Modelling Microscopic Freeway Traffic Using Cusp Catastrophe Theory

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Abstract— This paper proposes a framework based on stochastic cusp catastrophe theory to model microscopic freeway traffic flow. The approach considers that each driver - regardless of being aggressive or timid - may shift his/her behaviour and behave aggressively or timidly during driving. Based on the proposed modelling approach, the spacing of a driver is a function of his speed and acceleration. Moreover, the driver’s behaviour comprises of two equilibrium states - being aggressive or timid - and a shift between these states is considered to occur as a catastrophe phenomenon. Different models are developed with varying sampling intervals, while some of them possess memory properties. Results show that the cusp catastrophe model may accurately describe microscopic traffic, especially when compared to linear or logistic models. Results for models with memory are even more promising. Based on the proposed method, preliminary results on the critical regions of breakdown conditions or attitude shifting are described and discussed.

Index Terms — Microscopic traffic, driver’s behaviour, catastrophe theory, stochastic cusp catastrophe model

I. INTRODUCTION

Freeway traffic flow modelling has attracted enormous attention in the last decades. A vast number of macroscopic and microscopic approaches have been proposed with the main scope to model the transitional nature of traffic flow (for a review see [1]-[3]). Catastrophe theory is considered to provide a fruitful methodological background for replicating the transitional behaviour of traffic flow. Catastrophe theory is a mathematical model which can explain the discontinuous transition of a system between two equilibrium states [6]. Any catastrophic phenomena observed in nature, subject to some limitations, can be classified separately to one of seven elementary catastrophes: fold, cusp, swallowtail, butterfly, hyperbolic umbilic, elliptic umbilic and parabolic umbilic.

Conceptually, catastrophe theory may explain the manner in which a discontinuity occurs and under which circumstances. The rationale to use catastrophe theory has originated by the general literature finding that the classic Lighthill–Whitham and Richards (LWR) model which exploits the advantages of continuous flow, cannot adequately explain the two-capacity or reverse-λ state in the fundamental traffic diagram [4]. The reverse-λ state indicates that when traffic reaches a critical density point, it may break down to an unstable state (flow decreases and density increases) or reach a bi-stable state of higher density and flow values than critical density point, until a discontinuity leads traffic to congestion. This discontinuity is evident in Edie’s hypothesis [5].

The cusp catastrophe model has been used almost exclusively in macroscopic traffic modelling. The cusp catastrophe relates three variables and can explain the catastrophic change (sudden increase or decrease) of one of the three variables (state variable) in relation to the other two control variables, namely bifurcation/splitting factor and normal/asymmetry factor [6]. Dendrinos [7] first presented a relationship between volume to capacity ratio and the operating speed on highways using the fold catastrophe. However, the full potential of cusp catastrophe models in macroscopic traffic variables has been researched by Navin [8] and later Hall [9], who showed that the use of catastrophe theory for explaining the discontinuity in high volumes of traffic is a feasible and well founded idea. Dillon and Hall [10] and Forbes and Hall [11] addressed the issue of the correct transformation of the axes in the fundamental traffic flow diagrams for improving the fit to catastrophe models. Later, Acha-Daza and Hall [12] tested the applicability of cusp catastrophe models with real data and statistical tests and concluded that cusp catastrophe in some cases can accurately predict changes in speed. However, they failed to explain why cusp catastrophe’s model didn’t fit the data in some other cases.

As Park and Adbel-Aty [13] stated, all the above studies focused on a deterministic representation of the catastrophe and may not be ideal to model the stochastic nature of traffic flow which may negatively affect the accuracy of the models developed. The stochastic features of freeway traffic have been systematically addressed in literature [14]. Moreover, catastrophe models have not been implemented to microscopic traffic flow. Intuitively, catastrophe theory may be beneficial for modelling the shifting behaviour of drivers between aggressiveness and timidity during driving, as these shifts may be considered catastrophic phenomena.

The present paper extends past research by applying cusp catastrophe models to microscopic traffic flow variables such as speed, spacing and acceleration. Taking into account the effectiveness of catastrophe models to past applications on behavioural science, the proposed approach attempts to provide models that may replicate the shifts of driver’s behaviour between aggressive and timid states during driving. Behavioural changes during driving are considered as catastrophe phenomena. Catastrophe models are developed and evaluated with respect to classical linear and logistic models. The use of aggregated data, as well as the incorporation of a memory mechanism is also evaluated with respect to the improvement of the fit of the models.
Furthermore, critical regions of breakdown conditions or attitude shifting are presented and discussed.

II. CUSP CATASTROPHE APPLIED TO MICROSCOPIC TRAFFIC

A. The Cusp Catastrophe Model

There are seven elementary catastrophes appearing in nature. For every elementary catastrophe there is a corresponding germ, a dynamic function [6]. Cusp catastrophe’s germ is the function \( x^4 \) which is not a structurally stable function and has a degenerate critical point (singularity) at \( \{0,0\} \). For the case of cusp catastrophe, stability is accomplished by adding an adequate number of parameters in order to stabilize the degenerate critical point, as seen in the following equation:

\[
V(x,u,v) = \frac{1}{4}x^4 + \frac{1}{2}ux^2 + vx
\]  

where \( V(x,u,v) \) is the system’s potential, \( x \) is the state variable and \( u,v \) are the control factors (bifurcation/splitting factor and normal/asymmetry factor respectively). Equation 1 has the same critical point (singularity) as \( x^4 \). Intuitively, a singularity condenses a “total structure” to a “local structure” and its’ universal unfolding is the unfolding of every information that is hidden inside it. Equation 1 also represents the universal unfolding of \( x^4 \).

Potential plays a central role in catastrophe theory but cannot be accurately defined. The tendency of a stretched rope to shrink, the tendency of two chemical elements to react or, in the case of traffic engineering, the tendency of the driver to reach the desired speed or the tendency of a system of vehicles to move away from jammed conditions are all cases that are related to the concept of potential. A system’s potential is therefore linked to its’ equilibrium states. When the system reaches equilibrium, the potential reaches a minimum. The purpose of the equation 1 is the final outcome or numerical result which can show if a system is in equilibrium or not.

Zeeman achieved in creating a three dimensional model of the cusp catastrophe seen in Figure 1 [15].

![Fig. 1. The surface of a cusp catastrophe.](image)

As seen in Figure 1, the system moves on Surface M similarly to a ball. The upper surface of the folded (hatched) area and the surface below it are the two equilibrium states. When the specific ball reaches the edge of the folded area, it drops to the surface below. The ball in every point has different values of \( (y,u,v) \) representing different states of the system. When the ball (system) falls, then the state variable’s \( y \) value significantly drops while the other two variables \( (u,v) \) remain approximately the same. The reverse case is when the system reaches the edge of the folded area while being below the hatched area on surface M. In that case, state variable’s value increases in a catastrophic manner. Surface M is given by the partial differential of Equation 1:

\[
\frac{\partial V(x,u,v)}{\partial x} = x^3 + ux + v
\]

The shaded part of Surface M is the folding curve. The sum of these points is called catastrophe set or bifurcation set. Those points can be found by the following system of equations:

\[
\begin{align*}
\nabla_x V &= 0 \\
\Delta &= \det H(V) = 0 \\
\end{align*}
\]

where \( H(V) \) is the Hessian matrix of \( V \) (equation 1). By eliminating \( x \) from the above and solving the system of equations, the following function is deduced:

\[
4u^3 + 27v^2 = 0
\]

which is depicted as the hatched area in Surface C of Figure 1 and is the projection of the bifurcation set. Inside the bifurcation set is where every catastrophe occurs.

Different methods (conventions) may be used to explain how and when a catastrophe will take place. The major conventions that have been reviewed multiple times are the Delay and the Maxwell convention. According to the Maxwell convention, the transition between equilibrium states occurs immediately to the state of the lowest potential even if the local minimum of the potential has not disappeared. In graphic terms, the system’s state variable \( x \) is subject to a catastrophic change immediately when the system crosses the bisector of the bifurcation set meaning the bisector of the hatched area on Surface C in Figure 1. The Delay convention dictates that catastrophe occurs when the local minimum of the potential disappears completely and only one global minimum exists. Practically, according to Delay convention a catastrophe occurs when the system reaches the edge of the bifurcation set after crossing the bisector. The above are depicted in Figure 2; one can observe the physical example of the moving ball on the folded area for the Delay convention (Figure 2-a) and the Maxwell convention (Figure 2-b). Points 1-2 (Figure 2-a) and 1'-2' (Figure 2-b) are the correspondent points on the bifurcation set.

Cusp catastrophe encompasses five characteristics that are of major importance because if some, or all of them, are present then one can assume that a good fit of the data to the cusp catastrophe is feasible [16]. These characteristics are the following: bimodality, hysteresis, inaccessible behaviour, catastrophe and divergence. Catastrophe is the obvious characteristic of catastrophe theory and means the sudden transition of the system between equilibrium conditions (e.g., constrained and unconstrained traffic). Bimodality means that a set of the system’s parameters \( (u,v) \) may be the same for different values of the state variable; for example a system of vehicles with flow 2500 veh/h and density 40 veh/km can...
have different values of speed (consider the reversed-\(\lambda\) state). Due to inaccessible behaviour, the system cannot stabilize in conditions between the two equilibrium states and, due to hysteresis, the system changes from equilibrium state 1 to state 2 for different \((u, v)\) than in transition from state 2 to state 1. Finally, when the system is at the singularity point, there is equal probability to move to equilibrium state 1 or state 2 (e.g., if a macroscopic traffic system is immobilized in a freeway then it possible to be in congested or uncongested traffic). This is the characteristic of divergence.

![Figure 2](image)

**Figure 2.** Above (a): catastrophe occurs according to the Delay convention from point 1 to point 2. Below (b): catastrophe occurs according to the Maxwell convention from point 1′ to point 2′ or 2′ to 1′.

### B. Adaptation of cusp catastrophe to Microscopic Traffic

The assumption for a conceptually solid fit of microscopic traffic variables to the cusp catastrophe is mostly based on Newell’s [17] hypothesis of the asymmetric behaviour of drivers between accelerating and decelerating conditions. Forbes [18] observed that driver’s response to acceleration was slower than in deceleration while Newell [17] suggested that there are two different functions for accelerating and decelerating conditions in a speed-spacing diagram; accelerating and decelerating conditions are distinguished by shifts in the value of spacing which are rendered as the characteristic of catastrophe. In addition, the catastrophes between accelerating and decelerating condition occur in different points in terms of the acceleration and speed value. This is consistent with the characteristic of hysteresis. Figure 3 depicts the graphical relation of Newell’s hypothesis to the cusp catastrophe for a single vehicle. This is a realistic description of the mechanics of the catastrophe for microscopic traffic (equilibrium states are arbitrary - the equilibrium states used in this paper are explained later on).

As seen in Figure 3, the system (vehicle) is, first, accelerating, moving with a speed \(v_3, v_1 < v_3 < v_2\) and has a value of spacing \(d_2, d_1 < d_2\) which is increasing with time (point 1). When the vehicle reaches point 2, spacing decreases abruptly and in a catastrophic manner and the system moves to point 3. Between points 2 & 3, speed shows a minimal change thus acceleration is almost zero which is also evident from the inclination of the line between points 2 and 3. This phenomenon may have occurred due to breakdown in traffic (the vehicle under examination encountered a congestion), due to a sudden change of speed of the leading vehicle or due to a lane change of a slower moving vehicle in front of the one under examination. The vehicle from point 3 moves to point 4 where another catastrophe occurs and the system “jumps” to point 5. This catastrophic change may also occur for reasons similar but reversed to the above.

In the above example, four out of five characteristics of cusp catastrophe may be observed:

1. **Catastrophe:** the “jumps” from point 2 to point 3 and from point 4 to point 5 are catastrophic changes in the state variable.
2. **Bimodality:** While the system moves from point 1 to point 2 and from point 3 to 4, there are many cases where for the same values of speed and acceleration, the value of spacing is different. This is more evident from the Newell’s hypothesis graph on the right of Fig 2.
3. **Hysteresis:** “Jumps” between equilibrium states occur in different points depending on the state the system is already in. From unconstrained to constrained flow, the catastrophe occurs at point 2 and from constrained to unconstrained flow at point 4.
4. **Inaccessible behaviour:** The system does not take values between the equilibrium states. On Newell’s hypothesis graph, the system does not take any values inside the polygon 2-3-4-5.

The characteristic of divergence is not found on Figure 3, because, in this case, the vehicle has never zero acceleration and speed at the same point in time. Moreover, in Figure 3 the catastrophe is considered to occur in accordance with the Delay convention. The selection of convention is a major issue for applying catastrophe theory and also a subject of discussion in past research. Persaud and Hall [16] observed two values of speed for the same flow-density values in uncongested traffic. Because in the case of the Maxwell convention, the catastrophe occurs when the system crosses the bisector, the characteristic of bimodality doesn’t appear (any value of the control parameters correspond to one value of state variable – Figure 3). Thus, the Delay convention is the appropriate one. They also suggested that these bimodal conditions observed are not states of equilibrium. Forbes and Hall [10] and Acha-Daza and Hall [11] on the other hand suggested the use of Maxwell convention.

In microscopic traffic, the use of Maxwell convention dictates that the vehicle will change equilibrium states, when the acceleration reaches a zero value, while speed may take any value. On the other hand, the Delay convention dictates that the system’s equilibrium will change when the system reaches the edge of the folded area, practically when Equation 4 is fulfilled.

![Figure 3](image)

**Figure 3.** A simple example of cusp catastrophe. On the left is the cusp surface and on the right are the corresponding traffic states of the vehicle on the speed spacing relationship.
C. Equilibrium States Selection

A lane change of a lead vehicle with lower speed than the one examined, may cause a catastrophic change in the state variable (not including the change in spacing the moment immediately after the lane change). A sudden change in spacing signifies a change in the equilibrium of the system. However, accelerating and decelerating conditions cannot be accepted as equilibrium states, due to a paradox, which occurs with the characteristic of divergence. The characteristic of divergence suggests that when the system reaches a point of neutrality, then, there is equal possibility of moving to either state of equilibrium. In traffic, this point is when a vehicle is immobilized, i.e., in a traffic jam. Consequently, there is equal probability that the vehicle decelerates (turns up negative value in acceleration therefore moves backwards) or accelerates which is impossible from a practical point of view. Thus, accelerating and decelerating conditions cannot be considered as equilibrium states. On the other hand, aggressive and timid driving behaviour is in perfect accordance to the characteristics of cusp catastrophe.

Aggressive and timid drivers are classified by variables of acceleration/ deceleration, speed and spacing. Therefore, Newell’s hypothesis still applies for the occurrence of catastrophic phenomena in microscopic traffic flow variables. Furthermore, the categorization of drivers as aggressive and timid can not only adequately explain divergence, but can also explain the characteristic of inaccessible behaviour.

The physical implication of the aforementioned drivers’ classification is that, for every catastrophe in the state variable, a shift between aggressiveness and timidity occurs. Thus, no driver has to be classified a priori as aggressive or timid, since he/she is considered to be able to change equilibrium states. Additionally these equilibrium states are unique for each driver (and not for every category, aggressive or timid). The adoption of the above states as equilibrium states is based on the assumption that a driver’s behaviour is affected by the driving environment, as well as by his/her general attribute as a driver (meaning that an aggressive driver will, most likely, drive aggressively and vice versa).

D. Selection of Model Variables and Convention

In the present paper, spacing is considered as the state variable for the vehicle under study. Speed and acceleration are the control parameters. Acceleration as the normal factor, \( v \) and speed as the splitting factor, \( u \). The rationale for the selection of the factors is that speed can only take positive values, while acceleration can take both positive and negative values. Furthermore, the discrimination of aggressive and timid driving is based on speed and the change of speed meaning acceleration or deceleration.

By selecting the specific variables, the system’s potential can be quantified. The driver’s behaviour becomes a mathematical entity; thus, changes in the system are reflected by changes in the variables. Due to the definition of potential, the driver will always search for equilibrium. Consequently, if the driver is in the timid equilibrium plane, he/she will react to changes in the front vehicle’s speed mildly. Nevertheless, if the driver of the leading vehicle decelerates abruptly forcing the following vehicle to a sudden change in his/her driving characteristics, this can be translated as a catastrophic change from timid to aggressive equilibrium. It should be noted that it the potential shows when the system is in equilibrium and is not a measure of the “magnitude” of equilibrium. In other words, it cannot show how timid or aggressive is the driver but only if he/she is in one of the two equilibriums.

The transition of drivers between aggressiveness and timidity can only occur in points of zero acceleration/ deceleration (Maxwell convention). It shouldn’t be at any point of acceleration or deceleration that abstain significantly from zero because there is no reason to justify a change in driver’s reactions, considering that - by definition - a reaction means a different approach to a situation. Mathematically, at those points of zero acceleration/deceleration, the likelihood that the driver is aggressive or timid is equal for the two states. The equal possibility represents the debility of the observer to categorize the driver at the point of time when the driver chooses his/her reaction. It is a valid conclusion that the Delay convention is not the right one to be used because that would mean that catastrophe occurs at values of acceleration/ deceleration that differ from zero. Based on the above assumption, a driver will cross between equilibrium states (meaning, a driver will become timid if he/she was aggressive and vice versa) when acceleration is zero and at any value of speed.

According to Maxwell’s convention when the system reaches bimodality (a condition where two equal minima exist) and surpasses it by a bit, the system becomes metastable (in a condition of delicate balance). The system becomes susceptible to changing its condition to a state of more stable equilibrium (lesser potential). Maxwell convention dictates that the moment the system becomes metastable it moves to a more stable condition, the global minima [19].

E. Stochastic Cusp Catastrophe Modelling

Stochastic cusp catastrophe models are developed using the Cobb’s method [20] that converts cusp catastrophe's deterministic model into a stochastic one by adding a white noise Wiener process \( W(t) \) of \( \sigma^2 \) variance to the equation of motion of the dynamical system. Let \( y=y(t) \) be the system’s state variable, \( v \) be the normal factor, \( u \) be the splitting factor and \( V \) be the function of the system’s potential. Then [21]:

\[
\frac{\partial y}{\partial t} = -\frac{\partial V(y,v,u)}{\partial y}
\]

(5)

is an equation of motion for a dynamical system. This system will move towards states of equilibrium and will be at one when:

\[
-\frac{\partial V(y,v,u)}{\partial y} = 0.
\]

With the addition of the Wiener process (which is a continuous stochastic process) the equation becomes a stochastic differential equation (SDE):

\[
dY = \frac{\partial V(y,v,u)}{\partial y} dt + dW(t)
\]

(6)

This SDE is then affiliated with a probability density function that describes the allocation of system states at any moment in time. It can be expressed as follows:
\[ f(y) = \frac{\psi}{\sigma^2} \exp\left[\frac{\psi(y-\lambda)^2+\frac{1}{4}\psi(y-\lambda)^2-\frac{1}{2}(y-\lambda)^4}{\sigma^2}\right] \]  

(7)

where \( \psi \) is a normalizing constant and \( \lambda \) determines the origin of scale of the state variable.

Let there exist some measured values for the state variable \( y \) and, for the control parameters \( v, u \), and let them be \( Z_{i} = (Z_{i1}, Z_{i2}, \ldots, Z_{i\psi}), \ X_{i} = (X_{i1}, X_{i2}, \ldots, X_{i\psi}), \ Y_{i} = (Y_{i1}, Y_{i2}, \ldots, Y_{i\psi}) \) respectively. Then:

\[ y_{i} = w_{0} + w_{1}Z_{i1} + \ldots + w_{p}Z_{i\psi} \]
\[ v_{i} = a_{0} + a_{1}X_{i1} + \ldots + a_{q}X_{iq} \]
\[ u_{i} = \beta_{0} + \beta_{1}Y_{i1} + \ldots + \beta_{q}Y_{iq} \]  

(8)

where \( w_{0}, w_{1}, \ldots, w_{p}, a_{0}, a_{1}, \ldots, a_{q} \) and \( \beta_{0}, \beta_{1}, \ldots, \beta_{q} \) are first degree coefficients of a polynomial approach to the original smooth transformation.

Cobb’s method suffices in minimizing the following function \( L \) which is the negative log-likelihood function for a sample of observed values [20]:

\[ L(a, \beta, w, x, y, z) = \sum_{i=1}^{n} \log \psi_{i} - \sum_{i=1}^{n} \left( a_{i}y_{i} + \frac{1}{2}z_{i}^{2} + \frac{1}{4}y_{i}^{4} \right) \]  

(9)

The conversion of the deterministic catastrophe model to stochastic provides many advantages. First of all, a stochastic model is flexible enough in order to adapt to different conditions of the problem at hand. In addition, past attempts to fit any kind of data to the cusp catastrophe have failed or were successful in some parts of the research. With Cobb’s model, one cannot know the goodness of fit to the cusp catastrophe, but, if they fit best to the cusp catastrophe as compared to other simple, yet, fundamental models [20]. If the data fit better to cusp model, then it can be concluded that the cusp catastrophe is the best model to fit the data.

III. IMPLEMENTATION AND RESULTS

A. The Database

The data come from the program Next Generation Simulation (NGSIM) of Federal Highway Administration (FHWA) in the USA [24]. The available data consist of two different datasets; the first dataset includes the trajectories of the vehicles traveling on a 502.92 m segment of Interstate 80 at Emeryville, San Francisco, California with one entry ramp and an exit ramp just after the border of the test area. The second dataset refers to the microscopic characteristics of the available trajectories extracted every 0.1 sec. The data were collected at three different peak periods: 4:00pm-4:15pm, 5:00pm-5:15pm and 5:15pm-5:30pm at 13 of April 2005.

Using the above data, a 2-vehicle platoon analysis based on Laval [25] found that the hysteresis phenomenon observed in the macroscopic relationship between flow rate and density may be related to the behaviour of drivers which may be either timid or aggressive. Following this analysis a division of the available data into trajectories of aggressive and timid driving behaviour was made. The microscopic characteristics of the two subsets were extracted for the original dataset. The final dataset consists of speed, acceleration and spacing data for both aggressive and timid driving; these data will be used to fit cusp catastrophe models. The above consideration of vehicle trajectories in the prism of aggressive and timid driving behaviour is critical. In the present paper the application of cusp catastrophe enables to consider the case where a driver may shift between aggressive and timid driving despite of his predisposition to driving aggressively or timidly. This paper may also fit to Treiterer and Myer’s explanation of the hysteresis phenomenon as the difference between acceleration and deceleration which is comparable to Newell’s theory [22]. There might also be an extension to Zhang’s anticipation and relaxation model [23]. That is, if the driver anticipates the state of the traffic downstream, then a fit to the cusp catastrophe should not be achieved but a smoothed and continuous reaction is expected.

B. Models and Statistical Evaluation

For each vehicle in the dataset, five models are developed; Model \( M^{(0,0)}_{0.0} \) uses raw data (sampling interval equals to 0.1 sec), whereas the models \( M^{(0,0)}_{0.5} \) and \( M^{(0,0)}_{1.0} \) use aggregated information with sampling intervals equal to 0.5 sec and 1 sec respectively. Models \( M^{(1,5)}_{0.0} \) and \( M^{(1,10)}_{0.0} \) encompass memory. Memory is added by taking into consideration not only the traffic in the present interval \( t \) but also the traffic evolution, meaning the joint consideration of traffic in \( t \) and previous time intervals. Memory is generally defined by the dimension \( m \) and the delay \( \tau \) [26]. In this paper, models \( M^{(1,5)}_{0.0} \) and \( M^{(1,10)}_{0.0} \) encompass memory equal to 4 \((\tau=1, m=5)\) and 9 \((\tau=1, m=10)\) time intervals respectively. The specifications for the five different cusp catastrophe models are shown in Table 1.

<table>
<thead>
<tr>
<th>Model ( M^{(\tau,m)}_{(0,0)} )</th>
<th>Sampling Interval ( t ) (sec)</th>
<th>Memory proprieties ((\tau,m))</th>
<th>General modelling form</th>
</tr>
</thead>
<tbody>
<tr>
<td>( M^{(0,0)}_{0.1} )</td>
<td>0.1</td>
<td>( \tau=0, m=0 )</td>
<td>( y_{i} = w_{0} + w_{1}Z_{i1} )</td>
</tr>
<tr>
<td>( M^{(0,0)}_{0.5} )</td>
<td>0.5</td>
<td>( \tau=0, m=0 )</td>
<td>( y_{i} = w_{0} + w_{1}Z_{i1} )</td>
</tr>
<tr>
<td>( M^{(1,5)}_{0.1} )</td>
<td>0.5</td>
<td>( \tau=1, m=5 )</td>
<td>( y_{i} = w_{0} + w_{1}Z_{i1} + \ldots + w_{5}Z_{i5} )</td>
</tr>
<tr>
<td>( M^{(1,0)}_{1} )</td>
<td>1</td>
<td>( \tau=0, m=0 )</td>
<td>( y_{i} = w_{0} + w_{1}Z_{i1} )</td>
</tr>
<tr>
<td>( M^{(1,10)}_{0.1} )</td>
<td>1</td>
<td>( \tau=1, m=10 )</td>
<td>( y_{i} = w_{0} + w_{1}Z_{i1} + \ldots + w_{10}Z_{i10} )</td>
</tr>
</tbody>
</table>

Table 1: Cusp catastrophe models to be considered in the analysis.
The rationale in evaluating these five models is first to assess the change in the accuracy of fit of the cusp catastrophe model with the use of coarser traffic information. Previous research has shown that transportation modelling is sensitive to temporal aggregation; aggregation eliminates long memory characteristics and variance heterogeneity leading to smoothing traffic variations and creating a time series structure that has reduced sensitivity to changes in traffic [27]. Second, introducing memory to the cusp catastrophe models will enable the better approximation of the temporal dynamics of traffic flow [26].

The evaluation of the models and their fit to the available data is based on three criteria [28]; the first is based on the values of Akaike information criterion (AIC) [29], the corrected AIC for small sample sizes (AICc) [30] and BIC [31]; AIC penalizes the log-likelihood for the number of parameters of the model, whereas the BIC penalizes according to a function of the number of data points and the number of parameters. The model with the lowest AIC and BIC should be selected. The second criterion dictates that the values of the coefficients $w_0, w_1, \ldots, w_p$ (Equation 8) should be considerably different from zero [28]. The third criterion concerns the number of values of $(u_i, v_i)$ that lie in the bifurcation set; the number of values should be at least 10% [28].

The proposed modelling is compared to the linear and logistic regression models. The linear model (linear regression model) depicts a simpler response surface than that of the cusp model. Since changes in system's condition translate to position changes on that surface then it can be assumed that these changes occur rather smoothly in the linear model as opposed to the changes in the cusp model that are abrupt and catastrophic. The logistic model is an intermediate of the two previously mentioned models. The logistic curve doesn't have any degenerate critical points and can model sudden and significant changes of the state variable that coincide with a minute change in the independent variables [21]. Models are fitted using the Cusp v2.2 package programmed in R [21].

Results for the memory-less models ($M_{0.1}^{(0,0)}, M_{0.0}^{(0,0)}$, and $M_1^{(0,0)}$), are summarized in Table 2 for an aggressive and a timid driver respectively with regard to the AIC, AICc, BIC. Results for the models with memory ($M_{0.1}^{(1,5)}, M_{0.1}^{(1,10)}$) can be seen in Tables 3 for the same drivers as in Table 2 on test (unobserved) sample data. Further, memory seems to stabilize the performance of catastrophe models between aggressive and timid drivers.

### Table 2: Comparison of the different models for aggressive and timid driving.

<table>
<thead>
<tr>
<th>Models</th>
<th>Aggressive</th>
<th></th>
<th></th>
<th>Timid</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>AIC</td>
<td>AICc</td>
<td>BIC</td>
<td>AIC</td>
<td>AICc</td>
<td>BIC</td>
</tr>
<tr>
<td>Linear Model</td>
<td>4010.00</td>
<td>4010.10</td>
<td>4028.19</td>
<td>4423.62</td>
<td>4423.68</td>
<td>4441.73</td>
</tr>
<tr>
<td>Logistic Curve</td>
<td>3907.84</td>
<td>3907.93</td>
<td>3930.58</td>
<td>4008.77</td>
<td>4008.86</td>
<td>4031.40</td>
</tr>
<tr>
<td>Catastrophe Model</td>
<td>1031.47</td>
<td>1031.59</td>
<td>1058.75</td>
<td>745.56</td>
<td>745.69</td>
<td>772.72</td>
</tr>
<tr>
<td>$M_{0.1}^{(1,0)}$</td>
<td></td>
<td></td>
<td></td>
<td>4423.62</td>
<td>4423.68</td>
<td>4441.73</td>
</tr>
<tr>
<td>$M_{0.0}^{(0,0)}$</td>
<td></td>
<td></td>
<td></td>
<td>899.57</td>
<td>899.87</td>
<td>911.25</td>
</tr>
<tr>
<td>$M_1^{(0,0)}$</td>
<td></td>
<td></td>
<td></td>
<td>814.47</td>
<td>814.93</td>
<td>829.07</td>
</tr>
<tr>
<td>Linear Model</td>
<td>811.07</td>
<td>811.37</td>
<td>822.84</td>
<td>899.57</td>
<td>899.87</td>
<td>911.25</td>
</tr>
<tr>
<td>Logistic Curve</td>
<td>783.04</td>
<td>783.49</td>
<td>797.75</td>
<td>814.47</td>
<td>814.93</td>
<td>829.07</td>
</tr>
<tr>
<td>Catastrophe Model</td>
<td>220.33</td>
<td>220.96</td>
<td>237.98</td>
<td>164.81</td>
<td>165.45</td>
<td>182.33</td>
</tr>
<tr>
<td>$M_{0.1}^{(1,5)}$</td>
<td></td>
<td></td>
<td></td>
<td>455.97</td>
<td>456.60</td>
<td>464.91</td>
</tr>
<tr>
<td>$M_{0.1}^{(1,10)}$</td>
<td></td>
<td></td>
<td></td>
<td>411.78</td>
<td>412.00</td>
<td>422.00</td>
</tr>
<tr>
<td>Linear Model</td>
<td>399.14</td>
<td>399.76</td>
<td>408.14</td>
<td>455.97</td>
<td>456.60</td>
<td>464.91</td>
</tr>
<tr>
<td>Logistic Curve</td>
<td>388.23</td>
<td>389.17</td>
<td>399.48</td>
<td>410.82</td>
<td>411.78</td>
<td>422.00</td>
</tr>
<tr>
<td>Catastrophe Model</td>
<td>103.97</td>
<td>105.30</td>
<td>117.46</td>
<td>98.71</td>
<td>100.07</td>
<td>112.12</td>
</tr>
<tr>
<td>$M_{0.0}^{(1,5)}$</td>
<td></td>
<td></td>
<td></td>
<td>456.60</td>
<td>456.91</td>
<td>464.91</td>
</tr>
<tr>
<td>$M_{0.0}^{(1,10)}$</td>
<td></td>
<td></td>
<td></td>
<td>412.00</td>
<td>412.00</td>
<td>422.00</td>
</tr>
</tbody>
</table>

*Values in parenthesis are the values of $\tau$ and $m$. Value in subscript equals to the sampling rate.

### Table 3: Comparison of the different models with memory for aggressive and timid driving.

<table>
<thead>
<tr>
<th>Models</th>
<th>Aggressive</th>
<th></th>
<th></th>
<th>Timid</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>AIC</td>
<td>AICc</td>
<td>BIC</td>
<td>AIC</td>
<td>AICc</td>
<td>BIC</td>
</tr>
<tr>
<td>Linear Model</td>
<td>972.63</td>
<td>977.09</td>
<td>1019.59</td>
<td>1094.00</td>
<td>1098.57</td>
<td>1140.60</td>
</tr>
<tr>
<td>Logistic Curve</td>
<td>980.78</td>
<td>985.83</td>
<td>1030.66</td>
<td>1218.97</td>
<td>1224.16</td>
<td>1268.49</td>
</tr>
<tr>
<td>Catastrophe Model</td>
<td>324.17</td>
<td>329.87</td>
<td>376.99</td>
<td>343.58</td>
<td>349.43</td>
<td>396.01</td>
</tr>
<tr>
<td>$M_{0.1}^{(1,5)}$</td>
<td></td>
<td></td>
<td></td>
<td>1094.00</td>
<td>1098.57</td>
<td>1140.60</td>
</tr>
<tr>
<td>$M_{0.1}^{(1,10)}$</td>
<td></td>
<td></td>
<td></td>
<td>1224.16</td>
<td>1224.16</td>
<td>1268.49</td>
</tr>
<tr>
<td>Linear Model</td>
<td>441.66</td>
<td>495.28</td>
<td>510.91</td>
<td>549.50</td>
<td>604.62</td>
<td>618.31</td>
</tr>
<tr>
<td>Logistic Curve</td>
<td>168.72</td>
<td>227.39</td>
<td>240.21</td>
<td>348.43</td>
<td>408.77</td>
<td>419.46</td>
</tr>
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<td>Catastrophe Model</td>
<td>130.52</td>
<td>194.63</td>
<td>204.24</td>
<td>191.95</td>
<td>257.95</td>
<td>265.19</td>
</tr>
</tbody>
</table>

*Values in parenthesis are the values of $\tau$ and $m$. Value in subscript equals to the sampling rate.
Moreover, the fit of cusp catastrophe models to aggressive drivers is better than the one of timid. In order to establish a solid comparison between the cusp catastrophe models with memory and the base model \( M^{(0,0)}_{0.1} \), the root mean square error RMSE is calculated; results are presented in Table 4. As it can be observed, the memory improves the accuracy of the cusp catastrophe models. However, further research is needed on the optimization of the memory characteristics \((\tau,m)\) with respect to the predictability.

The memory models did not perform well against the memory-less models \( M^{(0,0)}_{0.5} \) and \( M^{(0,0)}_{1} \). This may be attributed to the level of aggregation. As it can be seen from Table 2, aggregation plays a significant role to the model’s accuracy. Choosing a different aggregation level for the information for the memory models, i.e., increasing the sampling interval will increase the accuracy. Similar results may be also traced in literature using autoregressive time series models for macroscopic traffic flow prediction [27].

### Table 4: Root mean square error for all models using raw data.

<table>
<thead>
<tr>
<th>Models</th>
<th>Aggressive</th>
<th>Timid</th>
</tr>
</thead>
<tbody>
<tr>
<td>( M^{(0,0)}_{0.1} )</td>
<td>0.491</td>
<td>0.547</td>
</tr>
<tr>
<td>( M^{(1,5)}_{0.1} )</td>
<td>0.327</td>
<td>0.511</td>
</tr>
<tr>
<td>( M^{(1,10)}_{0.1} )</td>
<td>0.304</td>
<td>0.309</td>
</tr>
</tbody>
</table>

Overall, the findings in Table 2 and Table 3 show that cusp catastrophe can fit the microscopic traffic data with high accuracy when compared to the logistic or the linear regression models. This applies to both aggressive and timid driving. A number of similarly specified tests were run verifying the statement. Computationally, the method requires less than a minute for all of the cases and less than 30 sec for most on an average computer and for testing more than 200 observations to calculate the parameters and provide the results for the statistical tests.

Nevertheless, in some of the tests the AIC/AICc/BIC values of linear regression model and cusp catastrophe were close showing that there are certain traffic conditions (e.g. in high volume or small value of spacing in high speed) where freeway traffic becomes metastable but preserves its linearity. This does not necessarily mean that the system remains in the same state of equilibrium, but, that it changes equilibrium states in a linear fashion and not abruptly. In this case, it can be assumed that the driver anticipates the change in traffic conditions, thus, smoothing his/her reactions.

Moreover, the fit to cusp catastrophe significantly improves with the use of coarser traffic information which is evident when the models \( M^{(0,0)}_{0.1} \), \( M^{(0,0)}_{0.5} \) and \( M^{(0,0)}_{1} \) are compared to each other; the differences between the cusp catastrophe models and the linear and logistic models are smoothed out with aggregation.

It should be noted that the values of the statistical tests show if a data set fits to the cusp catastrophe model given a certain convention (Maxwell or Delay). However, one cannot exclude the fact that in some cases the linearity of the system was preserved even if the model showed a good fit to the nonlinear model of cusp catastrophe. In those cases the potential of the system moved between equilibriums smoothly and in a linear manner and not in a catastrophic way. This is - theoretically - in accordance with the fact that the system either moves to congested conditions in a linear manner immediately after the critical density point or moves first to point of higher flow and density than critical density point before “catastrophically” reaching congested conditions. The same pattern was observed in Acha-Daza and Hall [11] where the cusp catastrophe model under the Maxwell convention showed mixed results.

Two possible scenarios, which are not mutually exclusive, can be derived from the above: i) Even in a metastable state, the system does not necessarily become nonlinear, and ii) Maxwell convention is not the appropriate one. The Delay convention is admittedly also not the appropriate one as it was discussed above. Maxwell convention by its’ definition deletes the characteristic of bimodality (there can be only one value of spacing for every set of speed and acceleration values) which might not be the case here. Additionally, when the system is in the bisector of the bifurcation set, according to Maxwell convention, the driver immediately changes equilibrium, i.e., if the driver is timid he/she becomes aggressive the moment his acceleration becomes zero and vice versa. That is a posteriori not true. Consequently, a different convention should be used. As Gilmore [32] has underlined, in between these two extremes (Maxwell and Delay), infinite conventions are available.

Due to the infinity of the cases, a different approach is deemed necessary. It can be said that if a system crosses the bisector of the bifurcation set, then it reaches a critical region of spacing. In this region the system is susceptible to a catastrophic change in its equilibrium (Figure 4). This critical region lies between the Maxwell’s and the Delay’s convention catastrophe points.

The system will cross a separate region (reversed critical) before the one described above in order to reach the critical region. These regions depend on the equilibrium state that the system is already in and are reverse if the equilibrium changes. These regions of driver’s behaviour can be seen in Figure 5. Function \( y(x) \) is arbitrary. A hypothesis on Figure 5 is that the critical region is marginalized not by function \( y(x) \) and the bisector of the bifurcation set but by two functions like \( y(x) \).

In practice terms, Maxwell dictates that acceleration should be zero for a system to change equilibriums. Delay on the other hand, requires that Equation 4 is met in order for the catastrophe to occur. These are demanding cases in terms of accuracy. If a model doesn’t fit to the convention selected then the entire model must be discarded. In the proposed modelling the stiffness of the two extreme conventions is alleviated.

From both past and this paper’s findings, the applicability of catastrophe theory in traffic seems to be a valid assumption.
Nevertheless, catastrophe theory fits only in some circumstances, whereas in others it fails. Findings show that traffic does not necessarily change equilibrium in a non-linear fashion. This observation corresponds to some findings in macroscopic traffic [10]. This leads to the conclusion that while catastrophe theory cannot be used for prediction, it can be efficiently used for prevention. If the system remains linear, then management and prediction are easier than when the system is non-linear.

If the system is in the critical region then a catastrophe is imminent. For the catastrophe to be avoided, one must consider that while speed and acceleration depend solely on the subject vehicle, spacing depends also on the leading vehicle (if the leading vehicle is considered to act independently compared to the examined vehicle, then spacing is truly an independent variable for the subject vehicle). If the leading vehicle and the subject vehicle adjust their speed accordingly, then change of equilibrium can occur in a linear manner or not occur at all. Using Cobb’s stochastic method, this translates to a better (lower) AIC/AICc/BIC value in the linear regression than in the catastrophe model. In other words, metastable systems can be manipulated.

![Fig. 4. On the left (a) the system is moving from aggressive to timid driving and on the right (b) the system moves from timid to aggressive driving. The critical regions are the areas where the catastrophe occurs.](image)

![Fig. 5. Critical regions in the driver’s behaviour bifurcation set.](image)

Additionally, if the vehicle is in the critical region, for known values of speed and acceleration, Equation 2 can be solved to find the value of spacing where the potential is minimum (values of spacing that resulted from Equation 2 should be evaluated in Equation 1 to determine the value of potential).

**IV. CONCLUSION**

Recently, various models for incorporating driver’s behaviour have been introduced to microscopic traffic flow. Most of them emphasize the need to make a distinction between aggressive/timid drivers. This paper extends past research on behavioural microscopic traffic flow models by considering that each driver – regardless of being aggressive or timid – may shift his/her behaviour and behave aggressively or timidly during his/her driving. Such behaviour in the modelling is introduced by the theory of cusp catastrophe. Based on the proposed modelling approach, the spacing a driver chooses is a function of its speed and acceleration. Moreover, the driver’s behaviour encompasses two equilibrium states -being aggressive or timid - and any shift between these states is considered to occur as a catastrophe.

Overall, five different modelling formulations are tested in order to evaluate the effect of using aggregated data, as well as the importance of introducing the temporal evolution of traffic to the accuracy of the catastrophe models. Moreover, all catastrophe models are compared to classical linear and logistic approximations. Results demonstrated that the cusp catastrophe models perform better than linear or logistic models of traffic flow. The models using aggregated data performs significantly better than those using raw data, most likely due to the fact that traffic dynamics become smoother when using data in coarser intervals. Interestingly, the incorporation of a memory mechanism that introduces past information of speed and acceleration to the models is
beneficial to the quality of the fit of both aggressive and timid drivers.

From a methodological standpoint, results show that a fit to the cusp catastrophe is achievable. The proposed approach is based on measured traffic information that can be collected directly by each vehicle and does not incorporate any estimations on the manner traffic evolve over time. The use of microscopic data provides the necessary background for a self-reliant system used by a single vehicle. Yet, it is evident that a lot of ground has to be covered in order to provide a sophisticated model for use in an everyday vehicle. Issues such as systematic evaluation of the models’ accuracy using extended datasets, the extent and penetration of automation in driving, the link between microscopic and macroscopic traffic, as well as traffic stabilization should be thoroughly addressed [34]–[36].

The concept behind the application presented in this paper is to control the microscopic kinematic characteristics to eliminate catastrophic transitions between equilibrium states and allow only smooth transitions to occur. This can be achieved by controlling the speed of the vehicle to avoid falling into the critical regions of driver’s behaviour described in Figure 5. This may lead to a dynamic speed management algorithm for use in automated vehicles with implications to driving safety and comfort. The proposed methodology may be integrated to autonomous and self-driven vehicles. The vehicle, by measuring its’ own variables and the distance from the lead vehicle, will be able to adjust its’ speed and acceleration and will manage to efficiently drive through traffic at any circumstances. In comparison to cruise control and adaptive cruise control technologies, the vehicle will be able to increase its’ speed when an opportunity arises and to decrease it not only to avoid accidents but to normalize its’ speed which resulting in efficient fuel consumption. Additionally, the advantage of the proposed methodology is that the system may operate at any speed in comparison with most systems that operate at specific speed boundaries.

From a theoretical standpoint, catastrophe theory can explain the metastability of traffic, by running what-if analyses, i.e., what will be the new system’s equilibrium if traffic’s equilibrium changes. By using the classic traffic flow models, such as LWR, in parallel to catastrophe theory, a margin of values, e.g., for flow and density, can be created so that the discontinuity of traffic is anticipated. The manner the developed models will respond to other traffic phenomena such as offered gap acceptance is a subject for further research.

V. REFERENCES


Alexandros E. Papacharalampous was born in Athens, Greece, in 1989. He received his 5-year diploma in Civil Engineering with specialization in Traffic and Transportation Engineering from the National Technical University of Athens, Greece, in 2011.

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She has extensive research experience in applying computational intelligent modelling to various transportation applications including traffic engineering, safety, air transportation, maritime transportation and urban transportation networks. She has also worked on nonlinear dynamics and statistical applications to traffic engineering. Her professional experience includes projects and consultancies, in a national and European level mostly focusing on urban traffic flow management, public transport and traffic safety. Web-page: http://users.ntua.gr/elenivl/