Some problems on Cayley graphs

Elena Konstantinova

Sobolev Institute of Mathematics, Siberian Branch of Russian Academy of Sciences, Novosibirsk, Russia

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Abstract

This survey paper presents the historical development of some problems on Cayley graphs which are interesting to graph and group theorists such as Hamiltonicity or diameter problems, to computer scientists and molecular biologists such as pancake problem or sorting by reversals, to coding theorists such as the vertex reconstruction problem related to error-correcting codes but not related to Ulam’s problem.

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1. Introduction

The definition of Cayley graph was introduced by Arthur Cayley in 1878 to explain the concept of abstract groups which are described by a set of generators. In the last 50 years, the theory of Cayley graphs has been grown into a substantial branch in algebraic graph theory. It has relations with some classical problems in pure mathematics such as classification, isomorphism and enumeration of Cayley graphs (see for example surveys by Xu [1], Li [2] and the handbook by Babai [3]), and many practical problems which are studied by graph and group theorists, by computer scientists, molecular biologists and coding theorists. In this paper, we present such problems for...
Cayley graphs having interesting connections with applications. In molecular biology, Cayley graphs on the symmetric $Sym_n$ and hyperoctahedral $B_n$ groups appear since permutations and signed permutations are used to represent sequences of genes in chromosomes and genomes, and some operations on permutations represent evolutionary events. In the 1980’s it was shown that the difference in genomes is explained by a small number of reversals which are the operations reversing the order of a substring of a permutation [4]. The problem of determining the smallest number of reversals required to transform a given permutation into the identity permutation is called sorting by reversals [5]. It is investigated by many researchers in molecular biology. This problem is also related to the well-known combinatorial pancake problem [6] in which so-called pancake (unburnt and burnt) Cayley graphs on $Sym_n$ and $B_n$ play the leading role. Both these problems are connected to the classical problem of establishing the diameter of a Cayley graph which also arises in computer science in a natural way since Cayley graphs are used in the representation of interconnection networks. The vertices in these graphs correspond to processing elements, memory modules, and the edges correspond to communication lines. We also pay attention to the vertex reconstruction problem which comes from coding theory, and is related to error-correcting codes but not related to Ulam’s problem [7,8]. Initially this problem was considered for distance-regular graphs such as Hamming and Johnson graphs (the first one is also a Cayley graph), however, this problem is much more complicated for graphs which are not distance-regular. Cayley graphs of this kind arise on the symmetric and hyperoctahedral groups. To solve this problem it is essential to investigate their structural and combinatorial properties. For instance, it is important to know about cycles of different lengths in a graph, in particular, about the existence of a Hamiltonian cycle, that is, well-known Hamiltonicity problem [9]. The Hamiltonicity conjectures on vertex-transitive and Cayley graphs were picked out by two invited speakers at the First IPM Conference on Algebraic Graph Theory as the most important conjectures in Algebraic Graph Theory [10].

So, in this paper we have emphasized the application variety of Cayley graphs in solving combinatorial, graph-theoretical and applied problems. The main results in Hamiltonicity and diameter problems, the vertex reconstruction problem, the problem of sorting by reversals and pancake problems for various Cayley graphs are presented. We begin with basic definitions, notations, general results and some examples of Cayley graphs. We also present their combinatorial, structural and symmetry properties. All graphs considered in the paper are assumed to be finite.

2. Groups and graphs: definitions, notations, general results

Let $G$ be a finite group. The elements of a subset $S$ of a group $G$ are called generators of $G$, and $S$ is said to be a generating set, if every element of $G$ can be expressed as a finite product of generators. We also say that $G$ is generated by $S$. The identity of a group $G$ is denoted by $e$ and the operation is written as multiplication. A subset $S$ of $G$ is identity free if $e \notin S$ and it is symmetric (or closed under inverses) if $s \in S$ implies $s^{-1} \in S$. The last condition can be also denoted by $S = S^{-1}$, where $S^{-1} = \{s^{-1} : s \in S\}$.

A permutation $\pi$ on the set $X = \{1, \ldots, n\}$ is a bijective mapping from $X$ to $X$. We write a permutation $\pi$ in one-line notation as $\pi = [\pi_1, \pi_2, \ldots, \pi_n]$, where $\pi_i = \pi(i)$ are the images of the elements for every $i \in \{1, \ldots, n\}$. We denote by $Sym_n$ the group of all permutations acting on the set $\{1, \ldots, n\}$, also called the symmetric group. Its cardinality is $n!$. For a permutation $\pi$ let $\pi^{-1}$ be the inverse of $\pi$ and $\pi \pi^{-1} = \pi^{-1} \pi = I$, where $I = [1, 2, \ldots, n]$ is the identity permutation. A transposition $t_{i,j}$ interchanges positions $i$ and $j$ when acting on the right, i.e., $[\ldots, \pi_i, \ldots, \pi_j, \ldots]t_{i,j} = [\ldots, \pi_j, \ldots, \pi_i, \ldots]$. A reversal $r_{i,j}$ is the operation of reversing
segments \([i, j]\), \(1 \leq i < j \leq n\), of a permutation, i.e., \([\ldots, \pi_i, \pi_{i+1}, \ldots, \pi_j-1, \pi_j, \ldots]\) \(r_{i,j} = [\ldots, \pi_i, \pi_j, \pi_j-1, \pi_j, \pi_i, \ldots]\). The reversal distance \(d(\pi, \tau)\) between two permutations \(\pi\) and \(\tau\) is the least number \(d\) of reversals needed to transform \(\pi\) into \(\tau\), i.e., \(\pi r_{i_1,j_1} \cdots r_{i_d,j_d} = \tau\).

The hyperoctahedral group \(B_n\) is defined as the group of all permutations \(\pi^\sigma\) acting on the set \(\{\pm 1, \ldots, \pm n\}\) such that \(\pi^\sigma(-i) = -\pi^\sigma(i)\) for all \(i \in \{1, \ldots, n\}\). An element of \(B_n\) is a signed permutation, i.e., a permutation with a sign attached to every entry and determined by two pieces of information: \(|\pi(|i|)|\), which permutes \(\{1, \ldots, n\}\), and the sign of \(\pi^\sigma(i)\) for \(1 \leq i \leq n\). This gives a bijection between \(B_n\) and the wreath product \(\mathbb{Z}_2 \wr \text{Sym}_n\) of the “sign-change” cyclic group \(\mathbb{Z}_2\) with the symmetric group \(\text{Sym}_n\); thus \(|B_n| = 2^n n!\). We also use the compact one-line notation for a signed permutation \(\pi^\sigma\) as \([\pi_1, \bar{\pi}_2, \ldots, \bar{\pi}_i, \ldots, \pi_n]\), where a bar is written over each element with a negative sign. A sign-change transposition \(t_{ij}\), \(i \neq j\), switches two elements \(i\) and \(j\) and their signs, e.g., \([\ldots, \pi_i, \ldots, \bar{\pi}_j, \ldots, \pi_j, \ldots]t_{ij} = [\ldots, \pi_i, \ldots, \pi_j, \ldots, \bar{\pi}_j, \ldots]\), and a sign-change “transposition” \(t_{ii}^\sigma\) changes the sign of the \(i\)th element, e.g., \([\ldots, \pi_i, \ldots]t_{ii}^\sigma = [\ldots, \bar{\pi}_i, \ldots]\). A sign-change reversal \(r_{i,j}^\sigma\), \(1 \leq i \leq j \leq n\), is the operation of reversing segments \([i, j]\) of a signed permutation \(\pi^\sigma\) with flipping the signs of its elements, e.g., \([\ldots, \pi_i, \bar{\pi}_{i+1}, \ldots, \pi_j-1, \pi_j, \ldots]r_{i,j}^\sigma = [\ldots, \bar{\pi}_j, \bar{\pi}_{j-1}, \ldots, \pi_{i+1}, \pi_i, \ldots]\). The reversal distance \(\rho(\pi^\sigma, \tau^\sigma)\) between two signed permutations \(\pi^\sigma\) and \(\tau^\sigma\) is the least number \(\rho\) of sign-change reversals needed to transform \(\pi^\sigma\) into \(\tau^\sigma\), i.e., \(\pi^\sigma r_{i_1,j_1}^\sigma \cdots r_{i_d,j_d}^\sigma = \tau^\sigma\).

Let \(S \subseteq G\) be an identity free and symmetric generating set of a finite group \(G\). In the Cayley graph \(\Gamma = \text{Cay}(G, S) = (V, E)\) vertices correspond to the elements of the group, i.e., \(V = G\), and edges correspond to multiplication on the right by generators, i.e., \(E = \{(g, gs) : g \in G, s \in S\}\). The identity free condition means that there are no loops in \(\Gamma\), and the symmetry condition means that when there is an edge from \(g\) to \(gs\), there is also an edge from \(gs\) to \((gs)s^{-1} = g\). If the symmetry condition does not hold in the definition of the Cayley graph then we have Cayley digraphs which are not considered in this paper.

A permutation \(\sigma\) of the vertex set of a graph \(\Gamma\) is called an automorphism provided that \((u, v)\) is an edge of \(\Gamma\) iff \(\{\sigma(u), \sigma(v)\}\) is an edge of \(\Gamma\). A graph \(\Gamma\) is said to be vertex-transitive if for any two vertices \(u\) and \(v\) of \(\Gamma\), there is an automorphism \(\sigma\) of \(\Gamma\) satisfying \(\sigma(u) = v\). A graph \(\Gamma\) is said to be edge-transitive if for any pair of edges \(x\) and \(y\) of \(\Gamma\), there is an automorphism \(\sigma\) of \(\Gamma\) that maps \(x\) into \(y\). These symmetry properties require that every vertex or every edge in a graph \(\Gamma\) looks the same and these two properties are not interchangable. A graph \(\Gamma\) is said to be regular of degree \(k\) (or \(k\)-regular) if every vertex has degree \(k\). A regular graph of degree \(3\) is called cubic.

**Proposition 1.** Let \(S\) be a symmetric set of generators for a group \(G\). The Cayley graph \(\Gamma = \text{Cay}(G, S)\) has the following properties:

(i) it is a connected regular graph of degree equal to the cardinality of \(S\);
(ii) it is a vertex-transitive graph.

**Proposition 2.** Not every vertex-transitive graph is a Cayley graph.

The simplest example is the Petersen graph, that is a cubic graph of order 10, which is a vertex-transitive but not a Cayley graph. A systematic study of those orders \(n\) for which there exist non-Cayley vertex-transitive graphs was initiated by Marušić [11] and then continued by McKay and Praeger [12,13].
Denote by \(d(u, v)\) the path distance between the vertices \(u\) and \(v\) of a connected graph \(\Gamma = (V, E)\) and by \(d(\Gamma) = \max \{d(u, v) : u, v \in V\}\) the diameter of \(\Gamma\). Let \(S_r(v) = \{u \in V : d(v, u) = r\}\) and \(B_r(v) = \{u \in V : d(v, u) \leq r\}\) be the sphere and ball of radius \(r\) centered at the vertex \(v\), respectively. Then all vertices \(u \in B_r(v)\) are \(r\)-neighbors of the vertex \(v\). A simple connected graph \(\Gamma\) is distance-regular if there are integers \(b_i, c_i\) for every \(i \geq 0\) such that for any two vertices \(x\) and \(y\) at distance \(i = d(x, y)\) there are precisely \(c_i\) neighbors of \(y\) in \(S_{i-1}(x)\) and \(b_i\) neighbors of \(y\) in \(S_{i+1}(x)\). Evidently \(\Gamma\) is regular of valency \(k = b_0\). The numbers \(c_i, b_i\) and \(a_i = k - b_i - c_i, i = 0, \ldots, d\), where \(d = d(\Gamma)\) is the diameter of \(\Gamma\), are called the intersection numbers of \(\Gamma\) and the sequence \((b_0, b_1, \ldots, b_{d-1}; c_1, c_2, \ldots, c_d)\) is called the intersection array of \(\Gamma\). A simple connected graph \(\Gamma\) is distance-transitive if, for any two arbitrary-chosen pairs of vertices \((v, u)\) and \((v', u')\) at the same distance \(d(v, u) = d(v', u')\), there is an automorphism \(\sigma\) of \(\Gamma\) satisfying \(\sigma(v) = v'\) and \(\sigma(u) = u'\). It is obvious that any distance-transitive graph is vertex-transitive as well as distance-regular. The converse is not true in general (for details see [14]).

3. Some families of Cayley graphs

The hypercube \(H_n\) is the graph with vertex set \(\{x_1x_2\cdots x_n : x_i \in \{0, 1\}\}\) in which two vertices \((v_1v_2\cdots v_n)\) and \((u_1u_2\cdots u_n)\) are adjacent if and only if \(v_i = u_i\) for all but one \(i, 1 \leq i \leq n\). It has diameter and degree equal to \(n\) and can be considered as the cartesian product of \(n\) complete graphs \(K_2\). The hypercube \(H_n\) is the Cayley graph of the group \(\mathbb{Z}_2^n\) with \(n\) generators from the set \(S = \{(0, \ldots, 0, 1, 0, \ldots, 0), 0 \leq i \leq n - 1\}\). This graph is also the Cayley graph of the subgroup of \(\text{Sym}_{2n}\) of \(2^n\) elements generated by the \(n\) transpositions \((2i - 1, 2i), 1 \leq i \leq n\). It is a distance-transitive graph. This graph is considered in coding theory, computer science, algebraic graph theory.

The butterfly graph \(BF_n\) is the Cayley graph with vertex set \(V = \mathbb{Z}_n \times \mathbb{Z}_2^n, |V| = n2^n\), and with edges defined as follows. Any vertex \((i, x)\) in \(V\), where \(0 \leq i \leq n - 1\) and \(x = (x_0x_1\cdots x_{n-1})\), is connected to \((i + 1, x)\) and \((i + 1, x(i))\), where \(x(i)\) denotes the string which is derived from \(x\) by replacing \(x_i\) by \(1 - x_i\). All arithmetic on indices \(i\) is assumed to be modulo \(n\). Thus, \(BF_n\) is derived from \(H_n\) by replacing each vertex \(x\) by a cycle of length \(n\), however, the vertices of this cycle are connected to vertices of other cycles in a different way such that the degree is 4 (for \(n \geq 3\)). For example, \(BF_2 = H_3\) and \(BF_1 = K_2\). The diameter of \(BF_n\) is \([\frac{3n}{2}\]\). This graph is not edge-transitive, not distance-regular and hence not distance-transitive. This graph is also the Cayley graph on the subgroup of \(\text{Sym}_{2n}\) of \(n2^n\) elements generated by \((12\cdots 2n)^2\) and \((12\cdots 2n)^2(12)\). It is used as a model for interconnection networks in computer science.

The Hamming graph \(L_n(q)\) is defined on the Hamming space \(F_q^n\) (where \(F_q\) is the field of \(q\) elements) consisting of the \(q^n\) vectors of length \(n\) over the field \(F_q, q \geq 2\). This space is endowed with the Hamming distance \(d(x, y)\) which equals to the number of coordinate positions in which \(x\) and \(y\) differ. So, the Hamming graph has vertex set given by the vector space \(F_q^n\) where \((x, y)\) is an edge of \(L_n(q)\) if and only if \(d(x, y) = 1\). It is the Cayley graph on the additive group \(F_q^n\) when we take the generating set \(S = \{xe_i : x \in (F_q)^\times, 1 \leq i \leq n\},\) where \((F_q)^\times\) is the cartesian product of \(n\) copies of \(F_q\) and \(e_i = (0, \ldots, 0, 1, 0, \ldots, 0)\) are the standard basis vectors of \(F_q^n\). The Hamming graph \(L_n(q)\) is distance-transitive with \(b_j = (n - j)(q - 1)\) and \(c_j = j\) for \(0 \leq j \leq n\), where \(n\) is its diameter. The Hamming graph is well-known in coding theory.
The transposition graph $\text{Sym}_n(T)$ is defined on the symmetric group $\text{Sym}_n$ and generated by transpositions from the set $T = \{t_{i,j} \in \text{Sym}_n, 1 \leq i < j \leq n\}$. The distance in this graph is defined as the least number of transpositions transforming one permutation into another. The transposition graph $\text{Sym}_n(T), n \geq 3$, is a connected bipartite $\left(\frac{n}{2}\right)$-regular graph of order $n!$ and diameter $n - 1$ without subgraphs isomorphic to $K_{2,4}$, where $K_{p,q}$ is the complete bipartite graph with $p$ and $q$ vertices in the two parts, respectively. This graph is edge-transitive but not distance-regular and hence not distance-transitive.

The bubble-sort graph $\text{Sym}_n(t)$ on $\text{Sym}_n$ is generated by the transpositions from the set $t = \{t_{i,i+1} \in \text{Sym}_n, 1 \leq i < n\}$. These transpositions $t_{i,i+1}$ are 2-cycles interchanging $i$ and $i + 1$, and they also known as the set of the $n - 1$ Coxeter generators of $\text{Sym}_n$ (for details see [15]). The distance in this graph is determined in the usual way. The graph $\text{Sym}_n(t), n \geq 3$, is a connected bipartite $(n - 1)$-regular graph of order $n!$ and diameter $\left(\frac{n}{2}\right)$ without subgraphs isomorphic to $K_{2,3}$. It is not edge-transitive, not distance-regular and hence not distance-transitive.

The star graph $\text{Sym}_n(st)$ on $\text{Sym}_n$ is generated by the transpositions from the set $st = \{t_{i,i} \in \text{Sym}_n, 1 < i \leq n\}$. For $n \geq 3$, it is a connected bipartite $(n - 1)$-regular edge-transitive but not distance-regular and hence not distance-transitive graph of order $n!$ and diameter $\left\lfloor \frac{3(n-1)}{2} \right\rfloor$ without cycles of lengths 3, 4, 5, 7. This graph is one of the most investigated in the theory of interconnection networks since many parallel algorithms are efficiently mapped on this graph.

The reversal graph $\text{Sym}_n(R)$ is defined on $\text{Sym}_n$ and generated by the reversals from the set $R = \{r_{i,j} \in \text{Sym}_n, 1 \leq i < j \leq n\}$. The distance in this graph corresponds to the reversal distance between two permutations. For $n \geq 3$, this graph is a connected regular of degree $\left(\frac{n}{2}\right)$ and order $n!$ with diameter $n - 1$. It is not edge-transitive, not distance-regular and hence not distance-transitive. It does not contain triangles nor subgraphs isomorphic to $K_{2,4}$.

The (unburnt) pancake graph $\text{Sym}_n(PR)$ on $\text{Sym}_n$ is generated by the prefix-reversals from the set $PR = \{r_{1,i} \in \text{Sym}_n, 1 < i \leq n\}$. The distance in this graph is defined as the least number of the prefix-reversals transforming one permutation into another. The diameter of $\text{Sym}_n(PR)$ is called the prefix-reversal diameter. The pancake graph $\text{Sym}_n(PR), n \geq 3$, is a connected $(n - 1)$-regular graph of order $n!$ without cycles of lengths 3, 4, 5. It is not edge-transitive, not distance-regular and hence not distance-transitive for $n \geq 4$.

The transposition graph $B_n(T^\sigma)$ on the hyperoctahedral group $B_n$ is generated by the sign-change transpositions from the set $T^\sigma = \{t^\sigma_{ij} \in B_n, 1 \leq i < j \leq n\}$. The distance in this graph is defined as the least number of the sign-change transpositions transforming one permutation into another. The graph $B_n(T^\sigma), n \geq 2$, is a connected bipartite $\left(\frac{n+1}{2}\right)$-regular graph of order $2^n n!$ which does not contain subgraphs isomorphic to $K_{2,3}$.

The reversal graph $B_n(R^\sigma)$ is defined on $B_n$ and generated by the sign-change reversals from the set $R^\sigma = \{r^\sigma_{i,j} \in B_n, 1 \leq i < j \leq n\}$. The distance corresponds to the reversal distance between two signed permutations. The graph $B_n(R^\sigma), n \geq 2$, is a connected $\left(\frac{n+1}{2}\right)$-regular graph of order $2^n n!$ and diameter $n + 1$. It does not contain triangles nor subgraphs isomorphic to $K_{2,3}$. It is not edge-transitive, not distance-regular and hence not distance-transitive.

The burnt pancake graph $B_n(PR^\sigma)$ is defined on $B_n$ and generated by the sign-change prefix-reversals from the set $PR^\sigma = \{r^\sigma_{i} \in B_n, 1 \leq i \leq n\}$. The distance in this graph is defined as the minimal number of the sign-change prefix-reversals transforming one signed permutation into another. The diameter of $B_n(PR^\sigma)$ is called the burnt prefix-reversal diameter. The graph $B_n(PR^\sigma), n \geq 2$, is a connected $(2n - 1)$-regular graph of order $2^n n!$. It does not contain triangles
nor subgraphs isomorphic to \( K_{2,3} \). It is not edge-transitive, not distance-regular and hence not distance-transitive.

4. Hamiltonicity problem

Let \( \Gamma = (V, E) \) be a connected graph where \( V = \{v_1, v_2, \ldots, v_n\} \). A Hamiltonian cycle in \( \Gamma \) is a spanning cycle \((v_1, v_2, \ldots, v_n, v_1)\) and a Hamiltonian path in \( \Gamma \) is a path \((v_1, v_2, \ldots, v_n)\). We also say that a graph is Hamiltonian if it contains a Hamiltonian cycle. The Hamiltonicity problem, that is to check whether a graph is a Hamiltonian, was stated by Sir W.R. Hamilton in the 1850s as it was mentioned in the survey paper by Gould [16]. Finding Hamiltonian cycles in Cayley graphs was initiated in 1959 by Rapaport-Strasser [17]. For a finite group \( G \) with a generating set \( S, |S| \leq 3 \), presented by involutions, where an element \( \alpha \in G \) is called an involution, if \( \alpha^2 = 1 \), it was proved the following theorem.

**Theorem 3** [17]. Let \( G \) be a finite group, generated by three involutions \( \alpha, \beta, \gamma \) such that \( \alpha\beta = \beta\alpha \). Then the Cayley graph \( \Gamma = Cay(G, \{\alpha, \beta, \gamma\}) \) has a Hamiltonian cycle.

A finite group generated by two elements was considered by Rankin [18] in 1966 and the following result was obtained.

**Theorem 4** [18]. Let \( G \) be a finite group, generated by two elements \( \alpha, \beta \) such that \((\alpha\beta)^2 = 1\). Then the Cayley graph \( \Gamma = Cay(G, \{\alpha, \beta\}) \) has a Hamiltonian cycle.

Today studying the Hamiltonian property of graphs is a favorite problem for graph and group theorists. Hamiltonian paths and cycles play an important role in computer science (see [19,20]) and in combinatorial designs (see [21,22]). For example, it is well-known fact that the Hamiltonian property of the hypercube \( H_n \) is demonstrated by Gray codes. Testing whether a graph is Hamiltonian is one of the classical NP-complete problems [23]. There is a famous Hamiltonicity problem for vertex-transitive graphs which was posed by Lovász in 1970 and well-known as follows.

**Problem 1.** Does every connected vertex-transitive graph with more than two vertices have a Hamiltonian path?

To be more precisely he stated a research problem in [9] asking how one can “construct a finite connected undirected graph which is symmetric and has no simple path containing all the vertices. A graph is symmetric if for any two vertices \( x \) and \( y \) it has an automorphism mapping \( x \) onto \( y \).” However, traditionally (see [24]) the problem is formulated in the positive and considered as the Lovász conjecture that every vertex-transitive graph has a Hamiltonian path.

There are only four vertex-transitive graphs on more than two vertices which do not have a Hamiltonian cycle, and all of these graphs have a Hamiltonian path. They are the Petersen graph, the Coxeter graph (it is a unique cubic distance-regular graph with intersection array \( \{3, 2, 2, 1; 1, 1, 1, 2\} \) on 28 vertices and 42 edges) and the graphs obtained from each of these two graphs by replacing each vertex with a triangle and joining the vertices in a natural way. In particular, it is unknown of a vertex-transitive graph without a Hamiltonian path. Furthermore, it was noted that all of the above four graphs are not Cayley graphs. So several people made the following conjecture.
Conjecture 1. Every connected Cayley graph on a finite group has a Hamiltonian cycle.

However, there is no consensus among experts what the answer on the problem above will be. In particular, Mohar and Babai both made conjectures which are sharply critical of the Lovász problem. In 1996, Babai [3] made the following conjecture.

Conjecture 2 [3]. For some \( \varepsilon > 0 \), there exist infinitely many connected vertex-transitive graphs (even Cayley graphs) \( \Gamma \) without cycles of length \( \geq (1 - \varepsilon)|V(\Gamma)| \).

Later Mohar [25] investigated the matching polynomial \( \mu(\Gamma, x) \) of a graph \( \Gamma \) on \( n \) vertices defined as \( \mu(\Gamma, x) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k p(\Gamma, k) x^{n-2k} \), where \( p(\Gamma, k) \) is the number of \( k \)-matching in \( \Gamma \), and formulated the following conjecture.

Conjecture 3 [25]. For every integer \( r \) there exists a vertex-transitive graph whose matching polynomial has a root of multiplicity at least \( r \).

It is known (see [26]) that a graph whose matching polynomial has a nonsimple root has no a Hamiltonian path. Hence, if such a vertex-transitive graph exists then Lovász conjecture will be disproved.

All these conjectures are still open. Most results obtained so far about the first conjecture on Cayley graphs were surveyed in 1996 by Curran and Gallian [24] for abelian and dihedral groups, for groups of special orders, and certain extensions. For example, it was proved in 1983 by Marušič [11] that this conjecture is true for abelian groups. Let us recall that in an abelian group the order in which the binary operation is performed does not matter.

Theorem 5 [11]. A Cayley graph \( \Gamma = Cay(G, S) \) of an abelian group \( G \) with at least three vertices contains a Hamiltonian cycle.

A rare positive result for all finite groups was obtained in 2004 by Pak and Radoičić [27].

Theorem 6 [27]. Every finite group \( G \) of size \( |G| \geq 3 \) has a generating set \( S \) of size \( |S| \leq \log_2 |G| \), such that the corresponding Cayley graph \( \Gamma = Cay(G, S) \) has a Hamiltonian cycle.

The bound on \( S \) is reached on the group \( G = \mathbb{Z}_2^n \) for which the size of its smallest generating set is equal to \( \log_2 |G| \). For other groups the size of a generating set is much smaller. For example, for all finite simple groups it is equal to two. This result can be also considered as a corollary of the following natural conjecture.

Conjecture 4 [27]. There exists \( \varepsilon > 0 \), such that for every finite group \( G \) and every \( k \geq \varepsilon \log_2 |G| \), the probability \( P(G, k) \) that the Cayley graph \( \Gamma = Cay(G, S) \) with a random generating set \( S \) of size \( k \) contains a Hamiltonian cycle, satisfies \( P(G, k) \rightarrow 1 \) as \( |G| \rightarrow \infty \).

On one hand, this conjecture is much weaker then the Lovász conjecture. On the other hand, it also does not contradict Babai’s conjecture. A recent work by Krivelevich and Sudakov [28] shows that for every \( \varepsilon > 0 \) a Cayley graph \( \Gamma = Cay(G, S) \) with large enough \( |G| \), formed by choosing a set \( S \) of \( \varepsilon \log^5 |G| \) random generators in a group \( G \), is almost surely Hamiltonian. Thus, they reduce the bound in Conjecture 4 down to \( k \geq \varepsilon \log^5 |G| \).
There are also some results for Cayley graphs on the symmetric group $\text{Sym}_n$ generated by transpositions. These graphs have been proposed as models for the design and analysis of interconnection networks (see [19,20]). Moreover, Hamiltonian paths in Cayley graphs on $\text{Sym}_n$ provide an algorithm for creating the elements of $\text{Sym}_n$ from a particular generating set. The following result was proved by Kompel’makher and Liskovets [29] in 1975.

**Theorem 7** [29]. The graph $\text{Cay}(\text{Sym}_n, S)$ is Hamiltonian whenever $S$ is a generating set for $\text{Sym}_n$ consisting of transpositions.

It was generalized by Tchuente [30] in 1982 as follows.

**Theorem 8** [30]. Let $S$ be a generating set of transpositions for $\text{Sym}_n$. Then there is a Hamiltonian path in the graph $\text{Cay}(\text{Sym}_n, S)$ joining any permutations of opposite parity.

Thus, by these statements Cayley graphs on the symmetric group $\text{Sym}_n$ generated by any set of transpositions are Hamiltonian. Independently, a number of results were shown for particular sets of generators based on transpositions. In 1991 it was shown by Jwo et al. [31] that the star graph $\text{Sym}_n(st)$ is Hamiltonian, and by Jwo [32] that the bubble-sort graph $\text{Sym}_n(t)$ is also Hamiltonian. Hamiltonian properties of a Cayley graph generated by a transposition and a long cycle were considered in 1993 by Compton and Williamson [33].

Hamiltonicity of the pancake graph $\text{Sym}_n(PR)$ has been investigated independently by Zaks [34] in 1984 and by Jwo [32] in 1991.

**Theorem 9** [32,34]. The pancake graph $\text{Sym}_n(PR)$ is Hamiltonian for any $n \geq 3$.

Alspach et al. [35] have proposed in 1990 the definition of a Hamiltonian decomposition of a regular graph $\Gamma$. It is said that $\Gamma$ is Hamiltonian decomposable if either

(i) $\deg(\Gamma) = 2k$ and $E(\Gamma)$ can be partitioned into $k$ Hamiltonian cycles, or
(ii) $\deg(\Gamma) = 2k + 1$ and $E(\Gamma)$ can be partitioned into $k$ Hamiltonian cycles and a 1-factor, where a 1-factor of a graph is a collection of disjoint edges covering all vertices.

Hamiltonian decompositions of Cayley graphs on abelian groups were considered by Liu [36–38]. In particular, he defines a minimal generating set $S$ of a group $G$, where a set $S$ is minimal if $S$ generates $G$ but no proper subset of $S$ does, to be strongly minimal if for every $s \in S$, $s^2$ is not in the subgroup generated by the set $S - s$. Then the following results were proved in 1996 and 2003.

**Theorem 10** [37]. A Cayley graph $\Gamma = \text{Cay}(G, S)$ of a finite abelian group $G$ of odd order generated by a minimal generating set $S$ is Hamiltonian decomposable.

**Theorem 11** [38]. A Cayley graph $\Gamma = \text{Cay}(G, S)$ of a finite abelian group $G$ of even order at least 4 generated by a strongly minimal generating set $S$ is Hamiltonian decomposable.

The Hamiltonian decomposability of the $n$-dimensional cube $H_n$ was investigated in 1990 by Alspach et al. [35] and by Lakshmivarahan and Dhall [39], and the following result is known.
Theorem 12 [35,39]. The n-dimensional cube $H_n, n \geq 2$, is Hamiltonian decomposable.

The Hamiltonian decomposability of the butterfly graph $BF_n$ was investigated in 1994 by Barth and Raspaud [40], in 1995 by Wong [41], and the following result was obtained.

Theorem 13 [40,41]. The butterfly graph $BF_n$ is Hamiltonian decomposable.

5. The diameter problem, pancake problems, sorting by reversals

Cayley graphs tend to have a number of other desirable properties as well, including low diameter. There is the problem of establishing the diameter of a Cayley graph $\Gamma = Cay(G, S)$, that is the maximum, over $g \in G$, of the length of a shortest expression for $g$ as a product of generators. Computing the diameter of an arbitrary Cayley graph over a set of generators is NP-hard since the minimal word problem is known to be NP-hard in general. This result was shown in 1981 by Even and Goldreich [42]. General upper and lower bounds are very difficult to obtain. Moreover, there is a fundamental difference between Cayley graphs of abelian and non-abelian groups.

Babai et al. [43] have considered in 1989 the diameter of Cayley graphs on non-abelian finite simple groups and the following result was obtained.

Theorem 14 [43]. Every non-abelian finite simple group $G$ has a set of $\leq 7$ generators such that the resulting Cayley graph has diameter $O(\log_2 |G|)$.

However, this property does not hold for Cayley graphs of abelian groups as it was shown in 1993 by Annexstein and Baumslag [44]. On the other hand, in 1988 it was conjectured by Babai and Seress [45] for non-abelian groups that the diameter will always be small.

Conjecture 5 [45]. There exist a constant $c$ such that for every non-abelian finite simple group $G$, the diameter of every Cayley graph of $G$ is $\leq (\log_2 |G|)^c$.

If the conjecture is true, one would expect to find Cayley graphs of these groups with small diameter. But this problem is open even for the alternating groups $A_n$, consisting of all the even permutations of $\{1, \ldots, n\}$. The first step towards a solution this conjecture was made by Babai and Seress [45] for the symmetric $Sym_n$ and alternating $A_n$ groups.

Theorem 15 [45]. If $G$ is either $Sym_n$ or $A_n$ then the diameter of every Cayley graph of $G$ is $\leq \exp((n \ln n)^{1/2}(1 + o(1)))$.

Even for simple examples the exact diameter is still unknown and there are only bounds. For example, for the pancake graphs which are known because of the open combinatorial pancake problems. The original (unburnt) pancake problem was posed in 1975 in the American Mathematical Monthly [6] by Jacob E. Goodman writing under the name “Harry Dweighter” (or “Harried Waiter”) and it is stated as follows:

“The chef in our place is sloppy, and when he prepares a stack of pancakes they come out all different sizes. Therefore, when I deliver them to a customer, on the way to the table I
rearrange them (so that the smallest winds up on top, and so on, down to the largest on the bottom) by grabbing several pancakes from the top and flips them over, repeating this (varying the number I flip) as many times as necessary. If there are $n$ pancakes, what is the maximum number of flips (as a function of $n$) that I will ever have to use to rearrange them?"

It is clear that a stack of these $n$ pancakes can be represented by a permutation on $n$ elements and the problem is to find the minimum number of flips (prefix-reversals) needed to transform a permutation into the identity permutation. Clearly, this number of flips corresponds to the prefix-reversal diameter $d(Sym_n(PR))$ of the pancake graph. There is the following open problem.

**Problem 2.** What is the prefix-reversal diameter $d(Sym_n(PR))$ for $n > 13$?

In 1979 Gates and Papadimitriou presented in [46] the upper and lower bounds for the diameter of the pancake graph as

$$17n/16 \leq d(Sym_n(PR)) \leq 5/3(n + 1).$$

A lower bound was improved in 1997 by Heydari and Sudborough [47] such that

$$15n/14 \leq d(Sym_n(PR)).$$

They also computed the diameter up to 13. Exact values of $d = d(Sym_n(PR))$ are presented as follows:

\[
\begin{array}{cccccccccccc}
 n & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \\
 d & 1 & 3 & 4 & 5 & 7 & 8 & 9 & 10 & 11 & 13 & 14 & 15 \\
\end{array}
\]

Recently an improved upper bound was presented by Sudborough in cooperation with a team at University of Texas at Dallas (see [48]) as follows:

$$d(Sym_n(PR)) \leq 18n/11.$$  

An interesting variant of the pancake problem, known as the burnt pancake problem, concerns the pancakes that are two-sided (one side is burnt). Initially, the pancakes are arbitrary ordered and each pancake may have either side up. After sorting, the pancakes must not only be in size order, but must have their burnt sides face down. Two-sided pancakes can be represented by a signed permutation on $n$ elements with some elements negated. The problem is to find the minimum number of burnt flips (sign-change prefix-reversals) needed to transform a signed permutation into the positive identity permutation. It is clear that this number of burnt flips corresponds to the burnt prefix-reversal diameter $d(B_n(PR^\sigma))$ of the burnt pancake graph, and the problem is formulated as follows.

**Problem 3.** What is the burnt prefix-reversal diameter $d(B_n(PR^\sigma))$?

The upper and lower bounds for the burnt prefix-reversal diameter of the burnt pancake graph were shown in 1995 by Cohen and Blum [49]:

$$3n/2 \leq d(B_n(PR^\sigma)) \leq 2n - 2,$$

where the upper bound holds for $n \geq 10$. It was also conjectured there that the worst case for sorting signed permutations (burnt pancakes) is the negative identity permutation $-I = [-1, -2, \ldots, -n]$. Later Hyedari and Sudborough [47] showed that if the conjecture is true...
then the diameter of the burnt pancake graph is
\[ d(B_n(P R^\sigma)) \leq 3(n + 1)/2, \]
since \(-I\) can be sorted in \(3(n + 1)/2\) steps for all \(n = 3 \text{ (mod 4)}\) and \(n \geq 23\). Currently, exact values of \(d^\sigma = d(B_n(P R^\sigma))\) are known for \(n \leq 18 [47]\) and they are as follows:

<table>
<thead>
<tr>
<th>(n)</th>
<th>1</th>
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<th>3</th>
<th>4</th>
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<th>6</th>
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<th>11</th>
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<th>14</th>
<th>15</th>
<th>16</th>
<th>17</th>
<th>18</th>
</tr>
</thead>
<tbody>
<tr>
<td>(d^\sigma)</td>
<td>1</td>
<td>4</td>
<td>6</td>
<td>8</td>
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<td>23</td>
<td>24</td>
<td>26</td>
<td>28</td>
<td>29</td>
</tr>
</tbody>
</table>

In general, a pancake stack is an example of a data structure. In molecular biology and computer science the problems presented above are called \textit{sorting by prefix-reversals}. The pancake graph (on burnt or unburnt pancakes) has practical applications in parallel processing since it corresponds to the \(n\)-dimensional pancake network such that this network has processors labeled with each of the \(n!\) distinct permutations of length \(n\). Two processors are connected when the label of one is obtained from the other by some prefix-reversal. The diameter of this network corresponds to the worst communication delay for transmitting information in a system. It is known that this network has sublogarithmic diameter and degree as a function of the number of processors (vertices). The pancake sorting can also provide an effective routing algorithm between processors. There is a very nice survey by Heydemann [20] about Cayley graphs as interconnection networks, which can be recommended for more details.

Recent advances in genome identification have also brought to light questions in molecular biology very similar to the pancake problem. Differences in genomes are usually explained by accumulated differences built up in the genetic material due to random mutation and random mating. In 1986 another mechanism of evolution was discovered by Palmer and Herbon [4]. Comparing two genomes one can often find that these two genomes contain the same set of genes. But the order of the genes is different in different genomes. For example, it was found that both human X chromosome and mouse X chromosome contain eight genes which are identical. In human, the genes are ordered as \([4, 6, 1, 7, 2, 3, 5, 8]\) and in mouse, they are ordered as \([1, 2, 3, 4, 5, 6, 7, 8]\). It was also found that a set of genes in cabbage are ordered as \([1, −5, 4, −3, 2]\) and in turnip, they are ordered as \([1, 2, 3, 4, 5]\). The comparison of two genomes is significant because it provides us some insight as to how far away genetically these species are. If two genomes are similar to each other, they are genetically close. This has inspired some molecular biologists to look at the mechanisms which might shuffle the order of the genetic material. One way of doing this is the prefix-reversals or just reversals. Analyzing the transformation from one species to another is analogous to the problem of finding the shortest series of reversals to transform one into the other.

The analysis of genomes evolving by inversions leads to the combinatorial problem of \textit{sorting by reversals}. Reversal distance measures the amount of evolution that must have taken place at the chromosome level, assuming evolution proceeded by inversion. Mathematical analysis of the problem was initiated by Sankoff [50] in 1992, and then continued by another authors. There are two algorithmic subproblems. The first one is to find the reversal distance \(d(\tau_1, \tau_2)\) between two permutations \(\tau_1\) and \(\tau_2\). Notice that the reversal distance between \(\tau_1\) and \(\tau_2\) is equal to the reversal distance between \(\pi = \tau_2^{-1}\tau_1\) and the identity permutation \(I\). It was shown in 1995 by Kececioglu and Sankoff [51] and in 1996 by Bafna and Pevzner [5] that \(\max_{\pi \in \text{Sym}_n} d(\pi, I) = n − 1\). The path distance in the reversal Cayley graph \(\text{Sym}_n(R)\) corresponds to the reversal distance between two permutations. Hence, its diameter is \(n − 1\), and the only permutations needing these many reversals are the Gollan permutation \(\gamma_n\) and its inverse, where the Gollan permutation, in one-line
notation, is defined as follows:
\[ \gamma_n = \begin{cases} 
3, 1, 5, 2, 7, 4, \ldots, n - 3, n - 5, n - 1, n - 4, n, n - 2 & \text{if } n \text{ is even,} \\
3, 1, 5, 2, 7, 4, \ldots, n - 6, n - 2, n - 5, n - 3, n - 1 & \text{if } n \text{ is odd.}
\end{cases} \]

In the case of signed permutations we have to find the reversal distance \( \rho(\tau_1^\sigma, \tau_2^\sigma) \) between signed permutations \( \tau_1^\sigma \) and \( \tau_2^\sigma \), or between \( \pi^\sigma = (\tau_2^\sigma)^{-1}\tau_1^\sigma \) and the positive identity permutation \( I^+ = [+1, \ldots, +n] \). It was shown in 1994 by Knuth in Exercises 5.1.4-43 [52] that at most \( n + 1 \) sign-change reversals are needed to sort any signed permutation to the positive identity permutation, for all \( n > 3 \), i.e., \( \max_{\pi^\sigma \in B_n} \rho(\pi^\sigma, I^+) = n + 1 \). The path distance in the reversal Cayley graph \( B_n(R^\sigma) \) corresponds to the reversal distance between two signed permutations. This means that its diameter is \( n + 1 \) and the following permutations, written in one-line notation, are at this maximum distance from the identity permutation \( I^+ \):
\[ \pi^\sigma = \begin{cases} 
+n, +(n - 1), \ldots, +1 & \text{if } n \text{ is even,} \\
+2, +1, +3, +n, +(n - 1), \ldots, +4 & \text{if } n > 3 \text{ is odd.}
\end{cases} \]

In 2001, it was also shown by Bader et al. [53] that the reversal distance could be calculated in linear time for signed permutations.

The next subproblem here is how to reconstruct a sequence of reversals which realizes the distance. Its solutions are far from unique. In 1994 it was shown by Kececioglu and Sankoff [54] that the problem is NP-hard for the unsigned permutations, and it is polynomial for the signed permutations as it was shown by Hannenhalli and Pevzner [55] in 1999. The 1.5-approximation algorithm for sorting unsigned permutations was presented by Christie [56] in 1998. One of the most effective algorithms that sort signed permutations by reversals was presented by Kaplan and Verbinin [57] in 2003. For more details see the recently published books by Pevzner [58], Sankoff and El-Mabrouk [59].

### 6. Vertex reconstruction problem

The vertex reconstruction problem which is not related to Ulam’s problem has been introduced in 1997 by Levenshtein [60] as the problem of efficiently reconstructing an arbitrary sequence for combinatorial channels with errors of interest in coding theory such as substitutions, transpositions, deletions and insertions of symbols. Let \( \Gamma = (V, E) \) be a simple connected graph with vertex set \( V \) and edge set \( E \). Sequences (or any other information) are represented by the vertices of \( \Gamma \) and an edge \( \{v, u\} \) is viewed as a single distortion or error transforming one vertex into the other. For given \( r \geq 1 \) denote by \( N(\Gamma, r) \) the largest number \( N \) such that there exist a subset \( A \subseteq V \) of size \( N \) and two vertices \( v \neq u \) with \( A \subseteq B_r(v) \) and \( A \subseteq B_r(u) \). Thus any \( N + 1 \) distinct vertices are contained in \( B_r(v) \) for at most one vertex \( v \), while this statement is wrong for any \( N < N(\Gamma, r) \). This means that an arbitrary vertex of \( \Gamma \) can be reconstructed uniquely by any subset of \( N(\Gamma, r) + 1 \) or more distinct vertices at distance at most \( r \) from the vertex, if such a subset exists. The vertex reconstruction problem is, for a given graph \( \Gamma \) and integers \( r = 1, \ldots, d(\Gamma) \), to determine
\[ N(\Gamma, r) = \max_{v, u \in V, v \neq u} |B_r(v) \cap B_r(u)|. \] (1)

We also ask what is an effective algorithm to determine \( x \). This problem is motivated by the fact that transmission of certain information in the presence of noise is realized without encoding and redundancy and a unique possibility to reconstruct a message (vertex) consists of having a sufficiently large number of erroneous patterns of this message.
Initially, this problem was studied by Levenshtein for the Hamming graph which is distance-
regular as well as a Cayley graph.

**Theorem 16 [60,61].** For any \( n \geq 2, q \geq 2 \) and \( r \geq 1 \), we have

\[
N(L_n(q), r) = q \sum_{i=0}^{r-1} \binom{n-1}{i} (q-1)^i.
\]  

(2)

It was also shown (see [7]) that any vector \( x = (x_1, \ldots, x_n) \in F_q^n \) can be reconstructed from
any \( M = N(L_n(q), r) + 1 \) distinct vectors \( y_1, \ldots, y_M \) of \( B_r(x) \), written as columns of a matrix
of size \( n \times M \), applying the majority algorithm to rows of this matrix. The coordinate \( x_i \) of the
sought vector \( x \) is equal to that of \( q \) elements of \( i \)th row which occurs more often.

The problem of finding the value (1) is much more complicated for graphs which are not
distance-regular. Cayley graphs of this kind arise on \( \text{Sym}_n \) and \( B_n \) when the reconstruction of
permutations and signed permutations is considered for distortions by single transposition or
reversal errors. The vertex reconstruction problem for Cayley graphs on \( \text{Sym}_n \) and \( B_n \) generated
by transpositions was considered in 2006 by Konstantinova [62] (see also [63]).

**Theorem 17 [62,63].** For any \( n \geq 3 \), we have

\[
N(\text{Sym}_n(T), 1) = 3 \quad \text{and} \quad N(\text{Sym}_n(T), 2) = \frac{3}{2}(n-2)(n+1).
\]

A permutation \( \pi \) is reconstructible from \( k \) distinct permutations \( x_1, \ldots, x_k \in B_1(\pi) \), if there
does not exist a permutation \( \tau, \pi \neq \tau \), such that \( x_1, \ldots, x_k \in B_1(\tau) \). From the theorem above,
an arbitrary permutation is uniquely reconstructible from four of its distinct 1-neighbors. In the
case of at most two transposition errors the reconstruction of a permutation requires many more
of its distinct 2-neighbors.

The similar results were obtained for the bubble-sort and star graphs.

**Theorem 18 [62,63].** For any \( n \geq 3 \), we have

\[
N(\text{Sym}_n(t), 1) = N(\text{Sym}_n(st), 1) = 2,
\]

\[
N(\text{Sym}_n(t), 2) = N(\text{Sym}_n(st), 2) = 2(n-1).
\]

The following result was presented in [8] for the Cayley graph \( B_n(T^\sigma) \) on the hyperoctahedral

**Theorem 19 [8].** For any \( n \geq 2 \), we have \( N(B_n(T^\sigma), 1) = 2 \).

The question about the reconstruction of a signed permutation from its distinct 2-neighbors in
this graph is open and there is the following conjecture.

**Conjecture 6 [8].** For any \( n \geq 2 \), we have \( N(B_n(T^\sigma), 2) = n(n+1) \).

The vertex reconstruction problem was also considered for Cayley graphs on \( \text{Sym}_n \) and \( B_n \)
generated by reversals and prefix-reversals. For the reversal Cayley graph \( \text{Sym}_n(R) \) the following
results were obtained by Konstantinova [64] (see also [65]) in 2005.
Theorem 20 [64,65]. For any $n \geq 3$, we have

\[
\begin{align*}
N(\text{Sym}_n(R), 1) & = 3, \\
N(\text{Sym}_n(R), 2) & \geq \frac{3}{2}(n - 2)(n + 1).
\end{align*}
\]

Thus, in this case an arbitrary permutation is uniquely reconstructible from four of its distinct 1-neighbors. It is also shown that a permutation is reconstructible from three of its 1-neighbors with probability $p_3 \to 1$ as $n \to \infty$ and it is reconstructible from two of its 1-neighbors with probability $p_2 \sim \frac{1}{3}$ as $n \to \infty$ under the conditions that these permutations are uniformly distributed. Inequality (3) is attained for the permutations $\pi_k = [1, \ldots, k - 1, k + 1, k + 2, k, k + 3, \ldots, n]$, for any $k = 1, \ldots, n - 2$, when reversals on intervals $[k, k + 2]$ can be considered as transpositions (see Theorem 17). A simple reconstruction algorithm for the cases when four, three or two distinct permutations are considered to reconstruct an arbitrary permutation is presented in [65].

There is the following result for the pancake graph $\text{Sym}_n(PR)$.

Theorem 21 [8]. For any $n \geq 4$, we have

\[
\begin{align*}
N(\text{Sym}_n(PR), 1) & = 2, \\
N(\text{Sym}_n(PR), 2) & = 2(n - 1).
\end{align*}
\]

Comparing these statements with Theorem 18 one can see that there is one and the same result for the bubble-sort, star and pancake graphs.

In the case of the burnt pancake graph $B_n(PR^\sigma)$ it is known [8] that any signed permutation is uniquely reconstructible from three of its distinct 1-neighbors. For the reversal Cayley graph $B_n(R^\sigma)$ the following results were obtained.

Theorem 22 [64,66]. For any $n \geq 3$, we have

\[
\begin{align*}
N(B_n(R^\sigma), 1) & = 2 \quad \text{and} \quad N(B_n(R^\sigma), 2) \geq n(n + 1).
\end{align*}
\]

It was also proved in [66] that an arbitrary signed permutation is reconstructible from two distinct signed permutations with probability $p_2 \sim \frac{1}{3}$ as $n \to \infty$ under the conditions that these permutations are uniformly distributed.

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