On reconstruction of signed permutations distorted by reversal errors

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Abstract

The problem of reconstructing signed permutations on \(n\) elements from their erroneous patterns distorted by reversal errors is considered in this paper. A reversal is the operation of taking a segment of the signed permutation, reversing it, and flipping the signs of its elements. The reversal metric is defined as the least number of reversals transforming one signed permutation into another. It is proved that for any \(n \geq 2\) an arbitrary signed permutation is uniquely reconstructible from three distinct signed permutations at reversal distance at most one from the signed permutation. The proposed approach is based on the investigation of structural properties of a Cayley graph \(G_{2n}\) whose vertices form a subgroup of the symmetric group \(\text{Sym}_{2n}\). It is also proved that an arbitrary signed permutation is reconstructible from two distinct signed permutations with probability \(p_2 \sim \frac{1}{3}\) as \(n \to \infty\). In the case of at most two reversal errors it is shown that at least \(n(n+1)\) distinct erroneous patterns are required in order to reconstruct an arbitrary signed permutation.

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1. Introduction

The problem of efficient reconstruction of sequences was introduced and investigated in [3–5]. The sequences are considered as a vertex set \(V\) of a graph \(G = (V, E)\), where \((x, y) \in E\) if there exist single errors of the type under consideration which transform \(x\) into \(y\) and \(y\) into \(x\). The problem was solved for graphs corresponding to such errors as substitutions, transpositions, deletions and insertions of symbols which are of interest in coding theory. One of the metric problems which arises here is reconstructing an unknown vertex \(x\) from the minimal number of vertices of the metric ball \(B_r(x)\) of radius \(r\) centered at \(x \in V\). It is reduced to finding the value \(N(G, r) = \max_{x, y \in V, x \neq y} |B_r(x) \cap B_r(y)|\), since \(N(G, r) + 1\) is the minimal number of distinct vertices in a metric ball \(B_r(x)\) of an arbitrary vertex \(x\) which are sufficient to reconstruct the vertex \(x\) subject to the condition that at most \(r\) single errors have occurred.

The reconstruction of permutations \(\pi \in \text{Sym}_n\) distorted by single reversal errors was considered in [2]. It was proved that for any \(n \geq 3\) the minimal number of reversals in a metric ball \(B_1(\pi)\) which are sufficient to reconstruct...
Then we define a Cayley graph corresponding with the set $\text{Sym}_n$ and $i$ and molecular biology [6,7].

Let us show that $\text{Sym}_n$.

For example, let $PQ$.

Moreover, this mapping is a one-to-one mapping since (1) defines a pair of vectors of $\text{Sym}_n$ elements, where $\pi \in \text{Sym}_n$ and $\sigma = (\sigma_1, \ldots, \sigma_n)$, $\sigma_i \in \{+,-\}$, $i = 1, \ldots, n$. We shall also represent a signed permutation $\pi^\sigma$ as a vector $[\pi_1, \pi_2, \ldots, \pi_i, \ldots, \pi_n]$, where a bar is written over an element with a negative sign.

A reversal $R_{i,j}$, $1 \leq i \leq j \leq n$, is the operation of reversing segments $[i, j]$ of a signed permutation $\pi^\sigma$ with flipping the signs of its elements, i.e.

$$[\ldots, \pi_i, \pi_{i+1}, \ldots, \pi_{j-1}, \pi_j, \ldots]R_{i,j} = [\ldots, \pi_j, \pi_{j-1}, \ldots, \pi_{i+1}, \pi_i, \ldots].$$

For example, $\pi^\sigma = [-2, +3, +4, -1, +6, -5] \in B_6$ is given by the vectors $\pi = [2, 3, 4, 1, 6, 5]$ and $\sigma = (-, +, +, -, +, -, -)$. It is also given by $\pi^\sigma = [\overline{2}34\overline{6}]$ and we have, for example, $\pi^\sigma R_{2,4} = [\overline{2}14\overline{3}6\overline{5}]$.

Now let us consider the symmetric group $\text{Sym}_{2n}$ of permutations on $2n$ elements. For any fixed $i$, $1 \leq i \leq n$, the elements of the set $(2i−1, 2i)$ are called the *vicinal numbers*. Let us denote by $\text{Sym}_{2n}^\sigma$ the subset of permutations $P$ of $\text{Sym}_{2n}$ which transform any set of vicinal numbers to a set of vicinal numbers. This means that for any $P \in \text{Sym}_{2n}^\sigma$ and $i$, $1 \leq i \leq n$, there exists $h$, $1 \leq h \leq n$, such that

$$P(2i − 1) = 2h − 1 \quad \text{and} \quad P(2i) = 2h$$

or

$$P(2i − 1) = 2h \quad \text{and} \quad P(2i) = 2h − 1$$

and these $h$ are distinct for distinct $i$.

Now we define a one-to-one correspondence between $B_n$ and $\text{Sym}_{2n}^\sigma$. For any signed permutation $\pi^\sigma \in B_n$ we consider a permutation $P \in \text{Sym}_{2n}$ such that for any $i$, $1 \leq i \leq n$, we have

$$P(2i − 1) = 2\pi(i) − \frac{\sigma_i + 1}{2} \quad \text{and} \quad P(2i) = 2\pi(i) + \frac{\sigma_i - 1}{2}. \quad (1)$$

For any $\sigma_i = \pm 1$, $1 \leq i \leq n$, the numbers $2\pi(i) − \frac{\sigma_i + 1}{2}$ and $2\pi(i) + \frac{\sigma_i - 1}{2}$ are vicinal numbers and thus $P \in \text{Sym}_{2n}^\sigma$. Moreover, this mapping is a one-to-one mapping since (1) defines a pair of vectors $\pi$ and $\sigma$ in a unique way.

Let $T = PQ \in \text{Sym}_{2n}$ be the product of the permutations $P, Q \in \text{Sym}_{2n}$ such that $T(i) = P(Q(i))$, $1 \leq i \leq 2n$, and let us show that $\text{Sym}_{2n}^\sigma$ is a subgroup of the group $\text{Sym}_{2n}$.

**Lemma 1.** $PQ \in \text{Sym}_{2n}^\sigma$ for any $P, Q \in \text{Sym}_{2n}^\sigma$.

**Proof.** Let $T = PQ$ where

$$P = \left(\ldots, 2\pi(i) − \frac{\sigma_i + 1}{2}, 2\pi(i) + \frac{\sigma_i - 1}{2}, \ldots\right) \in \text{Sym}_{2n}^\sigma,$$

$$Q = \left(\ldots, 2q(j) − \frac{\gamma_j + 1}{2}, 2q(j) + \frac{\gamma_j - 1}{2}, \ldots\right) \in \text{Sym}_{2n}^\sigma.$$
Let us check that for any \( j, 1 \leq j \leq n \), we have

\[
T(i) = 2\pi(q(j)) - \frac{\sigma_{q(j)} \gamma_j + 1}{2} \quad \text{when } i = 2j - 1, \\
T(i) = 2\pi(q(j)) + \frac{\sigma_{q(j)} \gamma_j - 1}{2} \quad \text{when } i = 2j.
\]

(2)

(3)

From these formulas it will immediately follow that \( T \in \text{Sym}_n^2 \).

We consider two cases. Let \( i = 2j - 1 \). Then by (1) we have \( T(i) = P(Q(i)) = P(Q(2j - 1)) = P(2q(j) - \frac{\gamma_j + 1}{2}) \).

If \( \gamma_j = +1 \) then \( Q(2j - 1) = 2q(j) - 1 \) and

\[
P(2q(j) - 1) = 2\pi(q(j)) - \frac{\sigma_{q(j)} + 1}{2}
\]

that corresponds to the formula (2). If \( \gamma_j = -1 \) then \( Q(2j - 1) = 2q(j) \) and

\[
P(2q(j)) = 2\pi(q(j)) + \frac{\sigma_{q(j)} - 1}{2}
\]

that also corresponds to the formula (2).

Let \( i = 2j \). Then by (1) we have \( T(i) = P(Q(i)) = P(Q(2j)) = P(2q(j) + \frac{\gamma_j - 1}{2}) \).

If \( \gamma_j = +1 \) then \( Q(2j) = 2q(j) \) and

\[
P(2q(j)) = 2\pi(q(j)) + \frac{\sigma_{q(j)} - 1}{2}
\]

that corresponds to the formula (3). If \( \gamma_j = -1 \) then \( Q(2j) = 2q(j) - 1 \) and

\[
P(2q(j) - 1) = 2\pi(q(j)) - \frac{\sigma_{q(j)} + 1}{2}
\]

that also corresponds to the formula (3). The considered cases complete the proof. \( \square \)

It follows from Lemma 1 that \( \text{Sym}_n^2 \) is a subgroup of the group \( \text{Sym}_n^2 \) since \( I \in \text{Sym}_n^2 \), where \( I \) is the identity permutation of the length \( 2n \). We denote by \( (P)^{-1} \) the inverse of \( P \in \text{Sym}_n^2 \) and have \( P(P)^{-1} = (P)^{-1}P = I \). For any \( T, P, Q \in \text{Sym}_n^2 \) we also have

\[
(TP)Q = T(PQ)
\]

(4)

since \( \text{Sym}_n^2 \) forms a subgroup of the group \( \text{Sym}_n^2 \).

An even reversal \( R_{2i-1,2j}, 1 \leq i \leq j \leq n \), is the operation of reversing segments \([2i - 1, 2j]\) of a permutation from \( \text{Sym}_n^2 \), i.e.

\[
\ldots, \pi_{2i-2}, \pi_{2i-1}, \pi_{2j}, \ldots, \pi_{2j-1}, \pi_{2j+1}, \ldots] \rightarrow [\ldots, \pi_{2i-2}, \pi_{2j}, \pi_{2j-1}, \ldots, \pi_{2j+1}, \pi_{2i-1}, \pi_{2i-1}, \pi_{2j+1}, \ldots].
\]

Note that \( R_{2i-1,2j} \in \text{Sym}_n^2 \) and \( R_{2i-1,2j}R_{2i-1,2j} = I \) and hence \( (R_{2i-1,2j})^{-1} = R_{2i-1,2j} \). It is easily checked that an even reversal on a permutation \( P \in \text{Sym}_n^2 \) corresponds to a reversal \( R_{i,j} \) on a permutation \( \pi^P \in B_n \).

The reversal distance \( d(P, Q) \) between two permutations \( P, Q \in \text{Sym}_n^2 \) is the least number \( d \) of even reversals needed to transform \( P \) to \( Q \), i.e.

\[
P_{\ldots, 2i-1,2j} \cdots R_{2i-1,2j} = Q.
\]

(5)

The reversal distance satisfies the axioms of a metric. In particular, the symmetry \( d(P, Q) = d(Q, P) \) holds since \( (R_{2i-1,2j})^{-1} = R_{2i-1,2j} \); (5) implies that \( P = Q_{\ldots, 2i-1,2j} \cdots R_{2i-1,2j} \), and the triangle inequality follows from (4) and (5). By the same formulas, for any \( T, P, Q \in \text{Sym}_n^2 \), we have

\[
d(TP, TQ) = d(P, Q).
\]
This means that \( \text{Sym}_{2n}^q \) is the isometry group. In particular, we have \( d(P, Q) = d(I, (P)^{-1} Q) \) (but \( d(P, Q) = d(I, Q(P)^{-1}) \) is not true in general). It was shown by Knuth in one of the exercises in [1] that at most \( (n + 1) \) reversals (with flipping the signs of elements) are needed to sort any signed permutation to the identity permutation, for all \( n > 3 \). Since there is a one-to-one correspondence between \( B_n \) and \( \text{Sym}_{2n}^q \) and a reversal on \( B_n \) corresponds to an even reversal on \( \text{Sym}_{2n}^q \), we have

\[
\max_{P \in \text{Sym}_{2n}^q} d(P, I) = n + 1 \quad \text{for } n > 3.
\]

In the remainder of this paper it is assumed that

\[
P = [\pi_0, \pi_1, \pi_2, \ldots, \pi_{2i-1}, \pi_{2i}, \ldots, \pi_{2n-1}, \pi_{2n}, \pi_{2n+1}],
\]

where \( \pi_0 = 0 \) and \( \pi_{2n+1} = 2n + 1 \), and define

\[
\alpha_i = \alpha_i(P) = \pi_{2i+1} - \pi_{2i}, \quad i = 0, \ldots, n.
\]

We say that the permutation \( P \) has a \textit{breakpoint} between positions \( 2i \) and \( 2i + 1, i \in [0, n] \), if \( |\alpha_i(P)| \geq 2 \). Denote by \( b(P) \) the number of breakpoints of \( P \). Note that \( b(P) = 0 \) if and only if \( P = I \) and \( b(P) \geq 2 \) when \( P \neq I \). If \( P \) has \( b = b(P) \geq 2 \) breakpoints between positions \( 2i_h \) and \( 2i_h + 1 \), respectively, \( 0 \leq i_h \leq n, h = 1, \ldots, b \), then the interval \([0, 2n+1]\) is partitioned into \( b - 1 \) internal monotonicity intervals \([2i_h + 1, 2i_{h+1}], h = 1, \ldots, b - 1 \), consisting of an even number of successive integers in decreasing or increasing order and two external monotonicity intervals \([0, 2i_1] \) and \([2i_b + 1, 2n + 1] \) consisting of an odd number of successive integers in increasing order (the external intervals have only one number when \( i_1 = 0 \) or \( i_b = n \), respectively).

For any \( 1 \leq i \leq j \leq n \), we will denote the internal even intervals

\[
2i - 2i + 1, \ldots, 2j - 1, 2j \quad \text{and} \quad 2j, 2j - 1, \ldots, 2i, 2i - 1
\]

of a permutation \( P \in \text{Sym}_{2n}^q \) by \( \tilde{i}, \tilde{j} \), respectively. The external increasing odd intervals will be denoted by \( \hat{0}, \hat{i} \) and \( \hat{i} + 1, \hat{n} + 1 \), respectively. We will also use \( \tilde{T} \) and \( \hat{T} \) instead of \( \tilde{i}, \tilde{j} \) and \( \hat{i}, \hat{j} \) when \( i = j \), and use \( \tilde{0} \) and \( \hat{n} + 1 \) instead of \( \hat{0}, \hat{i} \) and \( \hat{i} + 1, \hat{n} + 1 \) when \( i_1 = 0 \) and \( i_b = n \), respectively.

We define the vector \( \beta(P) = (\beta_1, \ldots, \beta_b) \) (\textit{the up-and down-sequence of monotonicity of breakpoints}), where

\[
\beta_h = \begin{cases} 
+ & \text{if } \pi_{2i_h} < \pi_{2i_{h+1}}, \\
- & \text{if } \pi_{2i_h} > \pi_{2i_{h+1}},
\end{cases} \quad h = 1, \ldots, b.
\]

Note that \( 0 \leq \pi_{2i_1} < \pi_{2i_{1+1}} \) and \( \pi_{2i_b} > \pi_{2i_{b+1}} \leq 2n + 1 \), hence \( \beta_1 = \beta_b = + \). We also define the vector \( \mu(P) = (\mu_1, \ldots, \mu_{b-1}) \) (\textit{the up-and down-sequence of monotonicity of internal intervals}), where

\[
\mu_h = \begin{cases} 
+ & \text{if } \pi_{2i_{h+1}} > \pi_{2i_h+1}, \\
- & \text{if } \pi_{2i_{h+1}} < \pi_{2i_h+1},
\end{cases} \quad h = 1, \ldots, b - 1.
\]

The vectors \( \beta(P) \) and \( \mu(P) \) are uniquely defined by the permutation \( P \), and two permutations are distinct if at least one of these vectors differ.

As an example, for the permutation \( \pi^q = [+1, -3, -2, +4, -5] \) we have the permutation \( P = [0, 1, 2, 6, 5, 4, 3, 7, 8, 10, 9, 11] \) for which the interval \([0, 11]\) is partitioned into five monotonicity intervals \([0, 2], [3, 6], [7, 8], [9, 10], [11, b(P) = 4, \beta(P) = (+, +, +, +), \) and \( \mu(P) = (+, +, +, +). \) The permutation \( P \) is also represented as \( P = (0, \hat{2}, \hat{5}, \hat{6}, \hat{7}, \hat{8}, \hat{9}, \hat{10}, \hat{11}) \).

To estimate the change of the number of breakpoints as a result of applying an even reversal \( R_{2i-1,2j} \) to \( P \in \text{Sym}_{2n}^q \) we use the function \( \delta(x, y) \) of two integer variables \( x \) and \( y \) defined as follows:

\[
\delta(x, y) = \begin{cases} 
1 & \text{if } |x - y| \geq 2, \\
0 & \text{if } |x - y| \leq 1.
\end{cases}
\]

\textbf{Lemma 2.} Let \( P = [\pi_0, \pi_1, \ldots, \pi_{2n}, \pi_{2n+1}] \in \text{Sym}_{2n}^q \) and \( R_{2i-1,2j} \) is an even reversal, \( 1 \leq i \leq j \leq n \). Then

\[
b(P R_{2i-1,2j} P) - b(P) = \delta(\pi_{2i-2}, \pi_{2j}) + \delta(\pi_{2i-1}, \pi_{2j+1}) - \delta(\pi_{2i-2}, \pi_{2i-1}) - \delta(\pi_{2j}, \pi_{2j+1}).
\]
A ranked poset is called between positions 2 and 2j if an arbitrary permutation. To investigate the structural properties of the metric ball of radius \( r \) and \( l \), we have \( \delta = \frac{r}{r} \) and\( \Delta = \frac{l}{l} \). The path distance between vertices \( P \) and \( Q \) in \( G_{2n} \) coincides with the reversal distance \( d(P, Q) \) and the diameter of \( G_{2n} \) equals \( n+1 \), by (6). It is well-known fact that Cayley graphs are vertex-transitive. This means that the graph \( G_{2n} \) viewed from any vertex looks the same, for example, from the vertex corresponding to the identity permutation. To investigate the structural properties of \( G_{2n} \), it is useful to describe all classes of permutations at distance \( i \) from the identity permutation. In this case we say that all such permutations have rank \( i \) and we consider the partition of the vertex set \( V = \text{Sym}^2_{2n} \) into \( n+2 \) subsets of vertices of rank \( i \):

\[
V_i = \{ P : P \in \text{Sym}^2_{2n}, d(P, I) = i \}, \quad i = 0, 1, \ldots, n+1. \tag{7}
\]

Note that if \( P \in V_i \), then \( (P)^{-1} \in V_i \) since \( d(P, I) = d((P)^{-1}, I) \), by (4). In particular, \( |V_{i+1}| = 1 \), where \( V_{n+1} = \{[2n-1, 2n, 2n-3, 2n-2, \ldots, 1, 2]\} \) when \( n \) is even and \( V_{n+1} = \{[3, 4, 1, 2, 5, 6, 2n-1, 2n, 2n-3, 2n-2, \ldots, 7, 8]\} \) when \( n > 3 \) is odd [1]. It is evident that \( V_0 = \{I\} \).

The partition (7) induces a ranked partial ordered set (poset) on the vertex set \( V \) such that \( P \in V_i \) precedes \( Q \in V_j \) if and only if \( i \leq j \) and there exists a path in \( G_{2n} \) of length \( j-i \) connecting \( P \) with \( Q \). By (4), we have an isomorphic poset if an arbitrary permutation \( T \in \text{Sym}^2_{2n} \) is considered in the definition (7), instead of \( I \). It also follows that the size of the metric ball

\[
B_r(P) = \{ Q : Q \in \text{Sym}^2_{2n}, d(P, Q) \leq r \}
\]

of radius \( r \) with center in \( P \) does not depend on \( P \) and equals \( \sum_{i=0}^r |V_i| \). To investigate this structure, for any \( P \in V_i \), we define

\[
V_{i,i+1}(P) = \{ Q : Q \in V_{i+1}, d(P, Q) = 1 \} \quad \text{when} \ 0 \leq i \leq n,
\]

\[
V_{i,i-1}(P) = \{ Q : Q \in V_{i-1}, d(P, Q) = 1 \} \quad \text{when} \ 1 \leq i \leq n+1.
\]

A ranked poset is called regular if for every \( i \) we have that \( |V_{i,i+1}(P)| \) and \( |V_{i,i-1}(P)| \) do not depend on \( P \in V_i \). For example, the corresponding posets are regular when the problem of reconstructing sequences distorted by substitutions or transpositions of symbols is considered [4]. The difficulties of solving similar problems for the reversal errors are explained by the fact that “our” poset is not a regular poset. Note that the graph \( G_{2n} \) is regular in the standard sense and we have \( V_1 = V_0(I) = \{R_{2i-1,2j}, 1 \leq i \leq j \leq n\} \), \( |V_1| = \frac{(n-1)n}{2} + n = \frac{n(n+1)}{2} \).

The graph \( G_{2,3} \) is given in Fig. 1. It is easy to see that the ranked poset is not regular since, for example, \( |V_{2,3}([23\overline{1}])| = 4 \) and \( |V_{2,3}([3\overline{1}2])| = 5 \) as well as \( |V_{2,1}([2\overline{1}3])| = 2 \) and \( |V_{2,1}([3\overline{1}2])| = 1 \). We also have that the graph \( G_{2,3} \) is regular since the vertex degree \( \text{deg}(P) = 6 \) for any \( P \in G_{2,3} \) and \( |V_1| = 6 \).

Now we fix \( k \) and \( l \), \( 1 \leq k \leq l \leq n \), and investigate \( V_{1,2}(R_{2k-1,2l}) \). Note that any \( R_{2k-1,2l} \in V_1 \) with \( b(R_{2k-1,2l}) = 2 \). Below we show that for any \( Q \in V_{1,2}(R_{2k-1,2l}) \), \( b(Q) \) can be equal to three, or four. First let us find \( b(P_{2k-1,2l}) \), where \( P = R_{2k-1,2l}, 1 \leq k \leq l \leq n \), under the condition \( \delta(\pi_{2i-2}, \pi_{2i-1}) = \delta(\pi_{2j-2}, \pi_{2j-1}) = 0 \), that is \( \pi_{2i-1} = \pi_{2j} = 1 \).

**Lemma 3.** If \( |\pi_{2i-1}| = |\pi_{2j}| = 1 \), where \( 1 \leq i \leq j \leq n \), then we have \( b(R_{2k-1,2l}R_{2i-1,2j}) = 4 \).

**Proof.** Let \( P = \{\pi_0, \pi_1, \ldots, \pi_{2n}, \pi_{2n+1}\} = R_{2k-1,2l} \) and hence \( b(P) = 2 \). The condition \( |\pi_{2i-1}| = |\pi_{2j}| = 1 \), where \( 1 \leq i \leq j \leq n \), holds for the following cases within the monotonicity intervals. Since \( \pi_{2i} = h \) when \( 0 \leq h \leq 2k-2 \) or \( 2l+1 \leq h \leq 2n+1 \) and \( \pi_{2i} = 2l+1 \) when \( 2k-1 \leq h \leq 2l \). We also have \( \pi_{2i} = 1 \) if \( i \leq k \) or \( l \leq i \leq n \), and \( \pi_{2i} = -1 \) if \( k+1 \leq i \leq l \). We also have \( \pi_{2i} = 2l+1 \) if \( 1 \leq j \leq k \) or \( l+1 \leq j \leq n \), and \( \pi_{2j} = -1 \) if \( k \leq l \).

Thus, the condition of Lemma does not hold for \( i = k \) or \( j = l \). Now show that for \( i \neq k \) and \( j \neq l \) we have \( b(R_{2k-1,2l}R_{2i-1,2j}) = 4 \). The condition \( |\pi_{2i-1}| = |\pi_{2j}| = 1 \) means \( \delta(\pi_{2i-2}, \pi_{2i-1}) = \delta(\pi_{2j}, \pi_{2j+1}) = 0 \) and by Lemma 2...
we have $b(R_{2k-1,2l}R_{2l-1,2j}) = b(R_{2k-1,2l}) + \delta(\pi_{2l-2}, \pi_{2j}) + \delta(\pi_{2l-1}, \pi_{2j+1}) = \delta(\pi_{2l-2}, \pi_{2j}) + \delta(\pi_{2l-1}, \pi_{2j+1}) + 2$.
Moreover, we have each of the equalities $|\pi_{2l-2} - \pi_{2j}| = 1$ or $|\pi_{2l-1} - \pi_{2j+1}| = 1$ if and only if $i = k$ and $j = l$.
Therefore, for all other cases we have $|\pi_{2l-2} - \pi_{2j}| \geq 2$ and $|\pi_{2l-1} - \pi_{2j+1}| \geq 2$, and hence, $\delta(\pi_{2l-2}, \pi_{2j}) = 1$ and $\delta(\pi_{2l-1}, \pi_{2j+1}) = 1$, this means that $b(R_{2k-1,2l}R_{2l-1,2j}) = 4$. □

Lemma 4. The set $\{Q : Q \in V_{1,2}(R_{2k-1,2l}), b(Q) = 3\}$ consists of disjoint sets $U_h(k, l), h = 1, \ldots, 6$, of permutations $Q = R_{2k-1,2l}R_{2l-1,2j}$ defined as follows:

1. $U_1(k, l)$ with $i = k, k \leq j \leq l - 1$; $U_1(k, l)$ consists of

   $Q = (0, k - 1, i + k - j, l, k, l + k - j - 1, l + 1, n + 1)$

   for which $\mu(Q) = (+, -);$
2. \( U_2(k, l) \) with \( i = k, l + 1 \leq j \leq n; \) \( U_2(k, l) \) consists of
\[
Q = (0, k-1, i, j, k, l, j+1, n+1)
\]
for which \( \mu(Q) = (-, +, +) \);
3. \( U_3(k, l) \) with \( 1 \leq i \leq k-1, j = l; \) \( U_3(k, l) \) consists of
\[
Q = (0, i-1, k, l, i, k-1, l+1, n+1)
\]
for which \( \mu(Q) = (+, -, +) \);
4. \( U_4(k, l) \) with \( k + 1 \leq i \leq l, j = l; \) \( U_4(k, l) \) consists of
\[
Q = (0, k-1, i, k, l, i, k-1, l+1, n+1)
\]
for which \( \mu(Q) = (-, +, -) \);
5. \( U_5(k, l) \) with \( i = l + 1, l + 1 \leq j \leq n; \) \( U_5(k, l) \) consists of
\[
Q = (0, k-1, i, l+1, j, j+1, n+1)
\]
for which \( \mu(Q) = (-, -) \);
6. \( U_6(k, l) \) with \( 1 \leq i \leq k-1, j = k - 1; \) \( U_6(k, l) \) consists of
\[
Q = (0, i-1, i, k, l, j, j+1, n+1)
\]
for which \( \mu(Q) = (+, -, +) \).

**Proof.** If the condition \(|x_{i-1}| = |x_j| = 1\), where \( 1 \leq i \leq j \leq n\), of Lemma 3 is satisfied for \( P = R_{2k-1,2l}, 1 \leq k \leq l \leq n\), then \( b(R_{2k-1,2l}R_{2l-2j}) = 4\).

If the condition \(|x_{i-1}| = |x_j| = 1\), where \( 1 \leq i \leq j \leq n\), of Lemma 3 is not satisfied then we have \( i = k \) or \( i = l + 1\), when \(|x_{i-1}| = \vert\pi_{2i-1} - \pi_{2j-1}\vert \geq 2\), and have \( j = k - 1 \) or \( j = l\), when \(|x_j| = \vert\pi_{2j-1} - \pi_{2j}\vert \geq 2\). These four cases \( i = k, i = l + 1, j = k - 1, j = l\) are incompatible, excepting the cases \( i = k \) and \( j = l\) when we have \( Q = R_{2k-1,2l}R_{2l-1,2j} = 1\).

The case \( i = k \) gives rise to the set \( U_1(k, l) \), when \( k \leq j \leq l - 1\), and gives rise to the set \( U_2(k, l) \), when \( l + 1 \leq j \leq n\). Analogously the case \( j = l \) gives rise to the set \( U_3(k, l) \), when \( 1 \leq i \leq k - 1\), and gives rise to the set \( U_4(k, l) \), when \( k + 1 \leq i \leq l\). The case \( i = l + 1 \) is possible only if \( l + 1 \leq j \leq n\) and we obtain the set \( U_5(k, l) \). Analogously the case \( j = k - 1 \) is possible only if \( 1 \leq i \leq k - 1\) and we obtain the set \( U_6(k, l) \). It is easily seen that permutations \( Q \) of all six sets have three breakpoints. The sets \( U_4(h, k, l), h = 1, \ldots, 6\), are disjoint since the even reversals \( R_{2l-1,2j} \) are distinct for all cases. They consist of permutations \( Q \) with \( \beta(Q) = (+, +, +) \) in the first four cases and with \( \beta(Q) = (+, +, +) \) in the last two cases. \( \square \)

**Corollary 1.** Any permutation \( Q = R_{2k-1,2l}R_{2l-1,2j} \) where \( R_{2k-1,2l} \neq R_{2l-1,2j} \) has three or four breakpoints, and the graph \( G_{2n} \) does not contain triangles.

**Lemma 5.** The set \( \{Q : Q \in V_{1,2}(R_{2k-1,2l}), b(Q) = 4\} \) consists of disjoint sets \( W_0(k, l), h = 1, \ldots, 4\), of permutations \( Q = R_{2k-1,2l}R_{2l-1,2j} \) defined as follows:

1. \( W_1(k, l) \) with \( k + 1 \leq l + 1 < i \leq j \) or \( i + 1 \leq j + 1 < k \leq l\); if \( Q \in W_1(k, l) \), then \( \beta(Q) = (+, +, +, +), \mu(Q) = (-, +, -) \);
2. \( W_2(k, l) \) with \( k < i \leq j \) or \( i < k \leq l < j\); if \( Q \in W_2(k, l) \), then \( \beta(Q) = (+, - , - , +), \mu(Q) = (+, - , - , +) \);
3. \( W_3(k, l) \) with \( i < k - 1 \leq j \leq l\); if \( Q \in W_3(k, l) \), then \( \beta(Q) = (+, + , + , +), \mu(Q) = (+, - , - , -) \);
4. \( W_4(k, l) \) with \( k < i \leq l + 1 \leq j\); if \( Q \in W_4(k, l) \), then \( \beta(Q) = (+, + , + , +), \mu(Q) = (+, - , - , +) \).

**Proof.** By Corollary 1 any permutation \( Q = R_{2k-1,2l}R_{2l-1,2j} \) where \( R_{2k-1,2l} \neq R_{2l-1,2j} \) has three or four breakpoints. Therefore, to prove this statement we consider all cases to arrange numbers \( i, j, 1 \leq i \leq j \leq n \), subject to the fixed numbers \( k, l, 1 \leq k \leq l \leq n \), and exclude the cases \( i = k, i = l + 1, j = k - 1, j = l \) of Lemma 4. \( \square \)

Lemmas 4 and 5 represent all sets \( V_{1,2}(R_{2k-1,2l}) \). Our next goal is to find the number of edges connecting \( V_1 \) to a fixed vertex \( Q \in V_2 \) that is to find \( |V_{2,1}(Q)| \). We consider the lexicographic ordering on the permutations \( R_{2k-1,2l} \),
1 \leq k \leq l \leq n$, assuming that $R_{2k-1,2l} < R_{2k'-1,2l'}$ if $k < k'$ or $k = k'$ and $l < l'$. Any pair of edges $(R_{2k-1,2l}, Q)$ and $(R_{2k'-1,2l'}, Q)$ of the graph $G_{2n}$ implies the following expression for $Q$:

$$R_{2k-1,2l} R_{2i-1,2j} = R_{2k'-1,2l'} R_{2i'-1,2j'},$$  \hspace{1cm} (9)

where $R_{2i-1,2j} = R_{2k-1,2l}$ and $R_{2i'-1,2j'} = R_{2k'-1,2l'}$ (see (4)). We shall say that (9) is a representation of $Q$ if $R_{2k-1,2l} < R_{2k'-1,2l'}$ and that (9) is a minimal representation of $Q$ if $R_{2k-1,2l}$ is minimal in the lexicographic order permutation of $V_{2,1}(Q)$. Note that if $Q$ has only one representation then this representation is minimal and $|V_{2,1}(Q)| = 2$. In a general case, if $Q$ has $h$ minimal representations with $h = 0, 1, \ldots$, then $|V_{2,1}(Q)| = h + 1$.

**Lemma 6.** Given a permutation $Q \in V_2$ let $k, 1 \leq k \leq n$, be the minimal integer such that $R_{2k-1,2s} \in V_{2,1}(Q)$ for some $s, k \leq s \leq n$. Then

1. $|V_{2,1}(Q)| = 2$, if $Q$ has one of the following representations:

$$R_{2k-1,2l} R_{2k-1,2j} = R_{2(k+l-j)-1,2l} R_{2k-1,2l}, \hspace{1cm} k \leq j \leq l - 1,$$

$$R_{2k-1,2l} R_{2k-1,2j} = R_{2k-1,2j} R_{2(k+j-l)-1,2j}, \hspace{1cm} l + 1 \leq j \leq n,$$

$$R_{2k-1,2l} R_{2i-1,2j} = R_{2i-1,2j} R_{2k-1,2l}, \hspace{1cm} k \leq i \leq j,$$

$$R_{2k-1,2l} R_{2i-1,2j} = R_{2l-(j+k)-1,2(l-i+k)} R_{2k-1,2l}, \hspace{1cm} k < i \leq j < l;$$

if $Q_1 \neq Q_2$, $|V_{2,1}(Q_1)| = 2$ and $|V_{2,1}(Q_2)| = 2$ then

$$|V_{2,1}(Q_1) \cap V_{2,1}(Q_2)| \leq 1;$$  \hspace{1cm} (14)

2. $|V_{2,1}(Q)| = 1$, in the remaining cases.

**Proof.** Let $|V_{2,1}(Q)| > 1$ and $s$ be some integer such that $R_{2k-1,2s} \in V_{2,1}(Q)$. Then the permutation $Q$ has a representation (9) (with the minimal integer $k$). To prove the theorem we describe all such representations of $Q$, determine which of them are minimal and then find $|V_{2,1}(Q)|$. If $Q$ has a representation (9) then, by Lemmas 4 and 5, we have $Q \in U_h(k, l) \cap U_m(k', l')$, if $b(Q) = 3$, and $Q \in W_h(k, l) \cap W_m(k', l')$, if $b(Q) = 4$, for some $h$ and $m$.

First we consider representations of permutations with three breakpoints. Note that for $Q \in U_1$ and $Q \in U_3$ we have $\mu(Q) = (+, -)$ and for $Q \in U_2$ and $Q \in U_4$ we have $\mu(Q) = (-, +)$. Therefore,

$$\left(U_1(k, l) \cup U_3(k, l)\right) \cap \left(U_2(k', l') \cup U_4(k', l')\right) = \emptyset.$$  \hspace{1cm} (15)

Since $\mu(Q) = (-, -)$, when $Q \in U_5 \cup U_6$, by the same argument, we have

$$\left(U_5(k, l) \cup U_6(k, l)\right) \cap \left(\bigcup_{m=1}^{4} U_m(k', l')\right) = \emptyset.$$  \hspace{1cm} (16)

Moreover,

$$U_h(k, l) \cap U_h(k', l') = \emptyset \hspace{1cm} \text{for any} \hspace{1cm} h = 1, \ldots, 6,$$

since $k$ and $l$ are uniquely defined by any element $Q \in U_h(k, l)$. These arguments show that if $Q$ with $b(Q) = 3$ has a representation (9) and $Q \in U_h(k, l) \cap U_m(k', l')$, then $(h, m)$ or $(m, h)$ must belong to the set

$$A = \{(1, 3), (2, 4), (5, 6)\}.$$  \hspace{1cm} (17)

We shall prove that if $R_{2k-1,2l} \in V_{2,1}(Q)$ then there exists a unique $R_{2k'-1,2l'}$ such that $Q \in U_h(k, l) \cap U_m(k', l')$ with $(h, m) \in A$ and $Q$ has the representation (9) (with $R_{2k-1,2l} < R_{2k'-1,2l'}$). It will follow from our proof that for cases $(m, h) \in A$ we also have the expression (9) but it is not a representation of $Q$ since in these cases $R_{2k-1,2l} < R_{2k'-1,2l'}$ does not hold.


Case 1: \((h, m) = (1, 3)\). For \(Q \in U_1(k, l) \cap U_3(k', l')\), we have \(i = k, k \leq j \leq l - 1, i' = k, j' = l = l', k' = l + k - j\) and get the representation (10).

Case 2: \((h, m) = (2, 4)\). For \(Q \in U_2(k, l) \cap U_4(k', l')\), we have \(i = k, l + 1 \leq j \leq n, j' = l', k' = k, i' = k' + l' - l = j + l - 1\) and get the representation (11).

Case 3: \((h, m) = (5, 6)\). For \(Q \in U_5(k, l) \cap U_6(k', l')\), we have \(i = l + 1, i' = k, j' = k - 1 = l, l' = j, i = l + 1 \leq j \leq n\) and get the representation (12).

Thus, (10)–(12) give the representations of all permutations with three breakpoints. Moreover, (10)–(12) are unique (and hence minimal) representations of distinct permutations. This implies the statement of theorem for permutations \(Q\) with three breakpoints (except inequality (14)).

Now we consider representations of permutations \(Q\) with four breakpoints and use Lemmas 2 and 5. If \(Q \in W_1(k, l)\) or \(Q \in W_2(k, l)\), then there exist two even reversals, namely \(R_{2l-1,2j}\) and \(R_{2k-1,2l}\), which decrease by 2 the number of breakpoints and transform \(Q\) to an element of \(V_1\). This gives rise to representations (12)–(13) of such permutations \(Q\). If \(Q \in W_3(k, l)\) or \(Q \in W_4(k, l)\) then there exists only one even reversal, namely \(R_{2l-1,2j}\), which decreases by 2 the number of breakpoints of \(Q\). Thus, for a permutation \(Q \in V_2\) with four breakpoints we have \(|V_{2,1}(Q)| = 2\), if \(Q\) is represented by (12)–(13) and \(|V_{2,1}(Q)| = 1\), otherwise.

Finally we prove (14). Let \(Q_1\) has the unique minimal representation (9) which coincides with one of representations (10)–(13). Then we have \(i = k, k' = k + l - j, l' = l, i' = k, j' = j, l' = j, j \neq l\), respectively. Let \(Q_2\) also has the unique minimal representation \(R_{2l-1,2j}R_{2k-1,2l} = R_{2k'-1,2l'}R_{2l'-1,2l'}\) which coincides with one of representations (10)–(13). Then we have \(s = k, k' = k + l - t, l' = l, s = k, k' = k, l' = t, or k' = s, l' = t, t \neq l\), respectively. It is easily seen that two from nine possible cases of coincidence are not compatible and remaining seven cases give rise to \(s = i, t = j\) and hence would imply \(Q_1 = Q_2\) if (14) is not valid. 

We denote by \(K_{m,h}\) a complete bipartite graph whose parts consist of \(m\) and \(h\) vertices, \(1 \leq m \leq h\).

**Theorem 1.** The graph \(G_{2n}\), \(n \geq 2\), does not contains subgraphs isomorphic to \(K_{2,3}\); each of its vertex belongs to \(\frac{1}{12}(n - 1)n(n + 1)(n + 4)\) subgraphs isomorphic to \(K_{2,2}\).

**Proof.** Let \(G_{2n}\) contains subgraphs isomorphic to \(K_{2,3}\). Without loss of generality one can assume that \(I\) belongs to the smaller part of \(K_{2,3}\). Hence there exist three distinct vertices from \(V_1\) which are adjacent with \(I\). Another vertex of the smaller part belongs to \(V_2\) and it is also adjacent to the same three vertices. That contradicts to Lemma 6 and proves the statement that there are no subgraphs isomorphic to \(K_{2,3}\) in \(G_{2n}\).

To prove the second statement, we only need to consider the total number \(N_2\) of the permutations \(Q \in V_2\) such that \(|V_{2,1}(Q)| = 2\). By Lemma 6, this number coincides with the number of all representations (10)–(13). The number of representations (10) equals

\[
\sum_{k=1}^{n-1} \sum_{l=k+1}^{n} (l - k) = \sum_{k=1}^{n-1} \sum_{h=1}^{n-k} h
\]

and the number of representations (11) equals

\[
\sum_{k=1}^{n-1} \sum_{l=k}^{n-1} (n - l) = \sum_{k=1}^{n-1} \sum_{h=1}^{n-k} h.
\]

Therefore, the number of representations (10)–(11) of permutations \(Q\) equals

\[
2 \sum_{k=1}^{n-1} \sum_{h=1}^{n-k} h = \sum_{k=1}^{n-1} (n^2 - 2nk + k^2 + n - k) = \frac{n(n - 1)(n + 1)}{3}.
\]

The number of representations (12)–(13) of permutations \(Q\) (with three and four breakpoints) equals \(\binom{n}{3}\), where \(k = l < i = j\), equals \(\binom{n}{3}\) in each of three cases \(k = l < i < j, k < l < i = j, k < i = j < l\), and equals \(2\binom{n}{4}\) in the cases when all \(i, j, k, l\) differ. Therefore, the number of representations (12)–(13) of permutations \(Q\) equals

\[
\binom{n}{3} \times 3 + 2 \binom{n}{4} = \frac{(n - 1)n^2(n + 1)}{12}.
\]
The total number \( N_2 \) of permutations \( Q \in V_2 \) such that \(|V_{2,1}(Q)| = 2\) equals

\[
N_2 = \frac{n(n-1)(n+1)}{3} + \frac{(n-1)n^2(n+1)}{12} = \frac{(n-1)n(n+1)(n+4)}{12}.
\]

**Corollary 2.**

\[|V_2| = \frac{(n-1)n(n+1)^2}{6}.\]

**Proof.** By Theorem 1 there are \( N_2 \) permutations \( Q \in V_2 \) for which \(|V_{2,1}(Q)| = 2\). To prove Corollary 3 we use that \(|V_1| = \frac{(n+1)}{2}\) and the fact that the graph \( G_{2n} \) does not contain triangles, by Corollary 1, and hence \(|V_2| = |V_1|(|V_1| - 1) - N_2\). \( \square \)

For example, for \( n = 3 \) we have \( N_2 = 14 \) and \(|V_2| = 16\) that can be checked by Fig. 1.

**4. The minimal number of erroneous patterns sufficient for reconstruction**

In this section we present the minimal number of erroneous patterns of an arbitrary permutation \( P \in \text{Sym}_{2n}^g \), which are sufficient for its reconstruction. First of all, we prove the following statement.

**Theorem 2.** For any \( n \geq 2 \) we have

\[
\max_{P, Q \in \text{Sym}_{2n}^g, P \neq Q} |B_1(Q) \cap B_1(P)| = 2.
\]

**Proof.** Without loss of generality one can assume that \( P = I \). If \( Q \in V_1 \), then \(|B_1(I) \cap B_1(Q)| = 2\), since there are no triangles in \( G_{2n} \). If \( Q \in V_2, 3 \leq i \leq n+1 \), then \(|B_1(I) \cap B_1(Q)| = 0\), by the triangle inequality. At last, if \( Q \in V_2 \), then by Lemma 6 we have \(|V_{2,1}(Q)| \leq 2\) that completes the proof. \( \square \)

We say that a permutation \( P \in \text{Sym}_{2n}^g \) is **reconstructible** from distinct permutations \( S_1, \ldots, S_h \in B_1(P) \), if there does not exist a permutation \( Q \in \text{Sym}_{2n}^g, Q \neq P \), such that \( \{S_1, \ldots, S_h\} \in B_1(Q) \). From this definition and Theorem 2 we have the following result.

**Corollary 3.** Any permutation \( P \in \text{Sym}_{2n}^g \) is reconstructible from any three distinct permutations in \( B_1(P) \).

We denote by \( t_i \) the number of sets of \( i \) distinct permutations in \( B_1(P) \) from which \( P \) is reconstructible. We also denote by \( N \) the number of permutations at reversal distance at most one from the permutation \( P \). It follows from (8) that \( N = |B_1(P)| = \frac{n(n+1)}{2} + 1 \). Then \( p_i = \frac{t_i}{\binom{n}{i}} \) is the probability of the event that a permutation \( P \) is reconstructible from \( i \) distinct permutations in \( B_1(P) \) under the condition that these permutations are uniformly distributed. It is evident that \( p_1 = 0 \) and \( p_3 = 1 \), i.e., we never can reconstruct any permutation \( P \) from a single permutation in \( B_1(P) \) and we can always reconstruct an arbitrary permutation \( P \) from three distinct permutations in \( B_1(P) \).

**Theorem 3.** \( p_2 \sim \frac{1}{3} \) as \( n \to \infty \).

**Proof.** By the definition, \( p_2 = \frac{t_2}{\binom{n}{2}} \), where \( t_2 \) is the number of sets of two distinct permutations in \( B_1(P) \) from which \( P \) is reconstructible. Without loss of generality one can consider \( I \) instead of \( P \). There are \( \frac{n(n+1)}{2} \) distinct pairs \((I, Q)\), \( Q \in V_1 \), that do not allow to reconstruct \( P \) uniquely. Moreover, a pair of permutations from \( V_1 \) that adjacent to one and the same permutation \( Q \in V_2 \) do not allow to reconstruct \( P \). By Theorem 1 there are exactly \( N_2 = \frac{1}{12} (n^4 + 4n^3 - 2n^2 - 4n) \) such pairs of permutations from \( V_1 \). Therefore, \( \binom{n}{2} - t_2 = \frac{n(n+1)}{2} + N_2 = \frac{1}{12} (n^4 + 4n^3 + 5n^2 + 2n) \). Since \( \binom{n}{2} = \frac{1}{8} (n^4 + 2n^3 + 3n^2 + 2n) \), we have \( p_2 = \frac{\binom{n}{2} - t_2}{\binom{n}{2}} \sim \frac{1}{3} \) as \( n \to \infty \) \( \square \)
By using Theorems 2 and 3 it is easy to realize a reconstruction algorithm for the cases where three or two distinct signed permutations are sufficient to reconstruct an arbitrary signed permutation. The similar algorithm for unsigned permutations was described in [2].

Now we show that the reconstruction of a signed permutation in the case of at most two reversal errors requires in general many more its distinct erroneous patterns.

**Theorem 4.** For any \( n \geq 2 \) we have

\[
\max_{P, Q \in \text{Sym}_{2n}, P \neq Q} |B_2(Q) \cap B_2(P)| \geq n(n + 1).
\]

**Proof.** Let \( P = I \) and let \( Q \in V_2 \) such that \( |V_{2,1}(Q)| = 2 \), and \( Q \) has the minimal representation (9). Then we show that

\[
|B_1(R_{2k-1,2l}) \cup B_1(R_{2k'-1,2l'})| \leq n(n + 1).
\]

Indeed, the metric balls \( B_1(R_{2k-1,2l}) \) and \( B_1(R_{2k'-1,2l'}) \) belong to \( B_2(I) \cap B_2(Q) \) and have two joint points \( I \) and \( Q \), since the graph \( G_{2n} \) does not contain triangles nor subgraphs isomorphic to \( K_{2,3} \), by Corollary 1 and Theorem 1. Each of the metric balls has size \( \left\lfloor \frac{n+1}{2} \right\rfloor + 1 \), and this implies the required statement. \( \square \)

By Corollary 1 the graph \( G_{2n} \) does not contain triangles. Therefore Fig. 1 represents the subgraph of \( G_{2,3} \) induced by the metric ball \( B_3(I) \) where edges between vertices at distance 3 from \( I \) are omitted. Note that the inequality (15) is attained for \( n = 3 \) when \( Q = [231], R_{3,6} = [132], \) and \( R_{1,6} = [321], \) since \( B_2(R_{3,6}) \cup B_2(R_{1,6}) = \{[123], [123], [123], [233], [132], [321], [231], [231], [231], [231] \} \) and \( |B_2(R_{3,6}) \cup B_2(R_{1,6})| = 12 \).

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**References**