Analysis of discrete-to-discrete imaging models for iterative tomographic image reconstruction and compressive sensing

Jakob H. Jørgensen, Emil Y. Sidky, and Xiaochuan Pan

Abstract—Discrete-to-discrete imaging models for computed tomography (CT) are becoming increasingly ubiquitous as the interest in iterative image reconstruction algorithms has heightened. Despite this trend, the full intuition for algorithm and system design derives from analysis of continuous-to-continuous models such as the X-ray and Radon transform. While the similarity between these models justifies some crossover, questions such as what are sufficient sampling conditions can be quite different for the two models. This sampling issue is addressed extensively in the first half of the article using singular value decomposition analysis for determining sufficient number of views and detector bins. The question of full sampling for CT is particularly relevant to current attempts to adapt compressive sensing (CS) motivated methods to application in CT image reconstruction. The second half goes in depth on this subject and discusses the link between object sparsity and sufficient sampling for accurate reconstruction. Particularly, it is pointed out that, while CS motivated image reconstruction is object dependent, there is a need to consider the imaging task so that test phantoms are employed with a similar sparsity level as what might be encountered.

Index Terms—Computed Tomography, Discrete-to-discrete Imaging Models, Sampling Conditions, Total Variation, Compressive Sensing, Breast CT

I. INTRODUCTION

RECENTLY, iterative image reconstruction (IIR) algorithms have been developed for X-ray tomography [1], [2], [3], [4], [5], [6], [7], [8], [9], [10] based on the ideas discussed in the field of compressive sensing (CS) [11], [12]. One can argue about whether these algorithms are truly novel or not: edge-preserving regularization [13], [14], [15], [16], [17] has a clear link to sparsity in the object gradient, and algorithms specifically for object sparsity have been developed for blood vessel imaging with contrast agents [18]. Nevertheless, the interest in CS has re-invigorated IIR development for application to computed tomography (CT) by broadening the perspective on applying optimization-based methods for IIR algorithm development. There has also been much recent work on efficient algorithms involving variants of the $\ell_1$-norm [19], [20], [21]. Despite these recent developments in IIR, all the intuition for algorithm and system design derives from analysis of continuous-to-continuous models such as the X-ray and Radon transform. While the similarity between these models justifies some crossover, questions such as what are sufficient sampling conditions can be quite different for the two models. The purpose of this article is to characterize the discrete-to-discrete system models used in most IIR algorithms; specifically, we address sampling conditions necessary for inverting such models. This discussion is particularly relevant in the context of discussing reduced sampling with CS motivated methods.

A. Generic continuous-to-continuous and discrete-to-discrete models of the X-ray transform

Explicit image reconstruction algorithms for CT are largely based on inversion formulas for the continuous-to-continuous conebeam or X-ray transform model

$$g[\vec{s}, \vec{\theta}] = \int_0^\infty dt f(\vec{s} + t\vec{\theta}), \quad (1)$$

where $g$, the line integral over the object function $f$ from source location $\vec{s}$ in the direction $\vec{\theta}$, is considered data. For example, 2D filtered back-projection (FBP) inverts this model for the case where the source location $\vec{s}$ varies continuously on a circular trajectory surrounding the subject, and at each $\vec{s}$ the ray-direction $\vec{\theta}$ is varied continuously through the object in the plane of the source trajectory. While algorithm design for image reconstruction in CT scanners must take into account a number of physical factors not present in these imaging models, algorithm development does benefit from an understanding of the inversion of simplified imaging models. For CT, the properties of the X-ray, Eq. (1), and Radon transform have been thoroughly studied [22], [23]: for example, in 2D it is well-known that $\pi$ plus the fan-angle yields sufficient angular coverage for fan-beam image reconstruction [24], and in 3D, Tuy’s condition [25] describes scanning angular range sufficiency for the cone-beam transform. Both of these commonly used conditions and other important properties of the X-ray transform are derived from analyzing the approximate model given in Eq. (1).

For most IIR algorithms, this data model is discretized by expanding the object function in a finite expansion set, for example, in pixels/voxels. The resulting imaging model becomes discrete-to-discrete,

$$\vec{g} = X\vec{f}, \quad (2)$$

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where $\vec{q}$ represents a finite set of ray-integration samples, and $\vec{f}$ are coefficients of the object expansion, and $X$ is the system matrix modeling the ray integration. This discrete-to-discrete model is almost always solved implicitly, because the matrix $X$, even though sparse, is beyond large for CT applications: $X$ is in the domain of a giga-matrix for 2D imaging and a tera-matrix for 3D imaging. Consulting Chapter 15 of [26] provides a more in depth discussion on various types of imaging models.

B. What are the sampling conditions of the discrete-to-discrete model?

Iterative image reconstruction, which is gaining much attention lately due to the pressure to reduce scanning dose, almost always uses the discrete-to-discrete imaging model of Eq. (2). Even when IIR algorithms employ non-linear models, Eq. (2) is often part of these more realistic non-linear models, which include beam polychromaticity, scatter, or partial volume averaging. Despite its ubiquity, the properties of this core, linear model are generally poorly understood. Existence, uniqueness, and stability of solutions of Eq. (2) under different scattering conditions depend on the object expansion set and the precise model for the ray integration.

Because the discrete-to-discrete system matrix $X$ is in some way connected to the X-ray transform, some transfer of intuition is justified. But there are questions for which the two models need to be handled separately. One example, which will be covered extensively in this article, is: how many data samples are enough for accurate reconstruction? Or, what can be considered full sampling (FS)? Inversion of Eq. (1) subsumes knowledge of the continuous data function, while actual CT systems are digital and provide a discrete set of measurements. As a result, interpolation of the data is necessary to employ the inverse of this model. Discrete-to-discrete models have the advantage that data interpolation may not be necessary because the image representation is finite; the size of a FS data set will be finite and at least as large as the size of the image representation. As will be seen, the difficulty in characterizing FS conditions for $X$ is that stability of its inverse becomes important and the fact that $X$ comes in many forms. Singular value decomposition (SVD) is the tool used here to analyze the sampling conditions. Some early singular SVD analysis was performed in, for example, Ref. [27], and even some work in deriving an inverse to the discrete Radon transform appear in Refs. [28], [29], [30]. Those early works, however, could not have foretold the variety of expansion sets used: pixels, blobs [31], wavelets [32], [33], natural pixels [34], [35], or non-uniform representations with varying element sizes; nor the various approximations to ray-integration: Siddon’s method, ray-tracing with various interpolators, and distance-driven projection [36], to name a few. Even simple changes like increasing the pixel density dramatically affects sampling conditions; what might be FS for a $100 \times 100$ pixel array might not be FS for a $1000 \times 1000$ pixel array.

C. Sampling conditions and CS

In fact, the notion of FS is not even well-defined. This issue bears directly on CS, where a central theme is to reconstruct from fewer measurements than FS, i.e., undersampled reconstruction. Without a proper definition, it is difficult to report meaningful quantitative undersampling ratios. We address this issue in this article.

CS seeks to link sampling to object sparsity. If the object is sparse, in some representation, there is a potential to save on scanning effort. The question is: relative to what scanning effort is CS potentially saving. The oft-heard refrain from the CS community is that exploiting sparsity allows the engineer to design an imaging system that samples an object transform at less than the Nyquist rate. Employing the Nyquist limit as the definition of FS, however, is not useful for CT, because all CT systems would then yield insufficient sampling. CT objects are modeled closely by functions that contain edge discontinuities and the corresponding projections are therefore discontinuous in the first derivative. As a result the projection data are not band-limited, and the Nyquist theorem would then require continuous sampling of the X-ray projections. And no CT system is capable of continuous projection sampling. The confusion is cleared up in realizing that this statement mixes up the two models for CT imaging: Eqs. (1) and (2). Nyquist sampling or other forms of interpolation are needed only for the former model, while CS theory pertains to the latter, discrete-to-discrete imaging model. Thus, to report potential theoretical gains from exploiting sparsity, FS conditions for Eq. (2) need to be determined.

In this article, we ignore model error and focus on characterizing sufficient sampling conditions for accurate recovery of $\vec{f}$ in the discrete-to-discrete linear system model (2). All data sets used in simulations are generated directly by Eq. (2). For example, if a $10 \times 10$ pixel array is used for reconstruction, the underlying object really is composed of a $10 \times 10$ array of pixels.

D. Organization of the paper

In Sec. [II] we introduce notation and the specific form of the X-ray transform used as the CT imaging model. Sec. [III] explains a strategy of employing SVD analysis to characterize sampling conditions targeted at small scale systems amenable to full SVD. In Sec. [IV] we extend the results to system matrices of realistic size, where the full SVD is unavailable. In Sec. [V] we discuss strategies for reconstruction from undersampled data, including CS. In Sec. [VI] having proposed a definition of FS, we present a case study on overcoming undersampling with CS. This study is performed through simulations and solving CS-type optimization problems. Although our proposed definition of FS is object independent, any simulation study must use a test phantom, which makes the simulation results object dependent. As such, selection of the test phantom is crucial, and should reflect the imaging task for which the CT scanner is designed. We conduct the simulations within the context of breast CT.

II. TOOLS AND TERMINOLOGY

A. The discrete-to-discrete X-ray transform imaging model

The 2D discrete-to-discrete X-ray transform model can be adapted to many geometries, e.g., parallel beam, limited angle,
etc. In the present work, we focus on circular fan-beam CT with equi-angular projection spacing. CT typically acquires projects from multiple source locations which lie on a curve trajectory and the source location \(s(\lambda)\) is specified by a scalar parameter. The circular trajectory is the most common, and is what we use here,

\[
s(\lambda) = R_0 (\cos \lambda, \sin \lambda),
\]

where \(R_0\) the distance from the center-of-rotation to the X-ray source. The detector bin locations are given by

\[
\vec{b}(\lambda, u) = (R_0 - D)(\cos \lambda, \sin \lambda) + u(-\sin \lambda, \cos \lambda),
\]

where \(D\) is the source-to-detector-center distance, and \(u\) specifies a position on the detector. The ray direction for the detector-geometry independent data function is

\[
\theta(\lambda, u) = \frac{\vec{b}(\lambda, u) - \vec{s}(\lambda)}{||\vec{b}(\lambda, u) - \vec{s}(\lambda)||_2}. 
\]

We divide the \(2\pi\) arc into \(N_{\text{views}}\) arranged at equally spaced angular intervals, so that the source parameters follow

\[
\lambda_i = i\Delta\lambda, \quad \Delta \lambda = 2\pi / N_{\text{views}} \quad \text{and} \quad i \in [0, N_{\text{views}} - 1].
\]

The detector is subdivided into \(N_{\text{bins}}\),

\[
u_j = u_{\text{min}} + (j + 0.5) \Delta u,
\]

where \(D_L\) is the detector length, \(u_{\text{min}} = -D_L/2\), \(\Delta u = D_L/N_{\text{bins}}\), and \(j \in [0, N_{\text{bins}} - 1]\). The detector length is determined by requiring it to detect all rays passing through the largest circle inscribed within the image array, which is assumed to be square and consisting of \(N \times N\) square pixels. Due to the geometric mismatch of a Cartesian image grid and the circular, fan-beam CT scan, only this central circle of the array is visible from all views and accordingly we restrict the unknown pixel values to lie within this circle. As a result the number of unknown pixel values \(N_{\text{pix}}\) is

\[
N_{\text{pix}} \approx \frac{\pi}{4} N^2,
\]

where the actual value, which clearly has to be an integer, is given with each simulation below. A pixel is considered to be part of the circular image region, if its center lies within the circle. Effectively, the dimensions of the projector \(X\) are \(N_{\text{views}} \times N_{\text{bins}}\) rows (number of ray integrations) and \(N_{\text{pix}}\) columns (number of variable pixels). To obtain the individual matrix elements, we employ Siddon’s method [37], where \(X_{m,n}\) is the intersection length of the \(m\)th ray with the \(n\)th pixel. This description completely specifies a linear system, Eq. (2). However, one important issue within the CT community is that there is no standard method for discretizing the X-ray transform. Many expansion elements have been used in CT studies: blobs [31], wavelets [32], [33], natural pixels [34], [35] to name a few. Also, the matrix elements using only the pixel expansion set can be calculated in different ways that all tend toward the continuous model in the limit of shrinking pixel size. Different modeling choices will necessarily alter \(X\).

Recovering image coefficients from a set of data, involves inverting Eq. (2). There has been some work in direct inversion of the discrete Radon transform, but in general no direct inversion of the discrete X-ray transform covers all the specific models used in practice. Instead, for discrete-to-discrete X-ray transform models, Eq. (2) is solved by implicit methods.

In practice, a tomographic device will have its sampling described by continuously varying parameters, in this case angular scanning interval and detector length, and a discretization of these sampling intervals. How well the image coefficients can be recovered depends on both the sampling intervals and how finely discretized are these intervals. For the present study, we restrict attention to the discretization of sampling intervals and choose the angular interval to be \(2\pi\). With the scanning ranges corresponding to theoretically complete values, the question of FS boils down to: what values of \(N_{\text{views}}\) and \(N_{\text{bins}}\) do we need to recover an \(N_{\text{pix}}\) set of pixel coefficients? The most direct way to answer this question is through singular value decomposition (SVD) analysis of \(X\).

As an aside before going on to SVD analysis, we briefly discuss the more familiar Fourier Transform (FT) imaging model used, e.g., in MRI. For the FT model, samples are usually collected on a Cartesian grid, and the image is almost always represented using Dirac delta functions. The locations of these deltas determined by the FT sample spacing so that the corresponding system matrix \(F\) is unitary and \(F^{-1} = F^\dagger\). The form of \(F\) also lends itself conveniently to algorithmic acceleration, yielding the well-known fast Fourier transform (FFT). SVD of \(F\) yields unit magnitude singular values and a condition number of 1. Note there is no need to invoke the Nyquist sampling theorem to invert \(F\), because it is a discrete-to-discrete imaging model. This theorem is only needed to be able to say that the discrete set FT samples represents the continuous data function. If more general sampling patterns, or different image expansions, are employed with the FT model, the corresponding system matrix may no longer be unitary and a similar analysis to what we present below may prove useful. Coming back to line-integral imaging models, there has been some effort to come up with convenient sampling schemes for the discrete Radon transform that have simple inverses, analogous to the FFT, or preserve properties of the continuous Radon transform [28], [29], [30]. As far as we know, such schemes, however, have not been widely adopted.

B. The singular value decomposition

The necessary tool for analyzing the discrete X-ray transform is linear algebra, and more specifically the singular value decomposition (SVD), which decomposes the system matrix into \(X = USV^T\), where \(U\) and \(V\) are unitary matrices holding the left and right singular vectors in the columns, and \(\Sigma\) is a diagonal matrix of the same size as \(X\) with the singular values on the diagonal in non-increasing order, \(\sigma_{\text{max}}, \ldots, \sigma_{\text{min}}\). Of particular interest is the condition number of \(X\), \(\kappa(X) = \sigma_{\text{max}} / \sigma_{\text{min}}\), which quantifies the numerical instability of \(X\). Using the condition number it is possible to bound the propagation of data error into the reconstruction; if \(\Delta\vec{f}\) is a data perturbation vector, and \(\Delta\vec{f}\) is the resulting
perturbation in the image, when solving $X\tilde{f} = \tilde{g}$, then
\[
\frac{\|\Delta\tilde{f}\|_2}{\|f\|_2} \leq \kappa(X) \frac{\|\Delta\tilde{g}\|_2}{\|\tilde{g}\|_2}.
\] (7)

SVD analysis of $X$ can be challenging because realistic size system matrices are too large. Projection data sets can vary anywhere from $10^6$ to $10^9$ transmission measurements, and the image representation can have equally many pixels/voxels. For addressing sampling sufficiency, however, it can be instructive to investigate smaller systems which can be completely SVD analyzed. Some limited knowledge can also be obtained for larger systems: the extreme singular values and their corresponding right singular vectors.

For complete SVD analysis of smaller $X$, we employed the sparse matrix routines available with the open source Python packages SciPy [38] and sparsesvd [39]. For these smaller system matrices, $X$ is computed and stored. $X$ is then converted to compressed sparse column format and handed off to the sparse SVD computer algorithm, yielding the complete set of singular values and vectors.

C. Candidates for sampling conditions

For the sake of completeness we remind the reader of Hadamard’s conditions [40] for an inverse problem to be well-posed:

1) A solution to the problem exists,
2) The solution to the problem is unique,
3) The solution is continuously dependent on the data.

If any of the conditions is violated, the problem is called ill-posed. For the discrete X-ray transform in (2) the conditions are satisfied if and only if $X$ has an empty nullspace and the data is in the range of $X$. If $X$ satisfies these conditions, we will say that it is ensures Hadamard Full Sampling (FS). Clearly, as a minimum for Hadamard FS, the number of rows in $X$ must be equal to or larger than the number of columns, but in practice more samples may be needed to obtain linear independence of the columns. In terms of the SVD of $X$, Hadamard FS amounts to having a nonzero $\sigma_{\text{min}}$, or equivalently, a finite condition number.

When solving $X\tilde{f} = \tilde{g}$ on a computer with finite numerical precision, Hadamard FS may not be enough to ensure that a reliable solution can be obtained. If $X$ is ill-conditioned, a perturbation of the data, even just from truncating the ideal infinitely precise data, may propagate into the reconstruction, depending on the size of the condition number of $X$. We want to define a concept of numerical FS for $X$, which ensures that not only is $X$ Hadamard FS, but also numerical instability is under control, i.e., the condition number of $X$ is not (loosely speaking) too large. More generally, we use the term stability to describe how well a given imaging model or algorithm handles data inconsistencies such as noise.

III. SAMPLING CONDITIONS FOR CIRCULAR FAN-BEAM

A. Sampling conditions through SVD of the discrete X-ray transform

We start with a small $N = 32$ image array with a total of $N_{\text{pix}} = 812$ pixels in the circular image region. With the model described in Sec. II-A we generate system matrices $X$ for different numbers of views and detector bin sampling. As discussed in Sec. II-C we need at least as many samples as the number of unknowns to have Hadamard FS, so we investigate samplings for $N_{\text{views}} \in [32, 128]$ and $N_{\text{bins}} \in [32, 128]$.

We have performed a comprehensive study of the fan-beam full scan with uniform discretization, and the condition number as a function of $N_{\text{views}}$ and $N_{\text{bins}}$ is shown for some configurations in Fig. 1. The largest condition number for the considered sampling range is 825.5 occurring at $N_{\text{views}} = N_{\text{bins}} = 32$. Because the condition numbers are finite, each of the discretizations shown leads to a Hadamard FS system matrix. Accordingly, for Hadamard FS, already the $32 \times 32$ data samples would be sufficient much like the analogous sampling for the DFT of a $32 \times 32$ pixel array. But the condition number is quite large for the lower number of samples, implying that any data inconsistency could be amplified tremendously when solving (2) and as a result these sampling conditions might not be practically useful. The condition number of $X$ decays sharply with increasing $N_{\text{bins}}$ and slowly with $N_{\text{views}}$. In principle, a more useful definition of complete sampling could be defined as the number of samples beyond which the condition number no longer decreases. This could make sense
on inspection of the $N_{\text{bins}}$ dependence, but the decay with $N_{\text{views}}$ is too gradual for a well-defined sampling condition.

Based on these results, we, somewhat arbitrarily, propose that for $2\pi$ circular, fan-beam projection with a circular image array the FS condition is

$$N_{\text{bins}} = 2N \quad \text{and} \quad N_{\text{views}} = 2N. \quad (8)$$

We offer the following heuristic justifications:

1) $N_{\text{bins}} = 2N$ is chosen in order to have a safety margin avoiding the sharp increase in condition number at $\approx 1.5N$.

2) $N_{\text{views}} = 2N$ is chosen because the condition number decay is slow there and this value is symmetric with the bins-dimension. But it must be kept in mind that this particular value is a softer boundary than for the bins-dimension.

To lend credence to the proposed soft sampling condition we explore the middle of the singular value spectrum, investigate scaling with $N$, and examine an alternate discrete form of the X-ray transform.

### B. Connecting the condition number and the full SVD spectrum

As the properties of $X$ depend on the full set of singular values, the condition number plots of the previous section may not have much meaning if the variation of the lowest singular value with respect to $N_{\text{views}}$, is different from that of the other small singular values. To examine this issue, we fix $N_{\text{bins}} = 2N$ with $N = 32$ and vary $N_{\text{views}}$ in [16, 64] displaying all the singular values of $X$ in Fig. 2. Note that the smallest data sampling shown $N_{\text{bins}} = 64$ and $N_{\text{views}} = 16$, which has slightly more samples than unknowns, has all singular values greater than zero, and is thus Hadamard FS.

We see the same general trend for all choices of $N_{\text{views}}$, namely a relatively slow decay of the singular values during the middle of the spectrum and a slightly faster decay for the smallest, but no distinct separation of the smallest singular value from the other small values. This supports our approach of using only the condition number, instead of the full set of singular values, as an indicator of FS. This will prove useful when analyzing larger $X$, where only the condition number is available, as the full spectrum of singular values can not be computed in reasonable time. We also see a gradually growing smallest singular value when increasing $N_{\text{views}}$, whereas the largest singular value is closer to being constant. This explains the smooth decrease in condition number with increasing $N_{\text{views}}$ we saw in Fig. 1.

### C. Alternate models for $X$

One of the main motivations of this paper is to point out that each linear imaging model is different, and accordingly the above analysis should be performed even when seemingly minor changes are made to the model. While we do believe that there are some general features like the scaling with $N$, other aspects of the model may not be that general. These results could depend on the approximation to the continuous ray integration. The previous results apply to the line intersection weights used in Siddon’s method. If instead we model the X-ray transform with ray-tracing, using nearest neighbor interpolation, we obtain quite different condition numbers in Fig. 3. That the ray-tracing condition numbers are lower does not necessarily mean that this method is better than Siddon’s method, because the other side of the story is model error, or how the different approximations impact data inconsistency. One could imagine that other models such as area-weighted integration instead of the linear integration or different basis functions could alter the condition numbers of $X$ even more substantially.

Summarizing the well-posedness of the circular, fan-beam CT version of the discrete-to-discrete imaging model, we have existence of a unique solution when the data $\vec{g}$ are in the range of $X$, and $X$ has full column rank, i.e., is Hadamard FS, which tends to occur in practice when the number of ray samples is slightly higher than the number of pixels. Stability of the solution, however, as measured by the condition number of $X$, improves dramatically until the number of ray samples is $2N \times 2N$.

### IV. Extension to Systems of Realistic Size

#### A. SVD algorithms

For large $X$, where the complete SVD is impractical, we only aim for the condition number of $X$, for which we need the largest and smallest singular values. We employ two algorithms for obtaining these as well as their corresponding right singular vectors. The well-known power method, see, e.g., [41], can be used to find the largest eigenvalue and eigenvector of $X^T X$, or in a slightly modified version, to produce the largest singular value $\sigma_{\text{max}}$ and its right singular vector $\vec{v}_{\text{max}}$ of $X$, see the algorithm in Fig. 4. This algorithm is extremely effective, and for our investigations twenty iterations sufficed.

The problem of finding the smallest singular value is more difficult, and there is a large amount of literature on SVD of large matrices, see for example [42]. Because SVD is
To solve this constrained minimization we solve a series of
where the linear projection operator
hyperplane is described by the expression
right singular vector, we seek the vector
some explanation. To find the minimum singular value and its
X
specifically adapted for finding the lowest singular value of
the lowest value. The algorithm in Fig. 5 describes a method
variations on the Lanczos method, aim at obtaining multiple
singular vectors and values. Here, we are interested only in
X
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singular vectors and values. Here, we are interested only in
the lowest value. The algorithm in Fig. 5 describes a method
specifically adapted for finding the lowest singular value of
our tomographic system matrix \( X \). This algorithm requires
some explanation. To find the minimum singular value and its
right singular vector, we seek the vector \( \vec{v}_{\text{min}} \) having minimal
\( \| X \vec{v} \|_2 \) among all unit-length vectors \( \vec{v} \), i.e.,

\[
\vec{v}_{\text{min}} = \arg\min_{\vec{v}} \| X \vec{v} \|_2 \quad \text{subject to} \quad \| \vec{v} \|_2 = 1.
\]

To solve this constrained minimization we solve a series of
quadratic minimizations. Starting with an initial guess \( \vec{v}_0 \), we
find the image \( \vec{v}_{\text{hyp}} \), which, among all vectors \( \vec{v} \) restricted
to the hyperplane tangent to the unit sphere at the current
(\( n \)th) estimate \( \vec{v}^{(n)} \), is the one that minimizes
\( \| X \vec{v} \|_2 \). This hyperplane is described by the expression
\( \vec{v} = P_{\vec{v}^{(n)}} \vec{v} + \vec{v}^{(n)} \), where the linear projection operator
\( P_{\vec{v}} \), which projects an image \( \vec{f} \) onto the hyperplane through the origin and having \( \vec{v} \)
as a unit normal vector, is defined by

\[
P_{\vec{v}} \vec{f} \equiv \vec{f} - \vec{v} (\vec{f} \cdot \vec{v}).
\]

For algorithmic purposes it is useful to know that \( P_{\vec{v}} \) is its own
transpose. We arrive at the intermediate optimization problem

\[
\vec{v}_{\text{hyp}} = \arg\min_{\vec{v}} \| X (P_{\vec{v}^{(n)}} \vec{v} + \vec{v}^{(n)}) \|_2^2.
\]

The solution to Eq. (10) occurs at

\[
XP_{\vec{v}^{(n)}} \vec{v} = -X \vec{v}^{(n)},
\]

and this equation shows up in the algorithm in Fig. 5 at line 3
where in the pseudocode we have omitted explicit reference
to the iteration counter \( n \).

The next estimate is found by normalizing \( \vec{v}_{\text{hyp}} \),

\[
\vec{v}^{(n+1)} = \vec{v}_{\text{hyp}} / \| \vec{v}_{\text{hyp}} \|_2.
\]

The algorithm in Fig. 5 is not proved to converge, in fact
it is possible to make bad choices for \( \vec{v}_0 \), namely, one of
the other right singular vectors of \( X \); however, these choices are
pathological and in practice the error in solving the linear
equation at line 3 causes the algorithm to migrate to the right
singular vector with the lowest singular value. Nevertheless the
candidate solution should be checked against the optimality
conditions of Eq. (9). We did, as of yet, not specify the
line equation solver algorithm at line 3. For the results
presented here we employed a standard conjugate gradients
(CG) algorithm. Accurate solution to this intermediate linear
system is not necessary, and for the present simulations we
performed twenty iterations of CG at line 3.

With these algorithms, the condition number of \( X \) can be
obtained by taking the ratio of the singular values. While the
behavior of \( X \) depends on the full set of singular values,
some understanding can be gained from only obtaining the
condition number. Furthermore, inspection of the right singular
vector \( \vec{v}_{\text{min}} \) itself can help to characterize artifacts to which
the modeled system may be sensitive, in the case where \( \vec{v}_{\text{min}} \) is,
or is close to being, in the nullspace of \( X \).
V. DEALING WITH UNDERSAMPLING

A. Overall strategies

After investigating the SVD of $X$, a logical question to address is what to do if the condition number is unacceptably high or if $X$ is outright not Hadamard FS. Many existing strategies are available that provide solutions to this problem. Three basic strategies are: solution space restriction, interpolation, and object/data model redesign. Mathematically, the condition of $X$ can also be improved by altering the scanning configuration, but for the sake of this discussion we regard the system hardware as fixed.

The solution space of $\hat{X}$ is the set of images whose projection $X\hat{f}$ is within some distance to the available data $\hat{g}$. If $X$ has a non-trivial nullspace, or if $X$ is poorly conditioned, many images will minimize, or nearly minimize, the distance to the data. For these cases, restriction of the solution space by incorporating other information on the image can yield an optimization problem that has a unique and stable solution. Solution space restriction is really a special case of a set theoretic approach to CT image reconstruction [43], where one of the sets is the set of all images within a prescribed error tolerance of the available data.

The next category of strategies is to interpolate the available data to an additional set of measurements corresponding to another system matrix $X$ with better condition number. This strategy is the one most analogous to what transpires with FBP. The difference here is that the image representation is finite while with FBP the image model is continuous. As a result, it is not necessary to interpolate to a continuous set of measurements.

Another common approach is to alter the image representation. Increasing voxel size, for example, will improve the condition number of $X$. Of course there may be a price to pay in the ability to represent the underlying object function. This is a well-known conundrum of implicit methods for image reconstruction (see for example Sec. 15.1.2 of Ref. [26]); namely, a complex image representation may model the object and scanner better but lead to an ill-posed imaging model, while a simple representation might not model the system as well but allow for a well-posed imaging model.

B. Compressive sensing background

Compressive sensing (CS) is a rapidly developing mathematical framework [11], [12], [44] for reconstruction of objects from undersampled data, i.e., where multiple solutions exist. The key concept is the assumption that there is a transform that renders the object of interest sparse, i.e., having many zeros. In image processing the most well-known such transforms are the Discrete Cosine Transform (DCT), wavelets, and the image gradient magnitude. Sparsity, as measured by the number of nonzeros, is difficult to work with numerically, so in CS it is common to do reconstruction using the $\ell_1$-norm on the transform coefficients for enforcing sparsity, and under certain conditions equivalent results are obtained. With this approach, a unique solution can be obtained, namely the solution having the sparsest representation under the transform as measured the $\ell_1$-norm.
In CT the typical image consists of regions having a fairly constant gray-level value separated by relatively sharp boundaries, e.g., between bone and surrounding tissue. The magnitude of the spatial gradient of such images is zero within constant regions, so the gradient magnitude image can be very sparse. Using the $\ell_1$-norm on the gradient magnitude image is known as Total Variation (TV) regularization, and this is the CS approach we will consider. Early results claimed perfect reconstruction possible but only under ideal conditions, and in practice data inconsistencies as well as numerical issues prevent perfect reconstruction.

One of the main contributions from CS is conditions on the sampling matrix for being able to reconstruct a sparse signal: it is often possible to reconstruct the object using four times as many samples as nonzeros in the transform of the object even if there are many more unknowns. This result relies on the sampling matrix satisfying certain properties, e.g. incoherence, or being random, e.g., having Gaussian entries—properties that a CT system matrix is far from satisfying. Hence, it is not clear whether to expect similar results for CT.

C. Formulation as optimization problems

We wish to compare several strategies of restricting the solution space, i.e., specifying a unique solution among all $\vec{f}$ that satisfy $X\vec{f} = \vec{g}$. This leads to equality-constrained optimization problems, which can be challenging to solve accurately. In practice, we will allow a small deviation from perfect equality, by requiring the normalized data residual error to be bounded by some $\epsilon$, i.e., having the inequality-constraint

$$D(X\vec{f}, \vec{g}) \leq \epsilon \quad \text{where} \quad D(X\vec{f}, \vec{g}) = \sqrt{\frac{\|X\vec{f} - \vec{g}\|_2^2}{N_{\text{views}}N_{\text{bins}}}}. \quad (11)$$

The price we must pay for trading the equality-constraint for an inequality-constraint is the loss of the possibility for exact reconstruction. Although theoretically interesting, exact recovery has little practical meaning due to inconsistencies and noise in real data.

For the CS approach, we solve the constrained TV-minimization problem

$$\hat{f}^* = \arg\min_{\vec{f}} \|\nabla \vec{f}\|_1 \quad \text{such that} \quad D(X\vec{f}, \vec{g}) \leq \epsilon. \quad (12)$$

The notation $\nabla \vec{f}$ refers to a pixelwise numerical gradient of the image $\vec{f}$; in each pixel the $\ell_2$-norm magnitude of the gradient is computed, and $\|\nabla \vec{f}\|_1$ is then the sum of the gradient magnitude over the image. This is the TV of the image. This optimization problem may yield accurate image reconstruction from undersampled projection data, provided that the underlying object is sparse in the gradient magnitude. In order to have a quantitative sense of the possible reduction in sampling, we compare the CS results with two forms of constrained quadratic norm minimization,

$$\hat{f}^* = \arg\min_{\vec{f}} \|\vec{f}\|_2^2 \quad \text{such that} \quad D(X\vec{f}, \vec{g}) \leq \epsilon, \quad (13)$$

$$\hat{f}^* = \arg\min_{\vec{f}} \|\nabla \vec{f}\|_2^2 \quad \text{such that} \quad D(X\vec{f}, \vec{g}) \leq \epsilon, \quad (14)$$

where $\|\nabla \vec{f}\|_2^2$, the roughness, is the sum of the squared gradient magnitude over the image. These constrained, quadratic minimizations are equivalent to Tikhonov regularization,

$$\hat{f}^* = \arg\min_{\vec{f}} \alpha^2\|L\vec{f}\|_2^2 + \|X\vec{f} - \vec{g}\|_2^2, \quad (15)$$

where $L$ is either the identity or $\nabla$, in the sense that for a given $\epsilon$ there is a choice of the regularization parameter $\alpha$ such that the two optimization problems have the same solution. We use the constrained formulation in order to get comparable results from the three optimization problems; the data residual error $D(X\vec{f}, \vec{g})$ tolerance is constrained to be the same, $\epsilon$, for each problem. We include the image roughness results, because this is a commonly used penalty in IIR.

The solutions to Eq. (13) and Eq. (14) can be obtained with standard CG applied to the Lagrangians of these problems with the multiplier being adjusted until $D(X\vec{f}, \vec{g})$ is $\epsilon$. Accurate solution of Eq. (12) is non-trivial; although the objective is convex, it is not quadratic. Indeed, one of the major efforts of CS has been to derive efficient algorithms for solving Eq. (12), particularly for small values of $\epsilon$. The algorithm employed here is an accelerated first-order method, using only the objective and its gradient, and is explained in detail in Ref. [19]. An important technical detail for this algorithm is that it requires that the image TV term be differentiable. For the algorithm implementation we use the smoothed TV term $\sum_i \sqrt{\|\nabla \vec{f}_i\|_2^2 + \eta}$ with a very small $\eta = 10^{-10}$ to prevent impact on the reconstructed image. $\nabla$ is computed with forward finite differencing. In the simulations we use a small but non-zero $\epsilon = 10^{-5}$, and in Sec. VI-E we discuss the effect of using other values of $\epsilon$.

VI. A CASE STUDY IN THE CONTEXT OF BREAST CT SIMULATIONS

A. Background

Up until now, the article has described generic analysis of discrete system matrices modeling CT and has been concerned mainly with characterizing FS. With a definition of FS in place, we can start considering reconstruction in case of undersampling, that is when the FS-conditions are not met. For many applications of IIR, undersampling is an important consideration. Particularly for CT, where resolution requirements can be quite high, the number of image representation coefficients can be higher than the number of equations arising from the available data set. To deal with this undersampling, one of the three previously mentioned, general strategies needs to be employed. We are, in particular, interested in the solution space restriction, where prior knowledge about the imaging system and task can be injected to potentially gain in image quality. This gain in image quality, however, comes at the price of generality of the method. Accordingly, when both designing and evaluating the image reconstruction algorithm, it is imperative to have a particular imaging system and task in mind.

For the following, we take a closer look at the CS approach of exploiting sparsity in the image, which falls under the category of solution space restriction. The success of exploiting sparsity depends on the typical sparsity levels expected
to be encountered for the particular imaging protocol. To determine what is the potential gain in exploiting sparsity, we must restrict the set of test phantoms to those typical of the application, paying particular attention to modeling the sparsity level correctly. We focus the following results on application to a breast CT model.

Breast CT is being considered as a possible screening and diagnostic tool for breast cancer. The system requirements are challenging from an engineering standpoint, because this type of CT must operate with a total exposure similar to two full-field digital mammograms (FFDM) to limit the risk of radiation-induced cancer. FFDMs for a screening exam entail two X-ray projections, while breast CT acquires on the order of 500 X-ray projections. The exposure previously used for only two views is now divided up among 250 times more projections. Having developed CS-motivated IIR algorithms for CT, we see an opportunity for application also to breast CT. The potential to reconstruct volumes from fewer views than a typical CT scan might allow an increased exposure per view. We made a preliminary study [45] of the trade-off between number of views and exposure per view with a simulation that involved a fairly realistic, synthetic breast phantom [46] and a noise model that reflected the change in exposure when modeling different numbers of views while fixing total exposure at a realistically low level. Our initial results were a bit surprising: the noise level seemed to be a secondary concern, and the more important issue seemed to be to achieve the minimum number of views from which the object could be accurately reconstructed from noiseless data. It was this result that prompted us to take a closer look at sampling sufficiency with a discrete-to-discrete system matrix, which is used by pretty much every IIR algorithm.

As the purpose of this study is to uncover the sampling condition for perfect recovery of the image, the data are generated from the digital phantom with the system matrix used in the IIR algorithm and no data inconsistency or noise is introduced – the “inverse crime scenario” [47]. We fix $N_{\text{bins}}$ at 512 and vary the number of views, evenly distributed over a $2\pi$ scanning arc, in an interval [32, 512].

For this study, we revisit our breast phantom, displayed in Fig. 7 which consists of $N_{\text{pix}} = 51468$ pixels within the circular image region, contained in a $256 \times 256$ array. The breast phantom has a small region of interest (ROI) containing 5 tiny ellipses which model microcalcifications. The gray values range from 1.0 to 2.3 in units of the attenuation of water. The modeled tissues and corresponding gray values are fat at 1.0, fibroglandular tissue at 1.10, skin at 1.15, and micro-calcifications ranging from 1.9 to 2.3. The sparsity in the gradient magnitude image is 10,000 non-zero pixels, or roughly one fifth of the total number of pixels, only considering the pixels in the circular image region.

In the results we report the root mean square error (RMSE) between a reconstruction and the original discrete phantom over the whole image, $\delta$, as well as the RMSE of the ROI, which can be a more sensitive measure of reconstruction quality for the image features of interest, thereby having clinical relevance. If the RMSE of a reconstructed image is small compared to 0.05, the minimum gray level contrast in the phantom, then the image can appear to be exact.

C. Results

The full image and ROI error plots are shown for each optimization problem for a range of view numbers in Fig. 8. Selected images corresponding to these results are shown in Fig. 9. To help in the interpretation of the results, two reference lines are drawn in the graphs. The vertical line indicates the smallest number of views, $N_{\text{views}} = 101$, for which the number of samples $N_{\text{views}} \times 512$ is larger than $N_{\text{pix}}$, and cause $X$ to be Hadamard FS. The horizontal line indicates the minimum gray level contrast, 0.05, in the test phantom and provides a reference for the RMSE in order to determine

B. Experimental setup

We are now in a position to take a look at the question that motivated the studies of this article; namely what is considered FS, and how much does CS based image reconstruction potentially allow us to reduce the data sampling in CT.
whether the reconstructed images are close to the original phantom. If an image is close to the original it must have an RMSE much smaller than 0.05. On the other hand, an image might be far from the original and still have a low RMSE—one example of this occurs for a reconstructed image which is identical to the original at all but one pixel. This single pixel causes all the RMSE and prevents overall closeness to the original. However, such examples are pathological and we will assume that the reconstruction is close to the original in case of small RMSE.

For the $\ell_2$ magnitude and roughness, both full and ROI, we observe a gradual decay of the RMSE w.r.t. $N_{\text{views}}$ until about $N_{\text{views}} = 256$, where the curves begin to level off. This very slow decay is in agreement with the condition number decay as function of $N_{\text{views}}$ in Fig. 4 as well as the increasing $\sigma_{\min}$ in Fig. 2. Numerical FS occurs at $N_{\text{views}} = 512$. At this point both RMSE curves are very flat, confirming that moving to a higher number of views will only provide a minor gain in quality. This supports our choice of definition for numerical FS. For this particular phantom we could have taken almost as few as $N_{\text{views}} = 256$ for the FS definition. It must, however, be kept in mind that doing a single phantom study can only show reconstruction quality for that particular phantom, whereas the FS condition is independent of the data, and hence, valid for all images. Images might exist that would still have a very rapidly decreasing RMSE at $N_{\text{views}} = 256$, so having the FS definition at $N_{\text{views}} = 512$ gives a margin of safety for such more demanding images.

The TV-curve shows a much steeper transition yielding low RMSE images already at $N_{\text{views}} \geq 64$, a factor of 4 lower than the $\ell_2$ magnitude and roughness reconstructions, and also before the Hadamard FS border. For fewer views, quality quickly deteriorates, and the RMSE approaches that of the $\ell_2$ magnitude and roughness reconstructions at $N_{\text{views}} = 32$. At the other end of the range for the full image RMSE, the TV reconstruction is actually slightly inferior to the $\ell_2$ magnitude one. This is not too surprising, since the regularizing effect of having a non-zero $\epsilon$ causes a small bias of the solutions compared to the original image. The relative size of the biases are not known in advance, and we note that for the ROI curves no crossing is observed.

We conclude that the TV-regularized solution is not in all cases to prefer over the $\ell_2$ magnitude and roughness solutions: when $N_{\text{views}}$ approaches numerical FS there is enough data that not much is gained by choosing one solution over the other. For very few views, $N_{\text{views}} \leq 32$, there is so little data that all considered methods fail to provide an accurate reconstruction. But in the middle there is a window, $N_{\text{views}} \in [64, 256]$ for the present phantom, where the TV-regularized solution is far superior in terms of an almost constant and very low RMSE.

We note that the full image and ROI RMSE are simply two choices for attempting to quantify the reconstruction errors. Assessment of image quality is notoriously difficult, and needs to be associated with the particular task to be solved by the reconstructed image. For this reason, as we focus on accurate imaging of microcalcifications, we report the ROI RMSE. That the two panels of Fig. 8 have different results for the two chosen metrics shows that the comparison of reconstruction approaches will necessarily be affected by the choice of metric. Other metrics might lead to different conclusions.

Visual inspection is also important for quality assessment, although it cannot provide an objective quantitative error. The reconstructed images in Fig. 9 show very different types of artifacts for the $\ell_2$ magnitude and roughness images and the TV-images. The gradual improvement in the upper two rows is in sharp contrast to the rapid improvement in the third row, as also seen from the RMSE curves. However, the microcalcifications can be identified in all reconstructions, although more clearly with more views. It may be argued that 32 views would suffice if we are solely interested in the microcalcifications and disregard the prominent artifacts of the background image. However, for accurate overall reconstruction, more views are definitely needed. The ROI of the TV-solution shown with a narrow gray scale window in the bottom row emphasizes the very accurate image obtained at $N_{\text{views}} = 64$, at which point both $\ell_2$ magnitude and roughness reconstructions still show...
D. Interpretation of $\epsilon$

With the previous example explained, we discuss the interpretation of $\epsilon$ in greater detail. First qualitatively: the case of $\epsilon = 0$ reveals where there is sufficient data for solving the imaging models in case of perfectly consistent data, but ignores the finite numerical precision of the computer. For system matrices, that barely satisfy Hadamard FS, large reconstruction errors can still occur, as can be seen from the large $\delta$ values at $N_{\text{views}}$ around 101 (nearly 0.03 in the ROI). Regularization, such as the use of TV-minimization, is needed to handle the large condition number of $X$. Furthermore, strict requirement of $\epsilon = 0$ can be challenging to achieve for an algorithm, so for practical solution, it is useful to consider a small, but nonzero $\epsilon$. Automatically, a nonzero $\epsilon$ also admits clear background artifacts.
a solution in the case of inconsistent data, where \( \epsilon = 0 \) has no solution.

Let us investigate the influence of changing \( \epsilon \). In Fig. 10 we repeat the study using \( \epsilon = 10^{-3} \) and \( 10^{-4} \) in addition to the previously used \( 10^{-5} \). As the \( \ell_2 \) magnitude and roughness curves were very similar we show only the \( \ell_2 \) magnitude curve here. We see that for many views the RMSE for both TV and \( \ell_2 \) magnitude are reduced by a factor of almost precisely a factor of 10 in each 10-factor reduction of \( \epsilon \). For few views there is almost no variation with \( \epsilon \). It seems reasonable to conclude that in the limit when \( \epsilon \) goes to 0, the TV-curve will approach a very sharp corner at about \( N_{\text{views}} = 50 \) and an RMSE of 0 at larger \( N_{\text{views}} \). From the plot it is not as clear what the \( \ell_2 \) magnitude curve approaches, but theoretically at the Hadamard FS of \( N_{\text{views}} \geq 101 \) the \( \ell_2 \) magnitude solution approaches the unregularized least-squares solution with an image RMSE of 0.

The particular values of \( \epsilon \) and the associated RMSEs are directly related to the contrast of the image. A simple derivation shows that if the phantom image is scaled by a factor of \( a \) then also \( \epsilon \) needs to be scaled by \( a \), in order to obtain the same (but scaled) reconstructed image. More complicated contrast changes of the phantom call for nontrivial changes in \( \epsilon \) but this example motivates the relation between \( \epsilon \) and the image contrast.

Another extremely important point about the parameter \( \epsilon \) is that it has nothing to do with the level of data inconsistency that the various optimization problems can withstand to yield a reconstructed image of high quality. The use of \( \epsilon = 10^{-5} \) in the above example may give the false impression that noise or other inconsistency has to be this level for these optimization approaches to be used. This is incorrect: first, here, we are interested in determining when reconstructions are visually very close to the original, while images of practical utility can, and will, have visible artifacts; and second, the landscape of the optimization problems can be quite different for the ideal and inconsistent cases. The ideal case with non-zero \( \epsilon \) settles on an image with lower TV than the test phantom, which can quickly degrade the image quality since the test phantom already has low TV. The inconsistent case can lead to an image with very high TV if \( \epsilon \) is small. In the inconsistent case, if \( \epsilon \) is chosen such that the test phantom lies just within the data constraint, there is a chance that the resulting image is close to the test phantom. This behavior was demonstrated in Ref. [4], but whether this happens, depends strongly on the form of the inconsistency as well as sampling conditions of \( X \). For this reason we do not directly address the inconsistent case in this article; we focus only on sampling for discrete-to-discrete imaging models for CT with ideal data.

### E. Implications for CS

As described in section (V-B) the main point of CS is the connection between the number of samples and the number of nonzeros in the sparse transform of the image, here the gradient magnitude image. We examine this connection for the present simulation. As argued in the previous section, in the limit of \( \epsilon = 0 \) accurate TV-reconstruction will be obtained for \( N_{\text{views}} \geq 50 \). The number of samples in the 50-view data set is approximately 25,000 or roughly 2.5 times the sparsity in the phantom’s gradient magnitude image. This factor of 2.5 is surprisingly low considering that the system matrix \( X \) does not appear to be CS friendly [48]; the sampling is not random and the matrix lacks incoherence [49]. We caution, however, that this is just a preliminary result and further study is required. On the other hand, the phantom with its random structure would not seem to be favored over other phantoms of similar gradient magnitude sparsity.

Another important question to address is how much does the CS-approach of exploiting sparsity actually help compared to the more well-known Tikhonov and least squares approaches. This question, although more difficult to address, due to the soft sampling condition in \( N_{\text{views}} \), as well as possible dependence on other properties of the object being scanned besides sparsity, is important for practical system and image reconstruction algorithm design. As we saw in the previous section, for obtaining an RMSE of 0 the \( \ell_2 \) magnitude approach needs \( N_{\text{views}} \geq 101 \) views in the limit that \( \epsilon \) goes to zero, because the system here is Hadamard FS. Compared to the 50 views needed by TV-reconstruction, we see a reduction in the number of views by a factor of two. The illustrated
The study with the head phantom completes our explanation of the CS specific approach to dealing with undersampling in a discrete-to-discrete imaging model. Because constrained TV-minimization employs such a model, SVD-based analysis gives a more useful reference for defining FS conditions than the Nyquist theorem. Having stated this, it is clear that there is more work to do in precisely defining the amount of undersampling afforded by, in this case, exploiting gradient magnitude sparsity. The complication over traditional CS theory stems from the fact that measurement matrix conditioning plays an essential role for CT. At present, we find graphs, such as Fig. 8 and the left panel of Fig. 11 that compare a given undersampling strategy to Tikhonov image reconstruction, a useful roadmap.

Because the studies of this section are based on simulations, it is not possible to make general conclusions on the effectiveness of image reconstruction exploiting gradient magnitude sparsity. The results, however, do hint at some general properties which could be explored theoretically. For example, the ratio of 2.5 between the data samples and gradient sparsity may turn out to be a general rule for evenly sampled circular fan-beam CT. Analogous theory as developed for image sparsity in Ref. [51], needs to be developed for gradient magnitude sparsity for linear transforms including the discrete X-ray transform. Also, the connection between image contrast levels and stability towards inconsistencies of constrained TV-minimization appears to be important.

VII. SUMMARY AND CONCLUSIONS

Discrete-to-discrete imaging models lie at the heart of most iterative image reconstruction algorithms. For CT applications in particular, such imaging models lack standardization as various expansion elements and projector models are routinely employed. In order to obtain some sense of the sampling conditions necessary for inverting such system models, it is useful to explore the system matrix condition number for various sampling patterns where the number of samples outnumber the image representation elements. Additional information about the imaging model may be obtained by scaling the system down to a manageable size, where a full SVD analysis is possible.

The power of optimization based methods comes from providing a natural framework for incorporating prior knowledge about the subject. For purposes of this article, this
prior knowledge is employed to handle non-Hadamard FS situations, i.e., where the discrete-to-discrete linear image model is has fewer samples than unknowns, or equivalently, has a non-trivial nullspace. The applicability of any strategy for exploiting prior knowledge can have tremendous variability depending on how closely the true image matches the prior assumptions. As a result, it is important to take into account the purpose of the imaging device even at the stage of algorithm development.

As a specific example of this strategy, we detailed the potential exploitation of sparsity for breast CT with circular fan-beam sampling with equispaced view angles. For this particular system, the condition number plots indicate a rough sampling condition of $2N$ views by $2N$ detector bins for reconstructing a circular region of pixels bounded by an $N \times N$ pixel array, employing Siddon’s method for projection. $2N$ bins was chosen to provide a margin of safety to a steep increase in condition number at $1.5N$. The value of $2N$ views is a soft boundary as the condition number is improving slowly at this number of views.

In terms of CS, we found for both the breast example and a modified FORBILD head-phantom that using 2.5 times as many samples as nonzeros in the gradient magnitude image is sufficient for accurate TV-reconstruction. The more complex structured breast phantom, with about four times as many gradient magnitude nonzeros, required about four times as many samples. Thus, stating a generally sufficient number of views based on simulation with a simple phantom may be misleading.

Toward integrating IIR algorithms into actual CT systems, it is often assumed that model error is the key barrier to achieving high image quality. While this is clearly an important concern, we point out that image quality can also suffer when sampling is insufficient or a particular strategy for dealing with undersampling does not apply. For IIR, it is important to understand the discrete-to-discrete imaging model sampling conditions and, when the data is undersampled, to characterize the performance of the particular undersampling strategy under ideal conditions. Without this understanding, it is possible that a particular IIR algorithm may be operating in a region of parameter space where image recovery is not possible or highly unstable towards data inconsistencies. Having gained this fundamental knowledge, it then makes sense to consider the impact of model error.

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REFERENCES