : D^\gamma_t u_0(x, t) = 0,$$
$$p : D^\gamma_t u_1(x, t) = -a(x)u_0(x, t),$$
$$p^2 : D^\gamma_t u_2(x, t) = -a(x)u_1(x, t),$$
$$p^3 : D^\gamma_t u_3(x, t) = -a(x)u_2(x, t),$$

Taking Laplace transform of both sides of this system yields

$$p^0 : s^\gamma \tilde{u}_0(x, s) - s^{\gamma-1}u_0(x, 0) = 0,$$
$$p : s^\gamma \tilde{u}_1(x, s) - s^{\gamma-1}u_1(x, 0) = -a(x)\tilde{u}_0(x, s),$$
$$p^2 : s^\gamma \tilde{u}_2(x, s) - s^{\gamma-1}u_2(x, 0) = -a(x)\tilde{u}_1(x, s),$$
$$p^3 : s^\gamma \tilde{u}_3(x, s) - s^{\gamma-1}u_3(x, 0) = -a(x)\tilde{u}_2(x, s),$$

where $\tilde{u}_k(x, s)$ is the Laplace transform $\mathcal{L}(u_k(x, t), s), \ k = 0, 1, 2, \cdots$. Now setting the initial condition $u(x, 0) = f(x)$ and taking the inverse Laplace transform of both sides of (43) gives

$$p^0 : u_0(x, t) = f(x),$$
$$p : u_1(x, t) = -\frac{a(x)T^\gamma}{\Gamma(\gamma + 1)}f(x),$$
$$p^2 : u_2(x, t) = \frac{a(x)}{\Gamma(\gamma)}(T^\gamma - 1 * u_1(x, t)),$$
$$p^3 : u_3(x, t) = \frac{a(x)}{\Gamma(\gamma)}(T^\gamma - 1 * u_2(x, t)).$$

where $\Gamma$ is the gamma function and $*$ the convolution operator. In this way we compute few terms to approximate the solution of the fractional fragmentation Cauchy problem (28)-(29) and the asymptotic solution is given by (40). The existence of solutions to (28)-(29) is guaranteed by the condition (30).

### 3.2 Analysis of the full fractional fission Cauchy problem

It is well known, see [46], that the bounded perturbation theorem stated in [43] is no longer valid in solution operators theory for models of type (17)-(18), with $0 < \gamma < 1$, so we aim here to use other perturbation methods to prove existence results for the full model

$$D^\gamma_t (u(x,t)) = -a(x)u(x,t) + \int_x^\infty a(y)b(x|y)u(t,y)dy, \quad 0 < \gamma \leq 1, \ x, t > 0. \quad (44)$$
subject to the initial condition
\[ u(x, 0) = f(x) \quad x > 0 \] (45)

**Proposition 3.1.** Assume that the coefficients \( a \) and \( b \) satisfy the conditions (6) and (7) respectively, then there is a solution operator for the fractional model (44)-(45) that is positive and contractive.

**Proof.** We make use of Kato’s Theorem 2.2, the Hille-Yosida theorem and its analogue Theorem 2.4. Consider the dynamics with first order derivative
\[ \frac{d}{dt} u = Au + Bu \] (46)
where \( A \) and \( B \) satisfy the expressions (11) and (12), respectively. Since the coefficients \( a \) and \( b \) satisfy the conditions (6) and (7) then, making use of Fubini theorem together with the conservation law (7), we have for all \( u \in D(A) \)
\[
\int_{\mathbb{R}} (Au + Bu) d\mu = \int_{\mathbb{R}} \left( -a(x)u(x,t) + \int_{\mathbb{R}} a(y)b(x,y)u(y,t)dy \right) dx \\
= -\int_{0}^{\infty} a(x)u(x,t)dx + \int_{0}^{\infty} a(y)b(x,y)u(y,t)dy \\
= -\int_{0}^{\infty} a(x)u(x,t)dx + \int_{0}^{\infty} a(y)u(y,t) \left( \int_{0}^{\infty} b(x,y)dx \right) dy \\
= -\int_{0}^{\infty} a(x)u(x,t)dx + \int_{0}^{\infty} a(y)u(y,t)dy \\
= 0.
\] (47)
Moreover, \( D(B) = D(A) \) and \( Bu \geq 0 \) for \( u \in D(B)_+ \). By Kato’s Theorem, there exists the smallest substochastic semigroup, \( (G_K(t))_{t \geq 0} \), generated by an extension, \( K \), of \( A + B \). Exploiting the relaxation relation (27), we set
\[
G(t)f = \int_{0}^{\infty} \Theta_p(t,r)G_K(r)f dr
\] (48)
where \( \Theta_p(t,r) = \frac{1}{\Gamma(r)} \sum_{n=0}^{\infty} \frac{(-r)^n}{n! \Gamma(1-\gamma n)} \). We aim to show that \( G(t) \) is the solution operator for the fractional model (44)-(45), positive and contractive.

The second last relation of (27) and monotonicity of \( E_p(-x) \) imply that \( (G(t))_{t \geq 0} \) is positive. By Hille-Yosida theorem we have \( (0, \infty) \subset \rho(K) \) and \( \|G_K(r)\|_{X_1} \leq 1 \), \( r \geq 0 \) since \( G_K(t) \) is substochastic. Then using (27) and the fact, see [26, 52], that
\[
\int_{0}^{\infty} \left( \sum_{n=0}^{\infty} \frac{(-r)^n}{n! \Gamma(1-\gamma n)} \right) dr = 1
\] (49)
yields \( \|G(r)\|_{X_1} \leq \int_{0}^{\infty} \Theta_p(t,r)\|G_K(r)\|_{X_1} dr \leq 1 \), \( t \geq 0 \); hence \( (G(t))_{t \geq 0} \) is contractive. Finally, by the subordination principle developed in [10, 46], we have \( K \in G'((1,0)) \) since \( K \in G(1,0) \) (Hille-Yosida theorem). Thus by theorem 2.4, \( G(t) \) is the solution operator for the fractional model (44)-(45), that is positive and contractive.

### 4 Concluding remarks

In this paper, we have set conditions allowing us to analyze and show the existence of a solution operator, positive and contractive, to the fission model (17)-(18) with derivative of fractional order \( \gamma \), with \( 0 < \gamma \leq 1 \). We have made use of He’s homotopy perturbation and Kato’s type perturbation methods together with the theory of strongly continuous
solution operator, properties of Mittag-Leffler relaxation function and the subordination principle. Fragmentation equations with fractional order dynamics has never been studied before. Whence, this work is an opening which can serve as the first step toward the full characterization of the operator \(K\) under which the fractional model admits solution operators and maybe, leads to a better understanding of strange phenomenons like the phenomenon of shattering and the sudden appearance of an infinite number of particles in some systems with initial finite particles number.

References