Application of New Transform "Elzaki Transform" to Partial Differential Equations

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Abstract

The ELzaki transform of partial derivatives is derived, and its applicability demonstrated using four different partial differential equations. In this paper we find the particular solutions of these equations.

Keywords: Elzaki Transform- Partial Differential Equations.

Introduction

The differential equations have played a central role in every aspect of applied mathematics for every long time and with the advent of the computer, their importance has increased father.

Thus investigation and analysis of differential equations cruising in applications led to many deep mathematical problems; therefore, there are so many different techniques in order to solve differential equations.

In order to solve the differential equations, the integral transforms were extensively used and thus there are several words on the theory and applications of integral transforms such as the Laplace, Fourier, Mellin, Hankel and Sumudu, to name but a few. Recently, Tarig Elzaki introduced a new integral transform, named the ELzaki transform, and further applied it to the solution of ordinary and partial differential equations.

In this paper we derive the formulate for the ELzaki transform of partial derivatives and apply them in Solving five types of initial value problems. Our purpose here is to show the applicability of this interesting new transform and its effecting in solving such problems.

Definition and Derivations the ELzaki Transform of Derivatives

The ELzaki transform of the function $f(t)$ is defined as

$$E\left[f\left(t\right)\right]=T\left(v\right)=v\int_{0}^{\infty}f\left(t\right)e^{-t/v}dt \quad , \ t>0 \ , \ v\in\left(-k_{1}\,,\,k_{2}\right) \tag{1}$$

To obtain the ELzaki transform of partial derivatives we use integration by parts as follows:

$$E\left[\frac{\partial f}{\partial t}\left(x,t\right)\right]=\int_{0}^{\infty}v\frac{\partial f}{\partial t}e^{\frac{-t}{v}}dt=\lim_{p\to\infty}\int_{0}^{p}ve^{\frac{-t}{v}}\frac{\partial f}{\partial t}dt=$$

$$\lim_{p\to\infty}\left\{\left[ve^{\frac{-t}{v}}f\left(x,t\right)\right]_{0}^{p}-\int_{0}^{p}e^{\frac{-t}{v}}f\left(x,t\right)dt\right\}= \tag{2}$$

$$\frac{T\left(x,v\right)}{v}-vf\left(x,0\right)$$

We assume that f is piecewise continuous and is of exponential order.
Now

$$E\left[\frac{\partial f}{\partial x}\right]=\int_{0}^{\infty}ve^{\frac{-t}{v}}\frac{\partial f\left(x,t\right)}{\partial x}dt \ =$$

$$\frac{\partial}{\partial x}\int_{0}^{\infty}ve^{\frac{-t}{v}}f\left(x\,,\,t\right)dt\left(u\sin g\ the\ Leibnitz\ rule\right)$$

$$=\frac{\partial}{\partial x}\left[T\left(x,v\right)\right] \quad and$$

$$\tag{3}$$

$$E\left[\frac{\partial f}{\partial x}\right]=\frac{d}{dx}\left[T\left(x,v\right)\right]$$

Also we can find:

$$E\left[\frac{\partial^{2}f}{\partial x^{2}}\right]=\frac{d^{2}}{dx^{2}}\left[T\left(x,v\right)\right] \tag{4}$$

To find

$$E\left[\frac{\partial^{2}f}{\partial t^{2}}\left(x,t\right)\right]$$

Let

$$\frac{\partial f}{\partial t}=g \ , \ then$$

By using equation (2) we have

$$\mathrm{E}\left[\frac{\partial^2 f}{\partial t^2}(x, t)\right] = \mathrm{E}\left[\frac{\partial g(x, t)}{\partial t}\right] = \mathrm{E}\frac{\left[g(x, t)\right]}{v} - vg(x,0)$$

$$\mathrm{E}\left[\frac{\partial^2 f}{\partial t^2}(x, t)\right] = \frac{1}{v^2}T(x, v) - f(x, 0) - v\frac{\partial f}{\partial t}(x, 0) \qquad (5)$$

We can easily extend this result to the nth partial derivative by using mathematical induction.

Solution of Partial Differential Equations

In this section we solve first order Partial differential Equations and the Second order partial differential equation, wave equation, heat equation Laplace's and Telegraphers equation which are known as four

Fundamental equations in mathematical physics and occur in many branches of physics, in applied mathematics as well as in engineering.

Example 1:

Find the solution of the first order initial value problem:

$$\frac{\partial y}{\partial x} = 2\frac{\partial y}{\partial t} + y$$

$$\qquad (6)$$

$$y(x,0) = 6e^{-3x}$$

and y is bounded for $x > 0$, $t > 0$.

Let Y be the ELzaki transform of y. then, taking the ELzaki transform of (6) we have

$$\frac{dY(x,v)}{dx} - \left(\frac{2}{v}+1\right)Y(x,v) = -12\,ve^{-3x}$$

This is the linear ordinary differential equation

The integration factor is $p = \exp\left(\int -\left(\frac{2}{v}+1\right)dx\right) = e^{-\left(\frac{2}{v}+1\right)x}$

Therefore

$$Y(x,v) = \frac{12\,v^2}{2+4v}e^{-3x} + cve^{\left(\frac{2}{v}+1\right)x}$$

Since Y is bounded, c Should be zero. Taking the inverse ELzaki transform we have:

$$y(x, t) = 6e^{-2t}\cdot e^{-3x} = 6e^{-2t-3x}$$

Example II:

Consider the Laplace equation:

$$u_{xx} + u_{tt} = 0 , \quad u(x,0) = 0 \quad, \quad u_t(x,0) = \cos x \quad, \quad x,t > 0 \tag{7}$$

Let $T(v)$ be the ELzaki transform of u. Then, taking the ELzaki transform of equation (7) we have:

$$\frac{T(x,v)}{v^2} - u(x,0) - v\, u_t(x,0) + T''(x,v) = 0$$

$$v^2 T''(x,v) + T(x,v) = v^3 \cos x$$

This is the second order differential equation have the particular, solution in the form

$$T(x,v) = \frac{v^3 \cos x}{v^2 D^2 + 1} = \frac{v^3 \cos x}{1 - v^2}$$

$$\tag{8}$$

$$\text{Where} \qquad D^2 \equiv \frac{d^2}{dx^2}$$

If we take the inverse ELzaki transform for Eq.(8), we obtain solution of Eq(7) in the form.

$$u(x,t) = \cos x \, \sinh t$$

Example III:

Solve the wave equation:

$$u_{tt} - 4u_{xx} = 0 \quad, \quad u(x,0) = \sin \pi x \quad, \quad u_t(x,0) = 0 \quad, x,t > 0 \tag{9}$$

Taking the ELzaki transform for Eq (9) and making use of Conditions we obtain.

$$4v^2 T''(x,v) - T(x,v) = -v^2 \sin \pi x$$

$$T(x,v) = -\frac{v^2 \sin \pi x}{4v^2 D^2 - 1} = \frac{v^2 \sin \pi x}{1 + (2\pi)^2 v^2}$$

Now we take the inverse ELzaki transform to find the particular solution of (9) in the form

$$u(x,t) = \sin \pi x \, \cos 2\pi t$$

Example IV:
Consider the homogeneous heat equation in one dimension in a normalized form:

$$4\,u_t = u_{xx}\ ,\ \ u(x,0)=\sin\frac{\pi}{2}x\ ,\ \ x,t>0 \tag{10}$$

By using the ELzaki transform for Eq (10)
We can obtain

$$vT\,''(x,v)-4\,T\,(x,v)=-4\,v^2\sin\frac{\pi}{2}x$$

Solve for $T\,(x,v)$ we find that the particular solution is

$$T(x,v)=\frac{v^2\sin\frac{\pi}{2}x}{1+\frac{\pi^2 v}{16}} \tag{11}$$

And similarly if we take the inverse ELzaki transform for Eq (11), we obtain the Solution of (10) in the form.

$$u(x,t)=e^{-\frac{\pi^2}{16}t}\sin\frac{\pi}{2}x$$

Example V:
Consider the telegraphers equation:

$$u_{tt}(x,t)+2\alpha u_t(x,t)=\alpha^2 u_{xx}(x,t)\ ,\ 0<x<1\ ,\ t>0 \tag{12}$$

With the initial conditions:

$$u(x,0)=\cos x\ ,\ \ u_t(x,0)=0 \tag{13}$$

Solution:
Take Elzaki transform of Eq (12) we get:

$$\frac{T(v)}{v^2}-u(x,0)-vu'(x,0)+2\alpha\frac{T(v)}{v}-2\alpha vu(x,0)=\alpha^2 T''(v) \tag{14}$$

Substituting Eq (13) into Eq (14) we have:

$$\alpha^2 v^2 T''(v)-(1+2\alpha v)T(v)=-(2\alpha v^3+v^2)\cos x$$

$$or\quad T(v)=\frac{-(2\alpha v^3+v^2)\cos x}{\alpha^2 v^2 D^2-(1+2\alpha v)}=\frac{\cos x(2\alpha v^3+v^2)}{(1+\alpha v)^2}\quad\text{which is the}$$

particular solution of (12).

We take the inverse of Elzaki transform we find that:

$$u = \cos x\, T^{-1}\left[\frac{2\alpha v^3}{(1+\alpha v)^2} + \frac{v^2}{1+\alpha v} \right] = \cos x\left[2\alpha t e^{-\alpha t} + e^{-\alpha t} \right]$$

$$= (1+ 2\alpha t)\, e^{-\alpha t} \cos x$$

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