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# Cognitively Guided Instruction: A Knowledge Base for Reform in Primary Mathematics Instruction

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## Abstract

In this article we propose that an understanding of students' thinking can provide coherence to teachers' pedagogical content knowledge and their knowledge of subject matter, curriculum, and pedagogy. We describe a research-based model of children's thinking that teachers can use to interpret, transform, and reframe their informal or spontaneous knowledge about students' mathematical thinking. Our major thesis is that children enter school with a great deal of informal or intuitive knowledge of mathematics that can serve as the basis for developing much of the formal mathematics of the primary school curriculum. The development of abstract symbolic procedures is characterized as progressive abstractions of students' attempts to model action and relations depicted in problems. Although we focus on one facet of teachers' pedagogical content knowledge, we argue that understanding students' thinking provides a basis for teachers to reconceptualize their own knowledge more broadly.

Over the last 15–20 years a consensus has emerged that teaching is a complex problem-solving activity that cannot be understood only by looking at the activities that teachers engage in as they teach. As with any problem-solving activity, teachers' actions are governed to no small degree by the knowledge they bring to a situation. As a consequence, the analysis of teachers' knowledge has become a central concern for understanding the process of teaching, for evaluating teacher competence, and for bringing about fundamental change in how teachers teach (for reviews, see Carter, 1990; Clark & Peterson, 1986; Fennema & Franke, 1992; Fenstermacher, 1994; and Peterson, 1988).

Teachers' knowledge can be described along a number of dimensions. Three

important categories of knowledge are (a) knowledge of subject matter, (b) knowledge of pedagogy, and (c) pedagogical content knowledge (Grossman, 1990; Shulman, 1986; Wilson, Shulman, & Richert, 1987). Shulman (1986) coined the term *pedagogical content knowledge*, which he defined as consisting of (a) knowledge of ways of representing and explaining a subject to make it comprehensible and (b) knowledge of students' thinking, in particular, knowledge of the conceptions, preconceptions, and misconceptions students bring to the learning of a subject that make it easy or difficult to learn. Since Shulman introduced the notion of pedagogical content knowledge, it has been the focus of a number of studies and attempts to refine and redefine the construct (Fenstermacher, 1994; Marks, 1990). Although the construct of pedagogical content knowledge is evolving, the fundamental proposition remains that an important facet of teacher knowledge is concerned with teaching and learning of specific content (Grossman, 1990).

Although distinctions among classes of knowledge provide an analytic framework that is useful for analyzing teachers' knowledge, it is equally important to understand how knowledge can be integrated to give it coherence (Chi, Feltovich, & Glaser, 1981). In this article we argue that understanding students' mathematical thinking can provide a unifying framework for the development of teachers' knowledge. We describe how the knowledge base in a teacher development program called Cognitively Guided Instruction (CGI), which focuses on children's understanding of specific mathematical concepts, can provide a basis for teachers to develop their knowledge more broadly. We argue that understanding students' typical understanding and its evolution in specific content domains provides a framework for teachers to develop understanding of other facets of students' thinking. We propose that the knowledge not only provides a basis for understanding

students' thinking; it also can provide a framework for teachers' knowledge of mathematics and curriculum, and it provides a context in which teachers can interpret and apply general pedagogical knowledge.

In this article we describe the substance of the knowledge base that mediated the development of teacher knowledge in our current research and discuss how it is related to current analyses of teacher knowledge. How this knowledge was actually reflected in individual teachers' knowledge, beliefs, and practice and in their students' learning is discussed elsewhere (Carpenter & Fennema, 1992; Fennema et al., in press; Fennema, Carpenter, Franke, & Carey, 1992; Fennema, Franke, Carpenter, & Carey, 1993).

### Related Research

Several projects have focused on teachers' conceptions of mathematical learning as a basis for helping teachers make fundamental changes in instruction (e.g., the Summer-Math for Teachers Project—Schifter & Fosnot, 1993; Schifter & Simon, 1992; Simon & Schifter, 1991; the Purdue Problem-Centered Mathematics Project—Cobb et al., 1991; Wood, Cobb, & Yackel, 1991; and Cognitively Guided Instruction—Carpenter & Fennema, 1992; Fennema et al., in press). An underlying assumption all three projects share is that students construct knowledge rather than simply assimilate some part of what they are taught (Cobb, 1994; Davis, Maher, & Noddings, 1990). A related assumption is that significant changes in practice depend on teachers fundamentally altering their epistemological perspectives so that they appreciate that students construct knowledge and that they must adapt instruction accordingly. The assumption that students construct rather than assimilate knowledge underlies most current research and curricular recommendations in mathematics education (Davis et al., 1990; National Council of Teachers of Mathematics, 1989, 1991). These three pro-

jects go beyond the general assumption that students construct knowledge to help teachers understand exactly what that means in specific instances.

Although the projects are based on similar assumptions, they have taken somewhat different approaches to helping teachers reflect on students' thinking so that it can be used as a basis for instruction. In the Purdue Problem-Centered Mathematics Project, teachers are provided with problem-centered mathematical activities and teaching strategies to use in their classes that provide opportunities for them to attend to and reflect on students' thinking. Through immersion in a constructivist teaching environment, teachers start to question their conceptions of what it means to learn mathematics and come to develop their understanding of children's thinking. In the SummerMath for Teachers program (Schifter & Fosnot, 1993; Schifter & Simon, 1992; Simon & Schifter, 1991), teachers develop an appreciation of students' thinking as they reflect on their own learning in a learning environment based on constructivist principles and explore how the perspectives they are gaining apply to their own students.

In CGI, we have focused more directly on helping teachers understand children's thinking by helping them construct models of the development of children's thinking in well-defined content domains. Whereas in SummerMath learning mathematics serves as a context for teachers to develop an understanding of students' thinking, in CGI students' thinking provides a context for teachers to enhance their own understanding of mathematics. In contrast with the Purdue project, CGI teachers start with a more explicit framework for interpreting children's strategies for solving problems and construct their own instructional materials and practices based on their emerging understanding of students' thinking. For teachers in the Purdue project, the instructional activities provide the context to learn about students' thinking.

## **Cognitively Guided Instruction**

Consistent with our assumptions about children constructing knowledge of mathematics, we recognize that teachers construct their own understandings of students' thinking. Teachers have informal knowledge about students' mathematical thinking that is consistent with our analysis of students' thinking, but this knowledge is not well organized, and it generally has not played a prominent role as teachers make instructional decisions (Carpenter, Fennema, Peterson, & Carey, 1988). We help teachers to build on and focus this initial knowledge. Our analysis of the development of children's mathematical thinking can be thought of as scientific knowledge, as defined by Vygotsky (1962), that provides a basis for teachers to interpret, transform, and reframe their informal or spontaneous knowledge about students' mathematical thinking.

Our analysis of children's thinking provides a framework that guides our in-service activities, but it is not an outline for a series of formal presentations. We provide activities in which teachers have the opportunity to interact with these ideas, and we provide opportunities for teachers to interpret the knowledge through interactions with students. Thus, the analysis of children's mathematical thinking that follows is not a fixed body of knowledge that we expect teachers to assimilate. Rather, it provides a framework in which teachers construct and test their own models of students' thinking to guide their instructional practices.

In this article we focus on the development of children's conceptions of whole number operations involving single-digit and multidigit numbers. Our research also includes an analysis of the development of children's fraction knowledge (Baker, 1994; Baker, Carpenter, Fennema, & Franke, 1992) and their knowledge of geometry and measurement (Lehrer, Fennema, Carpenter, & Ansell, 1992). Our analysis of whole number concepts and operations is based on stu-

dents' informal solutions of word problems representing different addition, subtraction, multiplication, and division situations and their development of place-value concepts. We begin by characterizing critical differences in basic word problems that are reflected in how students think about and solve them. Students' strategies for solving problems are framed in terms of this analysis. The development of more abstract symbolic procedures is characterized as progressive abstractions of children's attempts to model action and relations depicted in problems.

Our major thesis is that children bring to school informal or intuitive knowledge of mathematics that can serve as the basis for developing much of the formal mathematics of the primary school curriculum. Without formal instruction on specific algorithms or procedures, children can construct viable solutions to a variety of problems. Basic operations of addition, subtraction, multiplication, and division can be defined in terms of these intuitive problem-solving strategies, and symbolic procedures can be portrayed as extensions of them.

The theme that ties together our analysis of students' mathematical thinking is that children intuitively solve word problems by modeling the action and relations described in them. By developing this theme, we are able to portray how basic concepts of addition, subtraction, multiplication, and division develop in children and how they can construct concepts of place value and multidigit computational procedures based on their intuitive mathematical knowledge.

Because our analysis of children's thinking focuses on their intuitive solutions of different types of problems, we start with an analysis of the problem space. One of the most useful ways of classifying problems focuses on the types of action or relation described in the problems. This taxonomy of problem types distinguishes between problems that children solve differently and provides a framework to identify the relative difficulty of problems. By starting with a

detailed analysis of problems, we can describe explicitly how the general strategy of modeling the action and relations described in problems is instantiated in particular cases.

### Addition and Subtraction

Four basic classes of addition and subtraction problems can be identified: problems involving (a) joining action, (b) separating action, (c) part-part-whole relations, and (d) comparison situations. Problems within a class involve the same type of action or relation, but within each class several distinct types of problems can be identified depending on which quantity is the unknown. Some of these distinctions among problems are illustrated by the protocols that follow. Additional examples appear in Appendix Table A1. A more complete description of distinctions among problem types can be found in Carpenter (1985) and Carpenter, Carey, and Kouba (1990). The research base for the analysis of students' strategies for solving addition and subtraction problems that follows is summarized in Carpenter (1985) and Fuson (1992).

Students use a variety of problem-solving strategies to solve different addition and subtraction problems. Consider how Rachel, a first-grade student, solved three problems that most adults would solve by subtracting:

*Teacher:* TJ had 13 chocolate chip cookies. At lunch he ate 5 of those cookies. How many cookies did TJ have left?

*Rachel:* [Puts out 13 counters, removes 5 of them, and counts the counters that remain.] There are 8.

*Teacher:* Good. Now here's the next one. Jenelle has 7 trolls in her collection. How many more trolls does she have to buy to have 11 trolls?

*Rachel:* [First puts out a set of 7 counters and adds counters until there is a total of 11. She then counts the counters she added to the initial set to find the answer.] Four.

*Teacher:* That's good. Here's one more. Willy has 12 crayons. Lucy has 7 crayons. How many more crayons does Willy have than Lucy?

*Rachel:* [Makes two sets of counters, one containing 12 counters and the other containing 7. She lines up the two sets in rows so that the set of 7 matches 7 counters in the set of 12 and counts the unmatched counters in the row of 12.] Five more.

The solutions to these three problems illustrate that for children, not all addition or subtraction problems are alike. Important distinctions among different types of addition and subtraction problems are reflected in the way students solve them. However, although Rachel used a different strategy for each problem, a common thread ties the strategies together. In each case, she modeled the action or relationship described in the problem. The first problem involved the action of removing 5 from 13, and that is how Rachel modeled the problem. In the second problem, the action was additive, and Rachel started with a set representing the initial quantity and added objects onto it. The third problem involved a comparison of two quantities, and Rachel used a strategy for comparing two sets.

Direct-modeling strategies are replaced initially by counting strategies, which are essentially abstractions of the direct-modeling strategies. Although counting strategies continue to reflect the action in problems, they are more efficient and require a more sophisticated conception of number than does direct modeling with manipulatives. In applying these strategies, students actually count the numbers in a counting sequence rather than constructing physical or pictorial representations of the problem, and they recognize that it is not necessary to reconstruct counting sequences representing both sets. For example, consider how one first-grade student solved an addition problem:

*Teacher:* James had 5 clay animals. During art he made 9 more clay animals. How many clay animals does James have now?

*Vanessa:* Nine [pause], 10, 11, 12, 13, 14. [With each count, Vanessa extends a finger. When she has five fingers up, she stops counting.] He has 14.

Children learn number facts both in and out of school and apply this knowledge to solve word problems. Certain number combinations are learned before others; and before they have completely mastered their addition tables, some students use a small set of memorized facts to derive solutions for problems involving other number combinations. Doubles ( $4 + 4$ ,  $7 + 7$ , etc.) are usually learned before other combinations, and sums of 10 ( $7 + 3$ ,  $4 + 6$ ) are often learned relatively early. The following example illustrates a first-grade student's use of a derived fact strategy. Examples of derived facts used with other problem types appear in Table A1.

*Teacher:* Tanya had 6 rings. Her sister gave her 7 more rings. How many rings does Tanya have now?

*Ben:* Thirteen.

*Teacher:* Wow, you got that fast. Can you tell me how you did it?

*Ben:* I knew that 7 and 7 was 14, and I took away 1, and it was 13.

### Multiplication and Division

We start the discussion of multiplication and division by distinguishing among three basic problems. The three problems are related but differ in what is known and what is unknown. In a *multiplication* problem, the number of sets and the number in each set are given, and the solution requires that one find the total number. In a *measurement division* problem, one must find the number of sets when the total number and the number in each set are given. In a *partitive division* problem, the total number and the number of sets are known. The solution requires that the number in each set be found.

The three problems are illustrated in the following examples of children's solution strategies and in Appendix Table A2. Additional types of multiplication and division problems are included in the workshops, including rate problems, multiplicative comparison problems, array and area problems, and Cartesian products. We also consider how children deal with remainders in division. A more detailed analysis of multiplication and division problems can be found in Greer (1992).

As with addition and subtraction problems, children initially solve multiplication and division problems by modeling directly the action and relations in the problems (Carpenter, Ansell, Franke, Fennema, & Weisbeck, 1993; Kouba, 1989). Consider how Rachel modeled the following measurement and partitive division problems:

*Teacher:* Rodney is having some kids over for jelly doughnuts. Seven doughnuts can fit on a plate. How many plates will he need for 28 doughnuts? (Measurement division)

*Rachel:* [Counts out 7 cubes.] Seven, that's one plate. [Counts out 7 more in a separate pile.] That's two plates. [Counts out 7 more in a third pile, and then counts all the cubes, finding that there are now 21 altogether. She makes a fourth pile of 7 cubes and again counts all the cubes to check that there are 28. She looks at the four piles.] Four plates.

*Teacher:* Ok. I see just how you did that. Let's try another one. Karina had 20 cupcakes. She put them into 4 boxes so that there were the same number of cupcakes in each box. How many cupcakes did Karina put in each box? (Partitive division)

*Rachel:* [Counts out 20 cubes and then deals them into four piles. When all of the cubes have been dealt into the four piles, she counts the cubes in one pile.] There are five.

As she did with the addition and subtraction problems, Rachel directly modeled the action described in the problems. In the first case, she made groups of a specified size and counted the groups to find the answer. In the second, she made a given number of groups with the same number of objects in each group and counted the objects in one of the groups to find the answer. The differences in the strategies used to solve the two problems reflect the different action described in the problems. Although adults may recognize both problems as division problems, young children initially think of them in terms of the actions or relationships portrayed in the problems.

Over time these direct modeling strategies are replaced by more efficient strategies based on counting, adding or subtracting, or using derived number facts (Kouba, 1989). Representative counting strategies are illustrated in the following exchange and in Table A2.

*Teacher:* Anna earns 4 dollars each week. How many weeks will it take her to earn 32 dollars to buy a kitten?

*Susan:* Hmmmm, 4, 8, 12, 16, 20, 24, 28, 32. [With each count Susan extended one finger. When she reached 32, she looked at the eight extended fingers.] Eight.

*Teacher:* Good. How about this one. There are 24 children in the class. We want to divide the class into six teams with the same number of children on each team. How many children will there be on each team?

*Susan:* Let's see, 3, 6, 9, 12, 15, 18. [With each count, Susan extends one finger. When she has extended six fingers, she pauses.] No, that's not big enough; let's try 4. Four, 8, 12, 16, 20, 24. [Again Susan extends a finger with each count. When she reaches 24 she sees that there are six fingers extended.] That's it; there would be four in each group.

Susan could solve the first problem directly by counting by the number in each

group, but for the second problem she did not know the number in the groups. Her problem was to find a number to count by so that when there were six numbers in the counting sequence the total would be 24. Her first guess was too small, so she tried a larger number.

Derived facts often are based on what students know about addition and subtraction and frequently involve doubling:

*Teacher:* Naomi has three pockets. On each pocket there are six buttons. How many buttons does Naomi have.

*Sarah:* Eighteen. If she had four pockets 6 plus 6 is 12, and 12 plus 12 is 24. But there were three pockets. You have to take 6 away from the 24. You could take 4 away from 24. It's 20. And you could take 2 more because 2 plus 4 is 6. Then it's 18.

### Multidigit Number Concepts and Procedures

The modeling strategies children construct to solve addition, subtraction, multiplication, and division problems are based on a relatively intuitive tendency to represent the action and relations described in problems. Place-value knowledge, in contrast, involves substantial explicit knowledge of conventions that cannot be discovered independently. Thus, although it is reasonable to expect that children may construct solutions for problem types that they have not previously encountered, it is not reasonable to expect them to be able to discover the conventions of our base 10 numeration system entirely on their own.

However, many children come to school with some implicit knowledge about the base 10 number system that they have picked up in learning to count, from experience with money, from interactions with adults, and from other activities involving numbers (Carpenter, Ansell, Levi, Franke, & Fennema, 1995). Although the patterns involved in number sequences beyond 10 do suggest some of the fundamental prin-

ciples underlying the base 10 number system, skill at counting beyond 10 does not necessarily mean that a child has any substantial knowledge about place value (Fuson, 1990).

Students must grasp a number of fundamental concepts in order to understand base 10 numbers. We focus on several key principles. The central principle is that collections of 10 (or 100, 1,000, etc.) can be counted (Steffe & Cobb, 1988). This means that one can talk about the number of tens (hundreds, thousands, etc.) just like one talks about the number of individual units. For example, a collection of 36 counters can be thought about as 36 individual counters or as three groups of 10 counters and six additional counters. To find out how many objects are in a group, a student can count all of the objects by ones or put the objects in groups of 10 and count the groups of 10 and the leftover objects.

The fundamental context for developing this notion of grouping by 10 can be found in multiplication and measurement division situations (Carpenter, Fennema, & Franke, 1994). For example, consider the following problems:

Our class has 5 boxes of doughnuts. There are 10 doughnuts in each box. We also have 3 extra doughnuts. How many doughnuts do we have altogether?

Jim picked 54 flowers. He put them into bunches with 10 flowers in each bunch. How many bunches of flowers did Jim make?

All that distinguishes these problems from other multiplication and division problems is that objects are collected into groups of 10. But that allows students to use principles of the base 10 number system to solve them. For example, consider students' responses to the above measurement division problem:

*Bob:* [Counts out 54 individual counters, makes groups with 10 counters in each group, and



- counts the number of groups.] Five.
- Anna:* [Uses linking cubes. (These are cubes that can be connected together. In this class the cubes are always stored with 10 cubes connected together so that students can use them when solving problems involving two-digit numbers. Anna has worked with the cubes previously, so she knows that there are 10 cubes in each rod.) She has a collection of some cubes joined into rods of 10 cubes and some loose cubes. She puts out five ten rods to represent the 50 cookies and four individual cubes to represent the extra four cookies. She looks at the collection for a moment, and then counts the ten rods.] Five.
- Tanya:* [Counts by ten.] 10, 20, 30, 40, 50. [With each count she extends a finger. When she reaches 50 she sees that she has raised five fingers and responds.] Five.
- Julio:* [Immediately responds.] Five, because there's five tens in 54. [Carpenter et al., 1994, p. 57]

These responses demonstrate a progression of understanding of base 10 number concepts. As with the solutions to basic addition, subtraction, multiplication, and division problems, children strive to use increasingly efficient strategies. In the process they develop increasingly sophisticated understandings of base 10 numbers.

Algorithms or formal procedures for computing answers to multidigit addition, subtraction, multiplication, and division problems depend on base 10 number concepts. Many educators assume that this means that it is necessary for students to develop base 10 number concepts before they can add or subtract two- and three-digit numbers. That assumption has not proven to be valid (Fuson, 1990, 1992). As long as children can count, they can solve problems involving two-digit numbers, even when they have limited notions of grouping by 10. Addition and subtraction problems with two- and three-digit num-

bers may actually provide a context and motivation for students to develop an understanding of base 10 numbers (Fuson, 1990, 1992).

We do not suggest that students should be taught formal computational algorithms before they understand base 10 numbers, but we propose that students who do not have a complete understanding of base 10 numbers can construct solutions to multidigit problems that are meaningful to them. As they develop increasingly efficient ways to solve these problems, their understanding of base 10 numbers increases concurrently with an understanding of how to apply this knowledge to add, subtract, multiply, and divide multidigit numbers. In other words, children can acquire the skills and concepts required to solve problems as they are solving the problems. Thus, solving problems like those we have discussed in the preceding sections with increasingly large numbers can provide a basis for learning place-value concepts.

There are direct parallels between the strategies children use for multidigit problems and the strategies they use for problems with smaller numbers. Children use counters to model the action in problems directly, and they invent counting strategies involving units of 10 that essentially are abstractions of these modeling strategies. Students also construct strategies in which they combine tens and ones separately that are similar in many ways to the traditional algorithms for adding and subtracting (Carpenter et al., 1995; Fuson et al., 1994).

Children initially solve problems with larger numbers using the same modeling strategies they use for problems with smaller numbers. They model the problems using individual counters, counting by one. As students integrate their problem-solving schemes with their emerging knowledge of grouping by tens, they begin to use units of 10 to model two- and three-digit numbers (Carpenter et al., 1994, 1995; Fuson, 1990).

As with problems with smaller numbers, modeling with tens gives way to more

symbolic solutions (Carpenter et al., 1994, 1995; Fuson et al., 1994). One type of invented symbolic procedure is somewhat analogous to the counting strategies used with smaller numbers in that the solution involves successively increasing or decreasing partial sums or differences. With the other major type of invented symbolic procedure, the tens and units are operated on separately and the results subsequently combined. These solutions are somewhat analogous to the derived fact solutions that students employ with single-digit numbers in that numbers are decomposed and re-composed to simplify calculation. Combining the tens and units separately is more closely related to the procedures used in the standard addition and subtraction algorithm than the procedures in successive incrementing.

The following episode from a third-grade class illustrates the direct modeling strategy and one type of invented algorithm as well as a transition strategy linking them. (Additional examples appear in Appendix Table A3.) The episode comes from a discussion of students' solution to a word problem involving the sum  $54 + 48$ . (See Fig. 1.) The students had worked on the problem at their desks for about 15 minutes and were sharing their strategies with the class.

- Ms. G:* Now everyone go over to Ellen's desk.  
*Ellen:* They don't need to go to my desk, I can tell them right here.  
*Ms. G:* But I want them to go to your desk; I want them to see exactly what you showed me, and then you can tell me how you could do it without us having to go to your desk.

The children move around Ellen's desk.

- Ellen:* [Makes 54 and 48 with tens and ones blocks (see Fig. 1).] I knew this was 54, so I went 64, 74, 84, 94 [Ellen moves one ten block for each count. Then she counts

the single cubes, moving a cube with each count.], 95, 96, . . . , 102.

- Ms. G:* Now class, what question am I going to ask her? Norman?

*Norman:* You didn't use the 54; did you have to make it?

- Ms. G:* Good Norman, that is just what I was going to ask her. Ellen, did you need to make that 54?

*Ellen:* No.

- Ms. G:* [Pulls the 54 away and covers it with her hand.] Ok, now show me how you can solve the problem without the 54.

*Ellen:* 64, 74, . . . [Repeats the above strategy, counting on without the 54.].

- Ms. G:* Ok, now you told me that you could do this without us moving to your desk. How would you have done that?

*Ellen:* Ok, I just put 54 in my head, and then I go 48 more. I go 54 [slight pause], 64, 74, 84, 94 [She puts up a finger with each count to keep track of the tens. At this point she has four fingers up. She puts down her fingers and puts them up again with each count as she continues counting by ones.], 95, 96, 97, . . . , 102. [Carpenter et al., 1995, pp. 3-4]

In this one exchange, we see three related but quite distinct strategies: directly modeling the problem using tens bars, abstracting the first quantity and counting the second quantity, and counting on by tens using fingers. The three strategies represent successive levels of abstraction. In the first strategy, the objects in the problem were represented directly with the blocks. In the second strategy, the quantity representing the first set was abstracted, and Ellen counted on starting with the number in the initial set, counting the blocks representing the set that was joined to the initial set. In the final strategy, the counting words no longer were linked to physical materials. The counting words themselves were counted by keeping track of the counts on fingers. The fingers did not act like the blocks did in the first two strategies; they

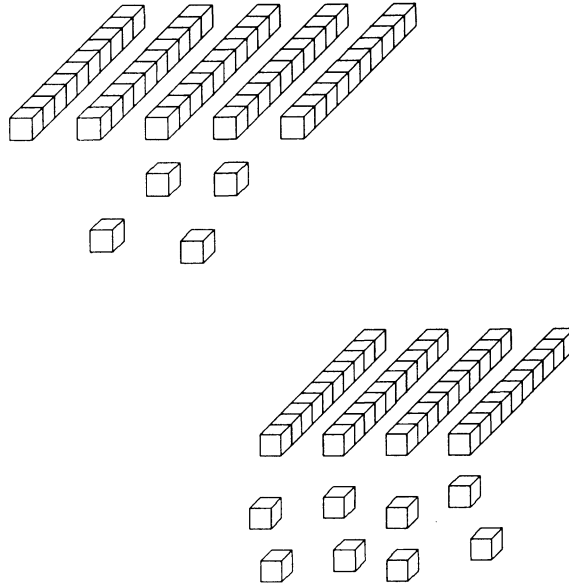


FIG. 1.—Ten block representation of  $54 + 48$

were not surrogates for the blocks. They played a very different role. As Ellen counted on by 10 from 54, she was not counting imaginary collections of 10. She was using her fingers to keep track of how many counts of 10 she had made. The counting sequence itself had become an object of reflection, and as such it could be counted (Steffe, von Glasersfeld, Richards, & Cobb, 1983).

Although the strategies represent quite different conceptions of the problem, there is a clear relation between them. Each strategy represents an abstraction of the one that precedes it. Furthermore, the verbal descriptions of the strategies are remarkably similar. In each case Ellen went through the same counting sequences; it is the referents for the counts that changed.

The above example suggests that students' invented algorithms are constructed through progressive abstraction of their modeling procedures with blocks. Ellen's final solution was for all intents and purposes a verbal description of what she did with the blocks. But it was more than that. It represented a solution that could actually

be carried out without the blocks as explicit referents. Other invented algorithms share the same features. Consider, for example, the invented algorithm that another student in Ms. G's class reported:

*Ms. G:* Did anyone solve the problem a different way? Brian.

*Brian:* First add the 5 tens and the 4 tens. That's 90. Then add the 4 and the 8; that makes 12. Then the 90 and the 12 make 102. [Carpenter et al., 1995, p. 6]

As did Ellen's solutions, this solution can be taken as a description of the way blocks could be combined and counted, or it can represent the thinking that was involved in solving the problem abstractly. Unlike Ellen, Brian did not first model this problem using blocks.

What we are proposing is that the manipulations of the blocks become objects of reflection. At some point the numbers involved in counting the blocks also become objects of reflection so that students can operate on the numbers independently of the blocks. A key factor in this process is the

continuing discussion of alternative strategies. Students regularly are called on to articulate their solutions, to describe in words what they have done with the blocks. In order to be able to describe their strategies, they need to reflect on them, to decide how to report them verbally. Initially, the descriptions are of procedures that have already been carried out. Eventually, the words that students use to describe their manipulations of blocks become the solutions themselves. Thus, the verbal description of modeling strategies provides a basis for connecting manipulations of tens blocks and invented algorithms using numbers only. The students do not imitate a strategy that they do not understand; they abstract the physical modeling procedures when they are comfortable doing so. In this way, the evolution from physical to symbolic procedures follows much the same course as the evolution from modeling to counting strategies with single-digit numbers.

### Summary

The foregoing analysis provides a coherent, principled framework for teachers to understand children's development of basic whole-number concepts. A small number of principles form the basis of the taxonomy of problems that link addition, subtraction, multiplication, and division. The taxonomy provides the basis for understanding how children think about and solve problems. Essentially, students model the structure of a given problem. Initially, they use physical objects or pictures to model the problem, but over time they abstract the modeling to use more efficient counting and derived-fact strategies. The development of multidigit number concepts and operations follows a similar pattern, with the same basic strategies extended to make use of groupings of 10, 100, 1,000, and so on.

### An Integrated Perspective of Teachers' Knowledge

Clearly, other variables that affect students' performance are not included in our anal-

ysis, and not all problems or all children's strategies fall neatly into distinct categories. We have attempted to help teachers construct models of students' thinking at a level of detail that is meaningful to teachers and useful to them in understanding their own students' thinking and making instructional decisions. We do not propose that the CGI analysis completely accounts for all children's mathematical thinking and problem solving, but it is a place for teachers to start. Although our analysis focuses on specific knowledge about students' mathematical thinking, it provides a general framework for understanding their thinking more broadly. In the following section, we examine how the knowledge base of CGI provides a unifying framework for understanding multiple facets of children's thinking.

### Teachers' Knowledge of Students' Thinking

Marks (1990) elaborated the notion of pedagogical content knowledge by considering five facets of teachers' knowledge of students' thinking: (a) students' typical understanding, (b) students' learning processes, (c) what is hard and what is easy for students, (d) the most common errors students make, and (e) particular students' understanding. Our analysis focuses on students' typical understanding; however, it differs from Marks's characterization in that the emphasis in CGI is not just on what problems students typically can solve but on how they solve them. Furthermore, although we focus on the students' development of understanding, our analysis provides a basis for thinking about the other four facets as well.

For example, Marks identified the two most important features of teachers' knowledge of students' learning processes as recognizing that (a) students must understand concepts rather than learn by rote and (b) abstract mathematical concepts and operations should be connected to experience with concrete objects. Our analysis of chil-

dren's thinking provides a basis for understanding what it means for students to learn with understanding. Students' understanding is characterized in terms of how students connect new ideas to existing knowledge. The nature of the knowledge students bring to the learning of mathematics and how they connect it with formal concepts and operations is portrayed explicitly.

Teachers also learn exactly how students initially use concrete materials to solve mathematical problems and how those operations on concrete materials evolve and are linked to more formal, abstract operations. It is generally acknowledged that the use of manipulative materials is not sufficient; how students conceive of the materials and use them is critical (Cobb, Yackel, & Wood, 1992; Thompson, 1994). The CGI framework provides a detailed analysis of how students use concrete materials to represent problems and the meanings they attribute to them.

The CGI framework also provides teachers a coherent basis for identifying what is difficult and what is easy for students and for dealing with the common errors they make. Simply put, problems that are difficult to model directly are generally more difficult to solve than problems that are easier to model. Certainly other variables are involved in problem difficulty, but this simple principle addresses some of the most critical criteria for selecting problems at the primary level.

The CGI analysis differs from many other characterizations of students' thinking that focus on identifying students' misconceptions and errors. In CGI, the emphasis is on what children can do rather than on what they cannot do. This leads to a very different approach to dealing with errors than an approach in which the goal is to identify students' misconceptions in order to fix them. For CGI teachers the goal is to work back from errors to find out what valid conceptions students do have so that instruction can help students build on their existing knowledge. Thus, it is important

for CGI teachers to be able to identify errors, but they do not represent the starting point for instruction.

The final element of Marks's analysis of teachers' knowledge of students' thinking deals with teachers' ability to identify individual students' thinking. A primary goal of CGI is to provide teachers a framework to assess their own students' understanding, and one of the important findings of our studies is that CGI teachers are significantly more successful in identifying problems their students can solve and the strategies that they use to solve them than non-CGI teachers (Carpenter, Fennema, Peterson, Chiang, & Loef, 1989). The CGI analysis provides a basis for selecting critical problems and for understanding what students' responses imply about their mathematical understanding. Because the framework is highly principled, it is possible to select a few key problems that will show what a student knows, and responses to individual problems fit together to provide a coherent profile of the strategies a student uses. As one CGI teacher put it: "I have always known that it is important to listen to kids, but before I didn't know what questions to ask or what to listen for" (Carpenter et al., 1989, p. 530).

#### Knowledge of Effective Representations and Explanations

The CGI model focuses on teachers' knowledge of students' thinking, but this knowledge also addresses another primary component of teachers' pedagogical content knowledge: knowledge of ways of representing and explaining a subject to make it comprehensible. With CGI the emphasis shifts from teachers finding ways of representing mathematical knowledge for students to students constructing their own representations based on their intuitive problem-solving strategies. The teacher is not perceived as the source of knowledge and does not provide ready-made explanations and representations.

This distinction can be seen in the difference between the way that multiplication and division are portrayed in CGI and in many traditional textbooks and methods texts, which often employ arrays as the primary representation for multiplication and division. Arrays have an advantage over the grouping strategies that students typically invent to represent multiplication and division problems in that they illustrate the commutative property of multiplication. However, students do not naturally use arrays as representations for problem situations that do not specifically describe arrays. Students' initial representations may not be the most efficient or provide the clearest insights about important mathematical concepts, but our goal is not to provide teachers with representations to teach directly to students. Rather, it is to help the teachers understand the ways students intuitively solve problems, so that they can help students build on that knowledge.

The selection of symbols for representing mathematical ideas also can be related to the analysis of the strategies students use to solve problems. Certain symbolic representations reflect the ways children naturally solve problems with objects or other invented representations. For example, young children generally solve a problem like the following by using an additive process, adding counters to a set of 7 counters until there are a total of 11 or counting on from 7 to 11:

Adam has 7 dollars. How many more dollars does he have to earn to have 11 dollars to buy a kitten?

A natural way for children to represent this problem is with the open sentence  $7 + \underline{\quad} = 11$  rather than with a standard subtraction number sentence (Bebout, 1990; Carey, 1992).

#### Subject-Matter Knowledge and Curricular Knowledge

Although the CGI analysis focuses on students' thinking, it also provides teachers

a context for developing subject-matter knowledge and acquiring and evaluating curricular knowledge. Elementary teachers often have relatively narrow conceptions of basic mathematical operations (Ball, 1990; Carpenter et al., 1988; Post, Harel, Behr, & Lesh, 1991; Tirosh & Graeber, 1990), and the problem taxonomies provide an expanded perspective of what it means to add, subtract, multiply, and divide. The strategies students use to solve these problems depend on the basic properties of these operations, and understanding children's strategies forces teachers to confront their own understanding of these properties from a new perspective. Because this perspective is closer to teachers' daily lives than the standard abstract statements of axioms and theorems often presented in content courses for elementary teachers, these operations take on a whole new meaning. Properties are not simply obvious statements about numerical relations; they allow students to construct increasingly powerful strategies for solving problems. For example, the counting-on-from larger strategy described earlier requires at least some implicit knowledge of commutativity of addition, and many multiplication derived facts involve the distributive property. Understanding the strategies students invent for solving multidigit numbers requires teachers to examine their own understanding of place value concepts and operations on multidigit numbers.

The curriculum in most CGI classes is derived from analyzing the development of students' mathematical thinking (Fennema et al., in press). Students spend most of their time solving problems and discussing alternative solutions. Using the classification of problem types as a guide, teachers make up many of the problems based on activities in which the class is engaged. Sometimes problems come out of activities in other subjects such as science or social studies, or are made up from contexts in story books. Sometimes the problems arise naturally in daily class activities like figuring out the

lunch count or sharing treats. When teachers do use prepared curriculum materials, the CGI framework provides a basis for evaluating how the materials might contribute to students' understanding of mathematics.

### General Pedagogical Knowledge

The CGI model does not address general pedagogical knowledge. We help teachers to develop deeper knowledge about students' thinking and rely on the teachers to use their general pedagogical knowledge to decide how to use that knowledge. However, as teachers think about how to integrate their emerging knowledge about children's thinking with their existing pedagogical knowledge, they examine and question their pedagogical knowledge. This frequently results in changes in teachers' general pedagogical knowledge that go beyond the teaching of mathematics (Fennema et al., 1992). Thus, CGI provides a context for reflecting on and evaluating pedagogical knowledge in general. In recognizing that students have knowledge worth listening to and building on, teachers evaluate their general philosophies about their role as the dispenser of knowledge, the nature of classroom interactions, the use of different forms of grouping, and the like. The detailed knowledge that teachers have about children's thinking in mathematics provides an explicit context for evaluating and reconceptualizing decisions about pedagogy. In a sense, because the knowledge is so explicit, it becomes paramount for many CGI teachers, and general pedagogical knowledge is shaped around it. Teachers'

thinking about pedagogy is refocused so that the primary considerations revolve around student thinking rather than teacher actions.

### Conclusion

In this article we have argued that CGI provides teachers a framework with which to construct a coherent, organized knowledge base that they can draw on to solve complex pedagogical problems they encounter in teaching primary school mathematics. We have not explicitly addressed the multiple components of teacher knowledge discussed in this article. In all our interactions with teachers both in formal workshops and individually, we have focused on understanding students' thinking. We seldom have discussed other issues related to teaching without putting them in the context of our understanding of children's thinking. We have tried consistently to help teachers relate questions about teaching to what they know about students' thinking or what they can learn about students from a particular problem or activity.

This clearly delineated knowledge appears to have provided a basis for a number of teachers to engage in what Richardson (1994) calls "practical inquiry." These teachers do not view their understanding of children's thinking as static knowledge that applies only in specific contexts; rather, they use the knowledge as a basis to construct and test their own theories about teaching and learning (Fennema et al., in press). In this article we have attempted to portray the nature of the knowledge that made that possible.

## Appendix Problems and Solution Strategies

TABLE A1. Primary Addition and Subtraction Strategies

Problem	Direct Modeling	Counting	Derived Facts
Robin had 5 toy cars. Her friends gave her 7 more toy cars for her birthday. How many toy cars did she have then?	Makes a set of 5 counters and a set of 7 counters. Pushes the two sets together and counts all the counters.	Counts "5 [pause], 6, 7, 8, 9, 10, 11, 12," extending a finger with each count. "The answer is 12." [The counting sequence may also begin with the larger number.]	"Take 1 from the 7 and give it to the 5. That makes $6 + 6$ , and that's 12."
Colleen had 12 guppies. She gave 5 guppies to Roger. How many guppies does Colleen have left?	Makes a set of 12 counters and removes 5 of them. Then counts the remaining counters.	Counts back "12, 11, 10, 9, 8 [pause], 7. It's 7." Uses fingers to keep track of the numbers of steps in the counting sequence.	"12 take away 2 is 10, and take away 3 more is 7."
Robin has 4 toy cars. How many more toy cars does she need to get for her birthday to have 11 toy cars altogether?	Makes a set of 4 counters. Makes a second set of counters, counting "5, 6, 7, 8, 9, 10, 11," until there is a total of 11 counters. Counts the 7 counters in the second set.	Counts "4 [pause], 5, 6, 7, 8, 9, 10, 11" extending a finger with each count. Counts the 7 extended fingers. "It's 7."	" $4 + 6$ is 10 and 1 more is 11. So it's 7."
Mark has 6 mice. Joy has 11 mice. Joy has how many more mice than Mark?	Makes a row of 6 counters and a row of 11 counters next to it. Counts the 5 counters in the row of 11 that are not matched with the set of 6.	There is no counting analog of the matching strategy.	" $6 + 6$ is 12, and 11 is 1 less, so it's 5."

TABLE A2. Primary Multiplication and Division Strategies

Problem	Direct Modeling	Counting	Derived Facts
Multiplication: There are 6 guppies in each fish bowl. There are 4 fish bowls. How many guppies are there altogether?	Makes a set of 6 counters and then another set of 6 counters, and a third and a fourth. Counts all 24 counters in the 4 sets.	Counts "6, 12, 18 [pause], 19, 20, 21, 22, 23, 24, [fingers are used to keep track of the 6 counts beyond 18]. It's 24."	" $6 + 6$ is 12 and $12 + 12$ is 24."



TABLE A2 (Continued)

Problem	Direct Modeling	Counting	Derived Facts
Measurement division: There are 24 children going on a picnic. Four children can ride in each car. How many cars are needed to take all the children to the picnic?	Counts out 24 counters. Makes groups of 4 counters until all the counters are used up. Counts the number of groups.	Counts "4, 8, 12, 16, 20, 24," extending a finger with each count. Counts the 6 fingers.	"5 fours is 20, so one more four is 24."
Partitive division: Mr. Franke baked 20 cookies. He gave all the cookies to 4 friends, being careful to give the same number of cookies to each friend. How many cookies did each friend get?	Counts out 20 counters, deals them one by one into 4 piles, and counts the counters in each pile.	Counts using trial and error. "5, 8, 12, 16. No, it will be too many groups. 5, 10, 15, 20. That's four groups. So it's 5."	"I know 4 fives is 20. So it's 5." This is really a recall strategy using knowledge of a multiplication fact.

TABLE A3. Multidigit Strategies

Problem	Direct Model by 10	Incremental	Combine Tens and Ones
There are 28 girls and 35 boys on the playground at recess. How many children were there on the playground at recess?	Puts out 2 ten bars and 8 individual one blocks. Puts out another pile of 3 tens bars and 5 one blocks. Pushes the piles together and counts the ten bars. Counts the one blocks. When 10 one blocks are counted, they look like a ten bar. Recounts the 6 ten bars including the new "ten bar" just made. Counts the remaining 3 ones.	"20 + 30 is 50 and 8 more is 58. Now I need 5 more. 2 more is 60 and 3 more than that is 63."	"20 + 30 is 50. 8 and 5 is like 8 + 2 and 3 more so it's 13. 50 + 13 is 63."
There were 51 geese in the farmer's field. Twenty-eight geese flew away. How many geese were left in the field?	Puts out 5 tens bars and 1 one block. Removes 2 ten bars. Trades 1 ten bar for 10 ones. Removes 8 ones. Counts the remaining 2 ten bars and 3 one blocks.	"50 take away 20 is 30, and put back the 1, it's 31, but now I have to take away the 8, and 31 - 8 is like 30 - 7, so it's 23."	"50 take away 20 is 30, and 8 take away 1 is 7. Now I have to take the 7 from 30 because that's how much more I have to take away. So it's 23."

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