Reconstruction of convex 2D discrete sets in polynomial time

Attila Kuba*, Emese Balogh

Department of Applied Informatics, University of Szeged, Árpád tér 2, H-6720 Szeged, Hungary

Abstract

The reconstruction problem is considered in those classes of discrete sets where the reconstruction can be performed from two projections in polynomial time. The reconstruction algorithms and complexity results are summarized in the case of $hv$-convex sets, $hv$-convex 8-connected sets, $hv$-convex polyominoes, and directed $h$-convex sets. As new results some properties of the feet and spines of the $hv$-convex 8-connected sets are proven and it is shown that the spine of such a set can be determined from the projections in linear time. Two algorithms are given to reconstruct $hv$-convex 8-connected sets. Finally, it is shown that the directed $h$-convex sets are uniquely reconstructible with respect to their row and column sum vectors. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

The problems of the reconstruction of two-dimensional (2D) discrete sets from their row and column sum vectors are considered. This is one of the most frequently studied problems of discrete tomography [12, 13]. Several theoretical questions are connected with reconstruction such as existence, consistence, and uniqueness (as a summary see [4, 10]). There are also reconstruction algorithms for different classes of discrete sets (e.g., [6, 15, 17]). For example, Kuba published an algorithm to reconstruct so-called two-directionally connected discrete sets [14]. Del Lungo et al., studied the reconstruction of different kinds of polyominoes [2, 9]. Recently, Chrobak and Dürr found reconstruction algorithms for special polyominoes. Since the reconstruction in certain classes can be NP-hard (see [19]), the classes, where the reconstruction can be performed in polynomial time, are the most important for the applications like electron microscopy [8], image processing [18], and radiology [16].

* Corresponding author.

E-mail addresses: kuba@inf.u-szeged.hu (A. Kuba), bmse@inf.u-szeged.hu (E. Balogh).

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This paper considers the reconstruction algorithms in those classes of 2D discrete sets where the reconstruction can be performed in polynomial time. The most frequently used properties are some kind of discrete versions of the convexity. The convexity itself is not sufficient to find a solution in polynomial time (see Section 3). For this reason further properties, such as connectedness and directedness, are supposed about the discrete sets to be reconstructed.

First, the necessary definitions and notations are introduced in Section 2. Then, the reconstruction in the class of horizontally and vertically convex 2D discrete sets is discussed (Section 3). Although the reconstruction cannot be performed in polynomial time in this class, the algorithm introduced here can be used in other classes efficiently, for example, in the class of horizontally and vertically convex 8-connected sets that Section 4 deals with. Two polynomial time algorithms are given for reconstructing such sets and as a new result it is shown that the spine (a subset of the solutions) in this class can be determined in linear time. In Section 5 we show how these results can be applied in the more special class of horizontally and vertically convex polyominoes. Finally, in Section 6 a reconstruction algorithm is given for the class of directed horizontally convex sets.

2. Definitions and notations

Let \( \mathbb{Z}^2 \) denote the 2D integer lattice, its elements will be called points or positions. The finite subsets of \( \mathbb{Z}^2 \) will be called discrete sets. The class of discrete sets will be denoted by \( \mathcal{F} \).

Let \( F \) be a discrete set. Then there is a discrete rectangle \( R \) of size \( m \times n \) (\( m \) and \( n \) are positive integers),

\[
R = \{1, \ldots, m\} \times \{1, \ldots, n\},
\]

such that \( R \) is the smallest discrete rectangle containing \( F \). The discrete set \( F \) can also be represented as a binary matrix \((f_{ij})_{m \times n}, f_{ij} \in \{0, 1\}\), such that

\[
f_{ij} = \begin{cases} 
1 & \text{if } (i,j) \in F, \\
0 & \text{otherwise}.
\end{cases}
\]

For any subset of \( R \) we define the \( i \)th row of the subset, \( 1 \leq i \leq m \), as its intersection with \( i \times \{1, \ldots, n\} \). Similarly, the \( j \)th column, \( 1 \leq j \leq n \), of the subset as its intersection with \( \{1, \ldots, m\} \times j \).

Let \( \mathbb{N} \) denote the set of positive integers. For any discrete set \( F \) we define its projections by the operations \( H \) and \( V \) as follows. \( H : \mathcal{F} \rightarrow \mathbb{N}^m, H(F) = (h_1, \ldots, h_m) \) where

\[
h_i = \sum_{j=1}^{n} f_{ij}, \quad i = 1, \ldots, m
\]
Fig. 1. A discrete set $F$, its elements are marked by dark grey squares. The projections of $F$ are the vectors $H$ and $V$. The cumulated vectors of $H$ and $V$ are denoted by $\tilde{H}$ and $\tilde{V}$ (their first elements, $\tilde{h}_0$ and $\tilde{v}_0$ are not indicated). The discrete set $F$ is 8-connected but not 4-connected.

and $\nu : \mathcal{F} \to \mathbb{N}^n$, $\nu(F) = V = (v_1, \ldots, v_n)$ where

$$v_j = \sum_{i=1}^{m} f_{ij}, \quad j = 1, \ldots, n.$$  

(3)

The vectors $H$ and $V$ will be called the projections or row and column sum vectors of $F$, respectively (see Fig. 1).

An $(H, V)$ pair of vectors is said to be compatible if there exist positive integers $m, n$, and $T$ such that

(i) $H \in \mathbb{N}^m$ and $V \in \mathbb{N}^n$;

(ii) $h_i \leq n$, for $1 \leq i \leq m$, and $v_j \leq m$, for $1 \leq j \leq n$;

(iii) $\sum_{i=1}^{m} h_i = \sum_{j=1}^{n} v_j = T$, i.e., the two vectors have the same total sum $T$.

Henceforth, we shall suppose that $H$ and $V$ are compatible.

The cumulated vectors of $H$ and $V$ will be denoted by $\tilde{H} = (\tilde{h}_0, \tilde{h}_1, \ldots, \tilde{h}_m)$ and $\tilde{V} = (\tilde{v}_0, \tilde{v}_1, \ldots, \tilde{v}_n)$, that is,

$$\tilde{h}_0 = 0, \quad \tilde{h}_i = \tilde{h}_{i-1} + h_i, \quad i = 1, \ldots, m,$$

(4)

and

$$\tilde{v}_0 = 0, \quad \tilde{v}_j = \tilde{v}_{j-1} + v_j, \quad j = 1, \ldots, n$$

(5)

(see Fig. 1).

Let $F$ and $F'$ be discrete sets. We say that $F$ and $F'$ are tomographically equivalent (w.r.t. the row and column sum vectors) if

$$\mathcal{H}(F) = \mathcal{H}(F') \quad \text{and} \quad \nu(F) = \nu(F').$$

(6)

The discrete set $F$ is unique (w.r.t. the row and column sum vectors) if there is no discrete set $F'$ ($\neq F$) being tomographically equivalent to $F$.

Let $\mathcal{G}$ be a class of discrete sets. The discrete set $F \in \mathcal{G}$ is determined (by its projections) if there is no tomographically equivalent set in the class $\mathcal{G}$. 

We are going to study the following problem in different classes of discrete sets.

**Reconstruction (**)\(^{G}\),**

**Instance:** Two vectors \(H \in \mathbb{N}^m\) and \(V \in \mathbb{N}^n\).

**Task:** Construct a discrete set \(F \in \mathcal{G}\) such that \(\mathcal{H}(F) = H\) and \(\mathcal{V}(F) = V\).

The classes of discrete sets to be studied in this paper will be introduced by the following definitions.

Let \(P = (p_1, p_2)\) and \(Q = (q_1, q_2)\) be two points in \(\mathbb{Z}^2\). Let us consider the Euclidean distance between \(P\) and \(Q\).

\[
\|P - Q\| = \sqrt{(p_1 - q_1)^2 + (p_2 - q_2)^2}.
\]

The points \(P\) and \(Q\) are said to be **4-adjacent** if \(\|P - Q\| \leq 1\) and **8-adjacent** if \(\|P - Q\| \leq \sqrt{2}\). (Explicitly, the 4-adjacent points of \((i, j)\) are \((i - 1, j)\), \((i, j - 1)\), \((i, j + 1)\) and \((i + 1, j)\), i.e. the next points in directions north, west, east and south, respectively, and the point \((i, j)\) itself). The sequence of distinct points \((i_0, j_0), \ldots, (i_k, j_k)\) is a \(4\)-path/\(8\)-path from point \((i_0, j_0)\) to point \((i_k, j_k)\) in a discrete set \(F\) if each point of the sequence is in \(F\) and \((i_l, j_l)\) is 4-adjacent/8-adjacent, respectively, to \((i_{l-1}, j_{l-1})\) for each \(l = 1, \ldots, k\). Two points are 4-connected/8-connected in the discrete set \(F\) if there is a 4-path/8-path, respectively, in \(F\) between them. A discrete set \(F\) is 4-connected/8-connected if any two points in \(F\) are 4-connected/8-connected, respectively, in \(F\).

The 4-connected set is also called **polyomino** (see, for example, [11]). The classes of 4-connected and 8-connected sets will be denoted by \((c_4)\) and \((c_8)\), respectively. From the definitions it follows that \((c_4) \subset (c_8)\) (see Fig. 1).

The discrete set \(F\) is **horizontally convex** (or, shortly, **h-convex**) if its rows are 4-connected. The class of **h-convex** sets will be denoted by \((h)\). Similarly, a discrete set \(F\) is **vertically convex** (or, shortly, **v-convex**) if its columns are 4-connected. The class of **v-convex** sets will be denoted by \((v)\). The **h-** and **v-convex** sets will be called **hv-convex** and their class will be denoted by \((h, v)\). Clearly, \((h, v) = (h) \cap (v)\). Using this notation, the **hv-convex** polyominoes and the **hv-convex** 8-connected discrete sets are the classes of \((c_4, h, v)\) and \((c_8, h, v)\), respectively.

The sequence of distinct points \((i_0, j_0), \ldots, (i_k, j_k)\) is an **EN-path** from point \((i_0, j_0)\) to point \((i_k, j_k)\) in a discrete set \(F\) if each point of the sequence is in \(F\) and \((i_l, j_l)\) is in east or north to \((i_{l-1}, j_{l-1})\) for each \(l = 1, \ldots, k\). The discrete set \(F\) is **directed** if there is a particular point of \(F\), called **source**, such that there is an EN-path in \(F\) from the source to any other point of \(F\). It follows from the definition that the source point of a directed set is necessarily the point \((m, 1)\). The class of directed sets will be denoted by \((d)\). Clearly, \((d) \subset (c_4)\).

### 3. Reconstruction of hv-convex sets

The reconstruction of hv-convex sets was studied by Kuba [14]. He gave a greedy type heuristic algorithm, by which the set \(F\) to be reconstructed could be approximated by two sequences of discrete sets, \(\{K^{(i)}\}\) and \(\{S^{(i)}\}\). The first sequence is
non-decreasing and it consists of hv-convex sets supposed to be the subsets of F, called core sets. The other sequence is non-increasing and it consists of discrete sets supposed to contain F, called envelope sets. As initial core and envelope sets we can use

\[ K^{(0)} = \emptyset \quad \text{and} \quad S^{(0)} = R. \]  

(8)

The core set in each iteration will be constructed from the (previous) envelope set \( S^{(l)} \) and from the projections \( H \) and \( V \) as follows. First, we create the discrete set \( K_h(S^{(l)}, H) \), the intersection of all h-convex sets contained by \( S^{(l)} \) and having horizontal row sum \( H \). (It can be computed easily row by row. For example, if the \( i \)th row of \( S^{(l)} \) consists of \((i, s_i'), (i, s_i'+1), \ldots, (i, s_i''') (1 \leq s_i' \leq s_i'' \leq n) \) and \( h_i \leq s_i''' - s_i' + 1 \leq 2 \times h_i - 1 \), then the \( i \)th row of \( K_h(S^{(l)}, H) \) is the set of \( \{(i, s_i'' - h_i + 1), (i, s_i' - h_i + 2), \ldots, (i, s_i' + h_i - 1)\} \).

Similarly, using the envelope \( S^{(l)} \) and the column sums \( V \) the set \( K_v(S^{(l)}, V) \) can be defined. Clearly, if \( F \subseteq S^{(l)} \) then \( K_h(S^{(l)}, H) \subseteq F \) and \( K_v(S^{(l)}, V) \subseteq F \). Therefore, we can select the intersection of all hv-convex sets containing \( K_h(S^{(l)}, H) \cup K_v(S^{(l)}, V) \) as the new core \( K^{(l+1)} \). (The intersection of all hv-convex sets containing a discrete set \( K \) is just \( K \) if \( K \) is hv-convex. Otherwise, it can be constructed by including new elements between the elements of \( K \) in the rows and columns alternatively until the new set of \( K \) is hv-convex.) The construction these core sets is illustrated in Fig. 2.

Analogously, the envelope set in each iteration will be constructed from the (previous) core set \( K^{(l)} \) and from the projections \( H \) and \( V \) as follows. First, we create the discrete set \( S_h(K^{(l)}, H) \), the union of all h-convex sets containing \( K^{(l)} \) and having horizontal row sum \( H \). (It can be computed easily row by row. For example, if the \( i \)th row of \( K^{(l)} \) consists of \((i, k_i'), (i, k_i'+1), \ldots, (i, k_i''') (1 \leq k_i' \leq k_i'' \leq n) \) and \( k_i''' - k_i' + 1 \leq h_i \), then the \( i \)th row of \( S_h(K^{(l)}, H) \) is the set of \( \{(i, k_i'' - h_i + 1), (i, k_i' - h_i + 2), \ldots, (i, k_i' + h_i - 1)\} \).

Similarly, using the core \( K^{(l)} \) and the columns sums \( V \) the set \( S_v(K^{(l)}, V) \) can be defined. Clearly, if \( K^{(l)} \subseteq F \) then \( F \subseteq S_h(K^{(l)}, H) \) and \( F \subseteq S_v(K^{(l)}, V) \). Therefore, we can select the set \( S_h(K^{(l)}, H) \cap S_v(K^{(l)}, V) \) as the new envelope, \( S^{(l)} \). The construction of these envelope sets is illustrated in Fig. 2.

During the iterations we have

\[ K^{(l)} \subseteq F \subseteq S^{(l)}. \]  

(9)

It is clear that \( K^{(l)} \subseteq K^{(l+1)} \) and \( S^{(l)} \supseteq S^{(l+1)} \). If we have \( K^{(l)} = S^{(l)} \) for some \( l \) then \( F = K^{(l)} = S^{(l)} \) is a solution. It may also happen that during the iterations we reach a situation when core or envelope do not change \( (K^{(l)} = K^{(l+1)} \) and \( S^{(l)} = S^{(l+1)} \) or \( K^{(l)} \not\subseteq S^{(l)} \). In the first case, we can take an arbitrary element of the set \( S^{(l)} \setminus K^{(l)} \), add it to the core and continue the algorithm with this new core set. In the second case we have a contradiction indicating that there is no solution between the current core and envelope. Then we can return to the arbitrary selected element (if we made such a formation trial earlier), take it away from the envelope set and continue the algorithm with this reduced envelope.

Unfortunately, as it turned out in [19], the reconstruction problem in the class of hv-convex sets is NP-hard, so this algorithm is not able to reconstruct hv-convex sets...
in the worst case in polynomial time. However, the algorithm can be used to reconstruct $hv$-convex 8-connected sets in polynomial time as it will be seen in the next section.

4. Reconstruction of $hv$-convex 8-connected sets

This class of discrete sets has been studied by Kuba [15] and recently by Brunetti et al. [5]. They showed that the algorithm suggested for reconstructing $hv$-convex polyominoes (Section 3) can be used even in this class. It is basically different in two points from the previous algorithm: selection of the initial core and handling the situation when the core and envelope cannot be changed in the way described in Section 3.

For the selection of the initial core the concepts of feet and spine of polyominoes are introduced by Barcucci et al. [2, 3].
4.1. The feet of $hv$-convex 8-connected sets

The north foot of $F$ denoted by $P_N$ is the set of columns of $F$ that have elements in the first row of the rectangle $R$. The column indices of $P_N$ determine a set of consecutive integers: $\{n_f, n_f + 1, \ldots, n_g\}$. Similar definition can be given for the south foot, $P_S$, taking the columns of $F$, $\{s_f, s_f + 1, \ldots, s_g\}$, which have an element in the last row of $R$. The east and west feet, $P_E$ and $P_W$, respectively, and their column indices, $\{e_f, e_f + 1, \ldots, e_g\}$ and $\{w_f, w_f + 1, \ldots, w_g\}$, can be defined analogously (see Fig. 3).

Let $F$ be an $hv$-convex 8-connected set with projections $(H,V)$. We denote by $l$ the index of the last element of the first nondecreasing maximal subsequence of $V$, formally,

$$l = \max\{1 \leq j \leq n \mid v_p \leq v_q \text{ for all } 1 \leq p < q \leq j\}. \quad (10)$$

Analogously, we denote by $r$ the index of the first element of the last non-increasing maximal subsequence of $V$, formally,

$$r = \min\{1 \leq j \leq n \mid v_p \geq v_q \text{ for all } 1 \leq p < q \leq j\}. \quad (11)$$

Let, furthermore,

$$l_1 = \min\{1 \leq j \leq l \mid v_j = v_l\} \quad \text{and} \quad r_1 = \max\{r \leq j \leq n \mid v_j = v_r\}, \quad (12)$$

and then

$$l_N = \min\{l_1 + h_1 - 1, l\}, \quad l_S = \min\{l_1 + h_m - 1, l\},$$

$$r_N = \max\{r_1 - h_1 + 1, r\}, \quad r_S = \max\{r_1 - h_m + 1, r\}.$$

Remark 1. In paper [3] the definition of $l_S$ was different from this one.

Then the following two propositions are true.
Fig. 4. The possible four cases of foot positions as they are discussed in Proposition 2. (a) Case 1: 
l = 7, r = 5. In this example, \( n_f = 3, n_g = 7, s_f = 5, \) and \( s_g = 9, \) that is, the figure shows the situation 
described by point (i). (b) Case 2: \( l = 7, r = 5. \) Then \( n_f = 5, n_g = 7, \) and the south foot is somewhere 
between 3 and 9. (c) Case 3: \( l = 7, r = 5. \) Then the north foot is somewhere between 2 and 10, and 
\( s_f = 5, s_g = 7. \) (d) Case 4: \( l = 7, r = 5. \) Then \( n_f = s_f = 5, n_g = s_g = 7. \)

**Proposition 1** (Brunetti et al. [5]). If there is an hv-convex 8-connected set \( F \) that 
satisfies \( F \in (H, V) \) with \( v_j < m \) for all \( j = 1, \ldots, n, \) then

1. if \( P_N \) is to the left of \( P_S \) then \( n_g \leq l_N \) and \( s_f \geq r_S, \)
2. if \( P_N \) is to the right of \( P_S \) then \( s_g \leq l_S \) and \( n_f \geq r_N. \)

**Proof.** See [5].

**Proposition 2** (Brunetti et al. [5]). If there is an hv-convex 8-connected set \( F \) that 
satisfies \( (H, V) \) with \( v_j = m \) for some \( j = 1, \ldots, n \) then the following four cases are 
possible (see Fig. 4):

1. If \( h_1 > l - r + 1 \) and \( h_m > l - r + 1 \) then
   (i) \( n_f = l - h_1 + 1, n_g = l, s_f = r, \) and \( s_g = r + h_m - 1, \) or
   (ii) \( n_f = r, n_g = r + h_1 - 1, s_f = l - h_m + 1, \) and \( s_g = l. \)

2. If \( h_1 = l - r + 1 \) and \( h_m > l - r + 1 \) then
   \( n_f = r, n_g = l, s_f \geq \max\{1, l - h_m + 1\}, \) and \( s_g \leq \min\{r + h_m - 1, n\}. \)

3. If \( h_1 > l - r + 1 \) and \( h_m = l - r + 1 \) then
   \( n_f \geq \max\{1, l - h_1 + 1\}, n_g \leq \min\{r + h_1 - 1, n\}, s_f = r, \) and \( s_g = l. \)

4. If \( h_1 = h_m = l - r + 1 \) then
   \( n_f = s_f = r, n_g = s_g = l. \)
Proof. See [5]. □

Remark 2. In paper [3] only the first three of the cases listed here are given (see Proposition VB). Furthermore, the conclusions of cases (b) and (c) of Proposition VB are slightly different from the conclusions of cases 2 and 3 in Proposition 2.

Considering the possible relative positions of the four feet, the following four foot configuration can arise:
1. \( n_f \leq s_g \) and \( e_f \leq w_g \),
2. \( n_f \leq s_g \) and \( w_f \leq e_g \),
3. \( s_f \leq n_g \) and \( e_f \leq w_g \),
4. \( s_f \leq n_g \) and \( w_f \leq e_g \).

Propositions 1 and 2 can be used to determine the limitations of the possible positions of the feet. If we suppose one of the foot configurations, then on the base of Propositions 1 and 2 we can determine two intervals containing the possible column indices of the north and the south feet. (Of course, analogous propositions and methods can be stated for the row indices of the east and west feet.) This operation will be called determination of the foot limitations [3].

4.2. The spine of hv-convex 8-connected sets

Certain elements in the middle of an hv-convex 8-connected set can be recognized easily from the row and column sums by the following.

Lemma 1 (Brunetti et al. [5]). Let \( F \) be an hv-convex 8-connected set with row and column sums \( (H, V) \). If for some \((i, j) \in R\)
\[
\tilde{v}_j > \tilde{h}_{i-1}, \quad \tilde{h}_i > \tilde{v}_{j-1}, \quad \tilde{h}_i > T - \tilde{v}_j, \quad T - \tilde{v}_{j-1} > \tilde{h}_{i-1},
\]
then \((i, j) \in F\).

Proof. See [5]. □

As an illustration of Lemma 1 see Fig. 5.

The spine of an hv-convex 8-connected set with row and column sums \( (H, V) \) is the set of positions \((i, j) \) in \( R \) where one of the following conditions are satisfied:
• \((n_f \leq j \leq s_g \) or \( w_f \leq i \leq e_g \)\) and \( \tilde{v}_j \geq \tilde{h}_{i-1} \), \( \tilde{h}_i \geq \tilde{v}_{j-1} \),
• \((s_f \leq j \leq n_g \) or \( e_f \leq i \leq w_g \)\) and \( \tilde{h}_i \geq T - \tilde{v}_j \), \( T - \tilde{v}_{j-1} \geq \tilde{h}_{i-1} \).

The spine of \( F \) will be denoted by \( F_p \).

Let \( F \) be an hv-convex 8-connected set whose projections are \((H, V)\). We can define its extended spines (Fig. 6):
\[
F_p' = \{(i, j) \mid \tilde{v}_j > \tilde{h}_{i-1} \text{ and } \tilde{h}_i > \tilde{v}_{j-1}\},
\]
\[
F_p'' = \{(i, j) \mid \tilde{h}_i > T - \tilde{v}_j \text{ and } \tilde{v}_j > T - \tilde{h}_i\}.
\]
Fig. 5. According to Lemma 1 $(4,4) \in F$ and $(4,5) \in F$.

Fig. 6. An hv-convex 8-connected discrete set $F$, its spine $F_p$ and its extended spines $F'_p$ and $F''_p$. The elements of the spine are marked by black circles. The elements of the extended spines $F'_p$ and $F''_p$ are marked by \ and /, respectively.

For a specific feet configuration we can determine the spine $F_p$ using the extended spines $F'_p$ and $F''_p$.

Proposition 3. Let $F$ be an hv-convex 8-connected set with projections $(H,V)$.
- If $n_f \leq j \leq s_y$ then $F_p' \cap \{(i,j) | n_f \leq j \leq s_y\} \subseteq F$.
- If $w_f \leq i \leq e_y$ then $F_p' \cap \{(i,j) | w_f \leq i \leq e_y\} \subseteq F$.
- If $s_f \leq j \leq n_y$ then $F_p'' \cap \{(i,j) | s_f \leq j \leq n_y\} \subseteq F$.
- If $e_f \leq i \leq w_y$ then $F_p'' \cap \{(i,j) | e_f \leq i \leq w_y\} \subseteq F$.

Proof. It is similar to the proof of Proposition IVA in [3].

In order to define an efficient algorithm for the construction of the spine we need to study some properties of the extended spines. We prove some properties for $F'_p$ and similar properties can be proven for $F''_p$.

Lemma 2. Let $F$ be an hv-convex 8-connected set whose projections are $(H,V)$, then the positions $(1,1)$ and $(m,n)$ belong to $F'_p$.

Proof. It follows directly from the definitions.

Lemma 3. Let $F$ be an hv-convex 8-connected set whose projections are $(H,V)$. If $(i+1, j) \in F'_p$ then $(i, j+1) \notin F'_p$ for any position in $R$. 
Proof. From \((i + 1, j) \in F'_p\) it follows that \(\tilde{v}_j \geq \tilde{h}_i\). From \((i, j + 1) \not\in F'_p\) we would get the contradiction that \(\tilde{h}_i \geq \tilde{v}_j\).

Lemma 4. Let \(F\) be an hv-convex 8-connected set whose projections are \((H, V)\) and let \((i, j) \in F'_p\) where \(i \in \{1, \ldots, m - 1\}\) and \(j \in \{1, \ldots, n - 1\}\). If \((i + 1, j) \not\in F'_p\) and \((i, j + 1) \not\in F'_p\) then \((i + 1, j + 1) \in F'_p\).

Proof. From the conditions \((i, j) \in F'_p\) and \((i + 1, j) \not\in F'_p\) it follows that
\[
\tilde{h}_i \leq \tilde{v}_j.
\]
Similarly, from the conditions \((i, j) \in F'_p\) and \((i, j + 1) \not\in F'_p\) it follows that
\[
\tilde{v}_j \leq \tilde{h}_i.
\]
From (14) and (15) it follows that
\[
\tilde{h}_{i+1} > \tilde{v}_j \quad \text{and} \quad \tilde{v}_{j+1} > \tilde{h}_i,
\]
which is equivalent to \((i + 1, j + 1) \in F'_p\). □

Using Lemmas 2, 3 and 4, we can construct the extended spine \(F'_p\) in the following way:
1. Let us start from position \((i, j) = (1, 1)\). From Lemma 2 we know that \((1, 1) \in F'_p\).
2. If \(\tilde{h}_i > \tilde{v}_j\) then we increase \(j\);
else
   if \(\tilde{h}_i = \tilde{v}_j\) then we increase \(j\);
   increase \(i\).
3. Then from Lemmas 3 and 4 we know that \((i, j) \in F'_p\).
4. We repeat the steps from Point 2 until we reach the cell \((m, n)\) which also belongs to \(F'_p\).

Then the extended spine \(F'_p\) can be reconstructed by the following algorithm.

Algorithm 1. Determining the extended spine \(F'_p\)

Input: Two compatible vectors, \(H \in \mathbb{N}^m\) and \(V \in \mathbb{N}^n\), and a matrix \(F'_p(1, 1)\) with initial value zero-matrix.

Output: \(F'_p\)—one of the extended spines.
\[
\begin{align*}
\tilde{h}_0 & := 0, \tilde{v}_0 := 0, \quad \tilde{h}_i := \tilde{h}_{i-1} + h_i \quad \text{for} \quad i = 1, \ldots, m; \\
\tilde{v}_j & := 0, \tilde{v}_j := \tilde{v}_{j-1} + v_j \quad \text{for} \quad j = 1, \ldots, n; \\
i & := 1, \quad F'_p(1, 1) := 1;
\end{align*}
\]
repeat
\[
\begin{align*}
\{ & \quad \text{if} \quad (\tilde{h}_j > \tilde{v}_j) \quad \text{then} \quad j := j + 1; \\
& \quad \text{else}
\}
\end{align*}
\]
\[
\begin{align*}
\{ & \quad \text{if } (\tilde{h}_i = \tilde{v}_j) \text{ then } j++; \\
& \quad i++;
\} \\
F'_p(i,j) := 1;
\}
\] until \(i = m \text{ and } j = n\).

The complexity of this algorithm is \(O(m + n)\). In order to construct the spine, first we have to determine the values of \(l, r, l_1, l_N, l_S, r_N, \) and \(r_S\). These constants can be computed in \(O(n)\) time. Then we have the foot limitations for each foot configuration. Since the spine can be obtained as a part of the corresponding extended spine, the complexity of the determination of the spine is \(O(m + n)\).

4.3. Reconstruction of hv-convex 8-connected sets iteratively

The algorithm starts with choosing the positions of the four feet. There are at most \(O(\min\{m^2, n^2\})\) possible cases. Some of them can be sorted out soon at the beginning of the reconstruction.) Then the hv-convex 8-connected set containing the selected feet (as an initial core) is constructed. Using the iterative procedure described in Section 3 the core set is increased and the envelope set is decreased (this procedure is called filling operations).

If the core and envelope sets cannot be changed, then the reconstruction is reformulated as a 2-satisfiability problem (also referred to as 2SAT), which can be solved in polynomial time \([1]\) and called in the reconstruction algorithm as the \textit{evaluation procedure} \([2]\). Such situations can be described by a 2SAT expression, i.e., a Boolean expression in conjunctive normal form with at most two literals in each clause. The 2SAT expression is satisfiable if and only if there is an hv-convex polyomino solution of the reconstruction problem.

As a summary we can describe the reconstruction algorithm as follows:

\textbf{Algorithm 2.} Reconstructing hv-convex 8-connected sets

\textbf{Input:} Two compatible vectors \(H \in \mathbb{N}^m\) and \(V \in \mathbb{N}^n\).

\textbf{Output:} 4-connected convex sets having projections \(H\) and \(V\) (if there is such a solution).

Compute the cumulated sums of \(H_i\) and \(V_j\) for \(i = 1, \ldots, m\) and \(j = 1, \ldots, n\).

Repeat for all four foot configurations.

\{
\begin{align*}
& \text{Compute the foot limitations.} \\
& \alpha := 0; \quad \beta := R.
\end{align*}
\}

Perform the building procedure.

If \(\alpha \subset \beta \land (\alpha \text{ is not 8-connected } \lor \text{ there is no cell of } \alpha \text{ in at least one row and one column of } R)\) then
\{ 
  KE := \alpha; \quad SH := \beta; 
\}

Repeat for all foot positions \((P_N \text{ and } P_S \text{ or } P_W \text{ and } P_E)\).
\{ 
  \alpha := KE \cup P_N \cup P_S \quad \text{(or } \alpha := KE \cup P_W \cup P_E); \quad \beta := SH; 
  \quad \text{If } \alpha \subseteq \beta \text{ then perform the building procedure.} 
\}

Remark 3. In Algorithm 2 \(\alpha\) and \(\beta\) denote, respectively, the core and envelope sets during the iterations. KE and SH are just to store the current core and envelope sets before selecting new foot position.

Algorithm 3. Building procedure

\textbf{Input}: Two vectors \(H \in \mathbb{N}^m\) and \(V \in \mathbb{N}^n\) and the kernel \(\alpha\), shell \(\beta\) and the interval \([j_f, j_l]\) (or \([i_f, i_l]\)) for foot position.

\textbf{Output}: Kernel \(\alpha\), shell \(\beta\) created by partial sum and filling operation.

\textit{Partial sum} operation.
Repeat \{ 
  Perform the filling operations.
\}
until \(\alpha \not\subset \beta\) or (\(\alpha\) and \(\beta\) are invariant).

If \(\alpha = \beta\) and they are 8-connected convex sets then \(S := \alpha\) is a solution.
If \(\alpha \subset \beta\) and \(\alpha\) is a 8-connected convex set having at least one cell in each row (or column) of \(R\) then recall the evaluation procedure.

The complexity of this reconstruction algorithm is \(O(mn\log(mn)\min\{m^2, n^2\})\) in the worst case if the algorithm terminates at the reconstruction of the first solution. As an example of the reconstruction process by Algorithm 2 see Fig. 7.

4.4. Reconstruction of \(hv\)-convex 8-connected sets by solving the corresponding 2SAT problem

Recently, Chrobak and Dürr have found a new algorithm [7] reconstructing \(hv\)-convex polyominoes in \(O(mn\min\{m^2, n^2\})\) time. The basic idea of their algorithm is to rewrite the whole reconstruction problem as a 2SAT problem.

In this section we show that the modification of the algorithm suggested by Chrobak and Dürr [7] for reconstructing \(hv\)-convex polyominoes can be used in the more general class of \((c_h, h, v)\). They showed that the reconstruction of a \(hv\)-convex polyomino was equivalent to the evaluation of a suitable constructed 2SAT expression. Now, we give
the description of the modified algorithm following the same idea as Chrobak and Dürr. Since the statements can be proven in the same way, we omit the proofs.

We say that the discrete set $A$ is an **upper-left corner region** in the discrete rectangle $R$ containing $F$ if $(i+1, j) \in A$ or $(i, j+1) \in A$ implies $(i, j) \in A$. The **upper-right, lower-left and lower-right regions**, $B$, $C$, and $D$, respectively, can be defined analogously (see Fig. 8). Let $\overline{F}$ denote the complement of $F$ (in $R$).

**Lemma 5** (Chrobak and Dürr [7]). $F \in (c_8, h, v)$ if and only if

$$
\overline{F} = A \cup B \cup C \cup D,
$$

where $A$, $B$, $C$ and $D$ are disjoint corner regions (upper-left, upper-right, lower-left and lower-right, respectively).

**Proof.** See [7]. □
Let \( F \in (c_8, h, v) \) be contained in the discrete rectangle \( R = \{1, \ldots, m\} \times \{1, \ldots, n\} \) and \( k, l \) be integers such that \( 1 \leq k, l \leq m \). We say that \( F \) is anchored at \((k, l)\) if \((k, 1), (l, n) \in F\). The basic idea of the reconstruction method is to construct a 2SAT expression \( F_{k,l}(H, V) \) with the property that \( F_{k,l}(H, V) \) is satisfiable if and only if there is an \( F \in (c_8, h, v) \) that is anchored at \((k, l)\). The suitable \( F_{k,l}(H, V) \) in the class (\( c_8, h, c \)) can be constructed as

\[
F_{k,l}(H, V) = Cor \land Dis \land Anc \land LBC \land UBR,
\]

where \( Cor, Dis, Anc, LBC, \) and \( UBR \) are sets of clauses describing the properties of “Corners”, “Disjointness”, “Anchors”, “Lower Bound on Column sums”, and “Upper Bound on Row sums”, respectively, in the following way.

\[
Cor = \bigwedge_{i,j} (a_{ij} \Rightarrow a_{i-1,j} \land a_{ij} \Rightarrow a_{i,j-1}) \land \\
\bigwedge_{i,j} (b_{ij} \Rightarrow b_{i-1,j} \land b_{ij} \Rightarrow b_{i,j+1}) \land \\
\bigwedge_{i,j} (c_{ij} \Rightarrow c_{i+1,j} \land c_{ij} \Rightarrow c_{i,j-1}) \land \\
\bigwedge_{i,j} (d_{ij} \Rightarrow d_{i+1,j} \land d_{ij} \Rightarrow d_{i,j+1}),
\]

\[
Dis = \bigwedge_{i,j} \{x_{ij} \Rightarrow \overline{y_{ij}} \mid \text{for symbols } X, Y \in \{A, B, C, D\}, X \neq Y\},
\]

\[
Anc = \overline{a_{k,1}} \land \overline{b_{k,1}} \land \overline{c_{k,1}} \land \overline{d_{k,1}} \land \overline{a_{l,n}} \land \overline{b_{l,n}} \land \overline{c_{l,n}} \land \overline{d_{l,n}},
\]

\[
LBC = \bigwedge_{i,j} (a_{ij} \Rightarrow \overline{c_{i+1,j}} \land a_{ij} \Rightarrow \overline{d_{i+1,j}}) \land \\
\bigwedge_{i,j} (b_{ij} \Rightarrow \overline{c_{i+1,j}} \land b_{ij} \Rightarrow \overline{d_{i+1,j}}) \land \\
\bigwedge_{j} (\overline{c_{i,j}} \land \overline{d_{i,j}}),
\]

\[
UBR = \bigwedge_{j} \left( \bigwedge_{i \leq \min\{k, l\}} (\overline{a_{ij}} \Rightarrow b_{i,j+h}) \land \bigwedge_{l \leq i \leq k} (\overline{a_{ij}} \Rightarrow d_{i,j+h}) \right) \land \\
\bigwedge_{j} \left( \bigwedge_{k \leq i \leq l} (\overline{c_{ij}} \Rightarrow b_{i,j+h}) \land \bigwedge_{\max\{k, l\} \leq i} (\overline{c_{ij}} \Rightarrow d_{i,j+h}) \right).
\]
Algorithm 4 (Chrobak and D"urr [7]). Reconstructing hv-convex 8-connected sets

**Input:** Two compatible vectors $H \in \mathbb{N}^m$ and $V \in \mathbb{N}^m$.

for $k, l = 1, \ldots, m$

\{

  if $F_{k,l}(H, V)$ is satisfiable then Output $F = \overline{A} \cup \overline{B} \cup \overline{C} \cup \overline{D}$ and halt.

\}

**Output:** Print “No solution”.

Theorem 1 (Chrobak and D"urr [7]). $F_{k,l}(H, V)$ is satisfiable if and only if there is an $F \in (c_8, h, v)$ having projections $H$ and $V$ and it is anchored at $(k, l)$.

**Proof.** See [7]. □

Theorem 2 (Chrobak and D"urr [7]). Algorithm 4 solves the reconstruction problem for hv-convex 8-connected sets in time $O(mn \min\{m^2, n^2\})$.

**Proof.** See [7]. □

5. Reconstruction of hv-convex polyominoes

This class of discrete sets was studied first by Del Lungo et al. [2, 9]. The method was improved [3, 5] and the latest version is able to reconstruct hv-convex polyominoes in time $O(mn \log(mn) \min\{m^2, n^2\})$.

Now we can study the differences of the construction of the spine between the 8-connected and 4-connected cases (see Fig. 9).

Lemma 6 (Barcucci et al. [3]). Let $F$ be an hv-convex polyomino with row and column sums $H$ and $V$. If for some $(i, j) \in R$

$$\tilde{v}_j \geq \tilde{h}_{i-1}, \quad \tilde{h}_i \geq \tilde{v}_{j-1}, \quad \tilde{h}_i \geq T - \tilde{v}_j, \quad T - \tilde{v}_{j-1} \geq \tilde{h}_{i-1},$$

then $(i, j) \in F$.

**Proof.** See [3]. □

Besides the properties showed for the 8-connected hv-convex sets, we have some more properties.

Lemma 7. Let $F$ be an hv-convex polyomino whose projections are $(H, V)$. If $\tilde{h}_i = \tilde{v}_j$ then $\{(i, j) \cup (i + 1, j) \cup (i, j + 1) \cup (i + 1, j + 1)\} \in F'$. 
Fig. 9. An $hv$-convex polyomino $F$, its spine $F_p$ and its extended spines $F_p'$ and $F_p''$.

**Proof.** If $\tilde{h}_j = \tilde{v}_j$ then
\[ \tilde{v}_j > \tilde{h}_i - h_i = \tilde{h}_{i-1} \quad \text{and} \quad \tilde{h}_i > \tilde{v}_j - v_j = \tilde{v}_{j-1}, \] (21)
that is, $(i,j) \in F_p'$. Furthermore, from
\[ \tilde{h}_i < \tilde{v}_j + v_{j+1} = \tilde{v}_{j+1} \quad \text{and} \quad \tilde{h}_{i+1} = \tilde{h}_i + h_i > \tilde{v}_j \] (22)
we get that $(i+1,j+1) \in F_p'$. Then from
\[ \tilde{v}_{j+1} > \tilde{v}_j = \tilde{h}_i > \tilde{h}_{i-1} \] (23)
we get $(i,j+1) \in F_p'$ and finally from
\[ \tilde{h}_{i+1} > \tilde{h}_i = \tilde{v}_j > \tilde{v}_{j+1} \] (24)
we have that $(i+1,j) \in F_p'$. □

**Lemma 8.** Let $F$ be an $hv$-convex polyomino with projections $(H,V)$. If $(i+1,j) \in F_p'$ then $(i,j+2) \notin F_p'$ for each $i = 1, \ldots, m - 1$ and $j = 1, \ldots, n - 1$.

**Proof.** Let us suppose that $(i,j+2) \in F_p'$, then $\tilde{h}_i \geq \tilde{v}_{j+1}$. If $(i+1,j) \in F_p'$ then $\tilde{v}_j \geq \tilde{h}_i$. From these two conditions we get that $\tilde{v}_j \geq \tilde{h}_i \geq \tilde{v}_{j+1}$, which is possible if $\tilde{v}_j = \tilde{h}_i = \tilde{v}_{j+1}$. However, $\tilde{v}_j = \tilde{v}_{j+1}$ is equivalent to $v_j = 0$ that is a contradiction, therefore, $(i,j+2) \notin F_p'$. □

Then we have the following.

**Algorithm 5.** Determining the extended spine $F_p'$ of an $hv$-convex polyomino

**Input:** Two compatible vectors, $H \in \mathbb{N}^m$ and $V \in \mathbb{N}^n$, and a matrix $F_p'$ with initial value zero-matrix.

**Output:** The extended spine $F_p'$. 
\[
\begin{align*}
\tilde{h}_0 &:= 0, \quad \tilde{h}_i := \tilde{h}_{i-1} + h_i \quad \text{for} \quad i = 1, \ldots, m; \\
\tilde{v}_0 &:= 0, \quad \tilde{v}_j := \tilde{v}_{j-1} + v_j \quad \text{for} \quad j = 1, \ldots, n; \\
i &:= j := 1, \quad F_p'(1,1) := 1;
\end{align*}
\]
repeat { 
    if ($\hat{h}_i = \tilde{v}_j$) 
        then { 
            $F'_p(i,j + 1) := F'_p(i + 1,j) := 1$; 
            $i++$; $j++$; 
        } 
        else { 
            if ($\tilde{v}_j < \hat{h}_i$) then $j++$; 
            else $i++$; 
        } 
    $F'_p(i,j) := 1$; 
} until ($i = m$ and $j = n$).

The complexity of this algorithm is $O(m + n)$.

The same algorithm as Algorithm 2 can be used to reconstruct $hv$-convex 4-connected sets.

A version of Algorithm 4 can be used for reconstruction in this class with the following differences.

Lemma 5 is not valid in the class $(c_4, h, v)$. In the class $(c_4, h, v)$ two further conditions should be satisfied (see Fig. 1).

**Lemma 9** (Chrobak and Dürr [7]). $F \in (c_4, h, v)$ if and only if

\[ \bar{F} = A \cup B \cup C \cup D, \]  

where $A$, $B$, $C$, and $D$ are disjoint corner regions (upper-left, upper-right, lower-left and lower-right, respectively) such that

(i) $(i, j) \in A$ implies $(i + 1, j + 1) \notin D$,

(ii) $(i, j) \in B$ implies $(i + 1, j - 1) \notin C$.

**Proof.** See [7]. □

**Remark 4.** In the case of $hv$-convex polyominoes one more set of classes was included describing the “Connectivity” (more exactly, the 4-connectivity) of $F$ as

\[ Con = \bigcap_{i,j} (a_{ij} \Rightarrow d_{i+1,j+1} \land b_{ij} \Rightarrow c_{i+1,j-1}). \]  

6. Reconstruction of directed $h$-convex sets

The class of directed $hv$-convex polyominoes was studied by Del Lungo et al. in [9].

In this section we consider a more general class, the class of directed $h$-convex sets, $(d, h)$ (see Fig. 10). The following reconstruction algorithm is a straight consequence of the definition of the class.
Algorithm 6. Reconstructing directed $h$-convex sets

**Input:** Two compatible vectors, $H \in \mathbb{N}^m$ and $V \in \mathbb{N}^n$.

**Output:** The binary matrix $B$.

Let $B = (0)_{m \times n}$ be the initial binary matrix representing the discrete set to be reconstructed.

for $j = 1, \ldots, n$

\{

if $v_j < \sum_{i=1}^m b_{ij}$ then halt (no solution);

otherwise let $i'$ be the maximal row index where $\sum_{i=1}^{j-1} b_{i'j} < h_i'$ (if there is no such row then halt (no solution)) and start $v_j - \sum_{i=1}^m b_{ij}$ number of rows (in rows $i', i'+1, \ldots$) of 1’s from the column $j$ consecutively.
\}

**Theorem 3.** If $F$ is a directed horizontally convex set then it is reconstructed by Algorithm 4.

**Proof.** It follows from the definition of directed $h$-convex sets.

It is also clear that there is at most one directed $h$-convex set with given projections. Finally, we can remark that similar algorithm and theorem can be given for directed $v$-convex sets.

7. Conclusion

The reconstruction problem is considered in different classes of discrete sets. We think that even from the viewpoint of eventual applications it is important to find more and more general class of discrete sets reconstructible from two projections in polynomial time. It is shown that, among the classes have been studied so far, the most general class where the reconstruction problem can be solved in polynomial time is the class of $hv$-convex 8-connected sets. Two methods are shown for reconstructing such sets (Algorithms 2 and 4). Although the worst case complexity of Algorithm 2 is higher than Algorithm 4 we think that it is interesting to compare the complexities of these algorithms from the viewpoint of average reconstruction time.
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