Higher-order conditions for strict local Pareto minima in terms of generalized lower and upper directional derivatives

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A B S T R A C T

We introduce lower and upper limits of vector-valued functions with respect to the usual positive cone in a finite-dimensional space. Using these concepts, we extend the definitions of \(m\)-th order lower and upper directional derivatives introduced in Studniarski (1986) [1] to vector-valued functions, and prove some necessary and sufficient conditions for strict local Pareto minimizers of order \(m\).

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1. Introduction

In [1] Studniarski introduced new generalized lower and upper directional derivatives of order \(m\) for an arbitrary extended-real-valued function \(f : \mathbb{R}^n \to \mathbb{R}\) (see formulas (22)–(23)). More recently, these derivatives were applied to obtain higher-order optimality conditions for some classes of scalar and vector optimization problems (see [2–5]), but this was done without extending the definitions themselves to vector-valued functions. However, Sun and Li [6] defined and used similar objects for set-valued maps.

In this paper we define the generalized lower and upper directional derivatives of order \(m\), which extend the notions from [1] to functions with values in finite-dimensional vector spaces. We also show that these derivatives can be used to formulate higher-order optimality conditions for strict local Pareto minima in a multiobjective optimization problem. In this way, we improve some results from [5] by relaxing the assumptions concerning the minimized function.

2. Infima and suprema of sets in extended Euclidean spaces

Let \(\mathbb{R} = \mathbb{R} \cup \{-\infty, \infty\}\) be the set of extended real numbers. The arithmetic operations in \(\mathbb{R}\) are extended to \(\mathbb{R}\) in an obvious manner, except for the combinations \(0 \cdot (-\infty), 0 \cdot \infty, -\infty + \infty\) and \(\infty - \infty\) which we regard as undefined rather than defining them in any special way (such as, for example, in [7, p. 15]). The weak inequality \(\leq\) in \(\mathbb{R}\) is extended to \(\mathbb{R}\) by assuming that the following (and only the following) inequalities hold for infinite elements:

\(-\infty \leq \alpha \leq \infty \quad \forall \alpha \in \mathbb{R},\]
\(-\infty \leq -\infty, \quad -\infty \leq \infty, \quad \infty \leq \infty. \quad (1)\]

\textbf{Definition 1.} For any positive integer \(p\), the extended Euclidean space \(\mathbb{R}^p\) is defined as the Cartesian product of \(p\) copies of \(\mathbb{R}\). The operations of addition and scalar multiplication in \(\mathbb{R}^p\) are performed componentwise whenever the respective
operations in \(\bar{\mathbb{R}}\) are defined.

Let \(I := \{1, \ldots, p\}\). For any two elements \(x = (x_1, \ldots, x_p), y = (y_1, \ldots, y_p)\) of \(\bar{\mathbb{R}}^p\), we write
\[
x \leq y \quad \text{if and only if} \quad x_i \leq y_i \quad \text{for all} \quad i \in I;
\]
\[
x \leq y \quad \text{if and only if} \quad (x_i \leq y_i \text{ for all } i \in I) \text{ and (}x_j < y_j \text{ for some } j\); \]
\[
x < y \quad \text{if and only if} \quad x_i < y_i \quad \text{for all } i \in I.
\]
The negation of (3) gives
\[
x \not\leq y \quad \text{if and only if} \quad x = y \text{ or } x_j > y_j \text{ for some } j.
\]

Using (1) and (2), it is easy to prove the following.

**Proposition 2.** \((\bar{\mathbb{R}}^p, \leq)\) is a partially ordered set (that is, the relation \(\leq\) is reflexive, transitive and antisymmetric on \(\bar{\mathbb{R}}^p\)).

Let \(M\) be a nonempty subset of \(\bar{\mathbb{R}}^p\). Then the standard definitions of a lower (upper) bound of \(M\) and the infimum \(\inf M\) (supremum \(\sup M\)), see, e.g., [8, Definition 2.1.7], can be used whenever these objects exist in \(\bar{\mathbb{R}}^p\). However, it is easy to check that
\[
(\infty, \ldots, \infty) = \inf \bar{\mathbb{R}}^p, \quad (-\infty, \ldots, -\infty) = \sup \bar{\mathbb{R}}^p.
\]
Moreover, Proposition 3 shows that \(\inf M\) and \(\sup M\) exist in the general case.

We now define the projections \(\pi_i : \mathbb{R}^p \to \mathbb{R}\) by
\[
\pi_i(x_1, \ldots, x_p) := x_i, \quad i \in I.
\]

**Proposition 3.** Let \(M\) be a nonempty subset of \(\bar{\mathbb{R}}^p\). For \(i \in I\), define
\[
il_i := \inf \{\pi_i(x) : x \in M\}, \quad u_i := \sup \{\pi_i(x) : x \in M\}.
\]

Then
\[
il := (\tilde{l_1}, \ldots, \tilde{l_p}) = \inf M, \quad u := (\tilde{u_1}, \ldots, \tilde{u_p}) = \sup M.
\]

**Proof.** We prove this statement for the case of infimum only. First, we verify that \(\tilde{l}\) is a lower bound of \(M\). Indeed, for every \(y \in M\) and \(i \in I\), we have
\[
il_i = \inf \{\pi_i(x) : x \in M\} \leq \pi_i(y) = y_i,
\]
which gives that \(\tilde{l} \leq y\). Now, suppose that \(s\) is another lower bound of \(M\). Then, for every \(y \in M\), we have \(s \leq y\), which means that, for every \(i \in I\),
\[
s_i \leq y_i = \pi_i(y).
\]
Taking the infimum (in \(\bar{\mathbb{R}}\)) of the right-hand side, we get
\[
s_i \leq \inf \{\pi_i(x) : x \in M\} = \tilde{l}_i.
\]
We have thus verified that \(s \leq \tilde{l}\), which proves that \(\tilde{l} = \inf M\). \(\square\)

**Corollary 4.** For every nonempty set \(M \subset \bar{\mathbb{R}}^p\) and for every \(i \in I\), we have
\[
\pi_i(\inf M) = \inf \pi_i(M), \quad \pi_i(\sup M) = \sup \pi_i(M).
\]

**Proof.** This follows directly from (6)–(8). \(\square\)

### 3. Lower and upper limits of vector functions

Let \(X\) be a real normed space. Below we define lower and upper limits for a function \(\varphi : X \to \bar{\mathbb{R}}^p\) in such a way that they generalize the well-known definitions for an extended-real-valued function [7, pp. 8,13].

**Definition 5.** Let \(E\) be a nonempty subset of \(X\), and let \(\bar{x}\) be a limit point of \(E\). The **lower** and **upper limits** of a function \(\varphi : E \to \bar{\mathbb{R}}^p\) at \(\bar{x}\) are the elements of \(\bar{\mathbb{R}}^p\) defined by
\[
\liminf_{E \ni x \to \bar{x}} \varphi(x) := \lim_{\delta \to 0^+} \left( \inf_{x \in B(\bar{x}, \delta) \cap E} \varphi(x) \right) = \sup_{\delta > 0} \left( \inf_{x \in B(\bar{x}, \delta) \cap E} \varphi(x) \right),
\]
\[
\limsup_{E \ni x \to \bar{x}} \varphi(x) := \lim_{\delta \to 0^+} \left( \sup_{x \in B(\bar{x}, \delta) \cap E} \varphi(x) \right) = \inf_{\delta > 0} \left( \sup_{x \in B(\bar{x}, \delta) \cap E} \varphi(x) \right),
\]
where \(B(\bar{x}, \delta) := \{x \in X : ||x - \bar{x}|| < \delta\}\).
Remark 6. The second equality in (10) follows from (9) and the fact that each component of $\inf_{x \in B(\bar{x}, \delta)} \varphi(x)$ is a nonincreasing function of $\delta > 0$. A similar explanation is valid for (11). These properties also imply that

$$\liminf_{E \ni x \to \bar{x}} \varphi(x) \leq \limsup_{E \ni x \to \bar{x}} \varphi(x).$$  \hspace{1cm} (12)

Proposition 7. For any function $\varphi = (\varphi_1, \ldots, \varphi_p) : X \to \mathbb{R}^p$ and $\bar{x} \in X$, we have

$$\liminf_{E \ni x \to \bar{x}} \varphi(x) = \left( \liminf_{E \ni x \to \bar{x}} \varphi_1(x), \ldots, \liminf_{E \ni x \to \bar{x}} \varphi_p(x) \right),$$  \hspace{1cm} (13)

$$\limsup_{E \ni x \to \bar{x}} \varphi(x) = \left( \limsup_{E \ni x \to \bar{x}} \varphi_1(x), \ldots, \limsup_{E \ni x \to \bar{x}} \varphi_p(x) \right).$$  \hspace{1cm} (14)

Proof. Applying Corollary 4 twice, we obtain, for any $i \in I$,

\[
\pi_i \left( \liminf_{E \ni x \to \bar{x}} \varphi(x) \right) = \pi_i \left( \sup_{\delta > 0} \left( \inf_{x \in B(\bar{x}, \delta) \cap E} \varphi_i(x) \right) \right) \\
= \sup_{\delta > 0} \left( \inf_{x \in B(\bar{x}, \delta) \cap E} \pi_i(\varphi_i(x)) \right) \\
= \sup_{\delta > 0} \left( \inf_{x \in B(\bar{x}, \delta) \cap E} \varphi_i(x) \right) = \liminf_{E \ni x \to \bar{x}} \varphi_i(x),
\]

which proves (13). The proof of (14) is analogous. \hspace{1cm} \Box

Proposition 8. Let $z \in \mathbb{R}^p$ be such that

$$\limsup_{E \ni x \to \bar{x}} \varphi(x) < z.$$  \hspace{1cm} (15)

Then there exists $\delta > 0$ such that

$$\varphi(x) < z \quad \text{for all } x \in B(\bar{x}, \delta) \cap E.$$  \hspace{1cm} (16)

Proof. Suppose that (15) holds. Then, according to (14), we have

$$\limsup_{E \ni x \to \bar{x}} \varphi_i(x) < z_i, \quad i \in I.$$

By (11), this means that

$$\inf_{\delta > 0} \left( \sup_{x \in B(\bar{x}, \delta) \cap E} \varphi_i(x) \right) < z_i, \quad i \in I.$$

Hence, for each $i \in I$, there exists $\delta_i > 0$ such that

$$\varphi_i(x) < z_i \quad \text{for all } x \in B(\bar{x}, \delta_i) \cap E.$$

Now, taking $\delta := \min\{\delta_i : i \in I\}$, we get (16). \hspace{1cm} \Box

4. Multiobjective optimization

Let $X$ and $Y$ be normed spaces. We shall deal with the following multiobjective optimization problem:

$$\min \{ f(x) : x \in S \},$$  \hspace{1cm} (17)

where

$$S := \{ x \in X : -g(x) \in D, x \in C \}.$$  \hspace{1cm} (18)

$f = (f_1, \ldots, f_p) : X \to \mathbb{R}^p$ and $g : X \to Y$. We assume that $C$ and $D$ are nonempty closed subsets of $X$ and $Y$, respectively, and $D$ is a convex cone, $D \neq Y$. The minimization in (17) is understood with respect to the partial order defined by (2), or, which is equivalent, with respect to the positive cone $\mathbb{R}_+^p := [0, \infty)^p$. Although here $f$ takes on finite vector values only, the theory developed in the preceding two sections will be useful for considering some kind of directional derivatives of $f$ and a vector indicator function of $S$ (see below).

We denote by $N(x)$ the collection of all neighborhoods of $x$. 

**Definition 9** ([2]). Let \( m \) be a positive integer, and let \( \bar{x} \in S \).

(a) We say that \( \bar{x} \) is a strict local Pareto minimizer of order \( m \) (or strict local efficient solution of order \( m \)) for (17), denoted \( \bar{x} \in \text{Str}L(m, f, S) \), if there exist \( \alpha > 0 \) and \( U \in \mathcal{N}(\bar{x}) \) such that

\[
(f(x) + \mathbb{R}^p) \cap B(f(\bar{x}), \alpha \|x - \bar{x}\|^m) = \emptyset \quad \text{for all } x \in S \cap U \setminus \{\bar{x}\}.
\]

(b) We say that \( \bar{x} \) is a super-strict local Pareto minimizer of order \( m \) (or super-strict local efficient solution of order \( m \)) for (17), denoted \( \bar{x} \in \text{SStr}L(m, f, S) \), if there exist \( \alpha > 0 \), \( U \in \mathcal{N}(\bar{x}) \), and at most \( p \) open cones \( A_i \) (without 0) \( i \in I' \subset I \), such that \( |V_i := x + A_i : i \in I' \) is a covering of \( S \cap U \setminus \{\bar{x}\} \) and

\[
f_i(x) > f_i(\bar{x}) + \alpha \|x - \bar{x}\|^m \quad \text{for all } x \in S \cap U \cap V_i.
\]

**Proposition 10** ([9, Proposition 2.11]). \( \bar{x} \in \text{Str}L(m, f, S) \) if and only if there exist \( \eta \in \text{int} \mathbb{R}^p \) and \( U \in \mathcal{N}(\bar{x}) \) such that

\[
f(x) \not\leq f(\bar{x}) + \eta \|x - \bar{x}\|^m \quad \text{for all } x \in S \cap U \setminus \{\bar{x}\},
\]

that is, for any \( x \in S \cap U \setminus \{\bar{x}\} \), the following cannot hold:

\[
\begin{align*}
f_i(x) &\leq f_i(\bar{x}) + \eta_i \|x - \bar{x}\|^m \quad \text{for all } i \in I, \\
f_j(x) &< f_j(\bar{x}) + \eta_j \|x - \bar{x}\|^m \quad \text{for some } j \in I.
\end{align*}
\]

Extending the definitions from [1] to vector-valued functions, we now introduce the following \( m \)-th order lower and upper directional derivatives:

\[
\begin{align*}
d^m f(\bar{x}; y) &:= \liminf_{(t,v) \rightarrow (0^+, y)} \frac{f(\bar{x} + tv) - f(\bar{x})}{t^m}, \\
d^m f(\bar{x}; y) &:= \limsup_{(t,v) \rightarrow (0^+, y)} \frac{f(\bar{x} + tv) - f(\bar{x})}{t^m},
\end{align*}
\]

where the limits are in the sense of **Definition 5**. More precisely, we have

\[
\begin{align*}
d^m f(\bar{x}; y) &:= \sup_{\delta > 0} \left( \inf_{\substack{t \in (0, \delta) \ \subset \mathbb{R}\, (y, \delta) \ \subset \mathbb{R}\, \subset (0^+, y) \ \subset \mathbb{R}\, \subset \mathbb{R}}} \frac{f(\bar{x} + tv) - f(\bar{x})}{t^m} \right), \\
d^m f(\bar{x}; y) &:= \inf_{\delta > 0} \left( \sup_{\substack{t \in (0, \delta) \ \subset \mathbb{R}\, (y, \delta) \ \subset \mathbb{R}}} \frac{f(\bar{x} + tv) - f(\bar{x})}{t^m} \right).
\end{align*}
\]

Applying **Proposition 7**, we get

\[
\begin{align*}
d^m f(\bar{x}; y) &\equiv (d^m f_1(\bar{x}; y), \ldots, d^m f_p(\bar{x}; y)), \\
d^m f(\bar{x}; y) &\equiv (d^m f_1(\bar{x}; y), \ldots, d^m f_p(\bar{x}; y)),
\end{align*}
\]

where the components of the right-hand sides are exactly the expressions studied in [1]. They may have infinite values, and therefore the derivatives (22)–(23) belong to \( \mathbb{R}^p \) in general. For \( m = 1 \), we will use the notation \( \partial f(\bar{x}; y) \) and \( \overline{\partial} f(\bar{x}; y) \) instead of \( d^1 f(\bar{x}; y) \) and \( \overline{d} f(\bar{x}; y) \).

We will also use the notation

\[
d^m f(\bar{x}; y) := \lim_{(t,v) \rightarrow (0^+, y)} \frac{f(\bar{x} + tv) - f(\bar{x})}{t^m},
\]

whenever this limit exists as a finite element in \( \overline{\mathbb{R}}^p \) (i.e., \( d^m f(\bar{x}; y) \in \mathbb{R}^p \)). The same notation can be used for the function \( g \) if the corresponding limit exists in \( Y \).

We denote by \( K(S, \bar{x}) \) the contingent cone to \( S \) at \( \bar{x} \):

\[
K(S, \bar{x}) := \{y \in X : \exists (t_n, y_n) \rightarrow (0^+, y) \text{ such that } x + t_n y_n \in S, \forall n\}.
\]

Let us introduce the following notation:

\[
\begin{align*}
K_f(S, \bar{x}) &:= K(S, \bar{x}) \cap \{y \in X : \partial f(\bar{x}; y) \leq 0\} \\
&= K(S, \bar{x}) \cap \bigcap_{i=1}^p \{y \in X : d f_i(\bar{x}; y) \leq 0\}.
\end{align*}
\]
Since $K(S, \bar{x})$ is a closed cone and each function $df_i(\bar{x}; \cdot)$ is positively homogeneous and lower semicontinuous, the set (30) is also a closed cone (containing 0).

The following vector indicator function of the set $S$ will be used:

$$\Delta(x|S) := (\delta(x|S), \ldots, \delta(x|S)) \in \bar{R}^p \quad \text{for } x \in X,$$

where

$$\delta(x|S) := \begin{cases} 
0 & \text{if } x \in S, \\
\infty & \text{if } x \notin S.
\end{cases}$$

We also define the function $f^S : X \to \bar{R}^p$ as follows: $f^S := f + \Delta(\cdot|S)$. Then

$$f^S = (f_1^S, \ldots, f_p^S), \quad \text{where } f_i^S := f_i + \delta(\cdot|S), \text{ } i \in I.$$

5. Necessary optimality conditions

To formulate our optimality conditions, we will need the extended positive cone $\bar{R}^p := [0, \infty]^p$.

**Theorem 11.** Let $\bar{x} \in \text{StrL}(m, \bar{f}, S)$.

(a) Suppose that $\text{int } D \neq \emptyset$ and $\text{dg}(\bar{x}; y)$ exists for all $y \in X$. Then there exists $\beta > 0$ such that

$$\bar{d}^m f(\bar{x}; y) \not\in B(0, \beta \|y\|^m) - \bar{R}^p_+ \quad \text{for all } y \in K(C, \bar{x}) \cap \{u \in X : \text{dg}(\bar{x}; u) \in -\text{int } D\}.$$  

(b) Let $Y = \bar{R}^g$ and $D = \bar{R}^g_+$. Then there exists $\beta > 0$ such that

$$\bar{d}^m f(\bar{x}; y) \not\in B(0, \beta \|y\|^m) - \bar{R}^g_+ \quad \text{for all } y \in K(C, \bar{x}) \cap \{u \in X : \text{dg}(\bar{x}; u) < 0\}.$$

**Proof.** (a) (Using the idea of Theorem 3.1(a) $\Rightarrow$ (b) in [10].) By assumption, there exist $\alpha > 0$ and $U \in \mathcal{N}(\bar{x})$ such that condition (19) holds, which is equivalent to

$$f(x) - f(\bar{x}) \not\in B(0, \alpha \|x - \bar{x}\|^m) - \bar{R}^p_+ \quad \text{for all } x \in S \cap U \setminus \{\bar{x}\}.$$  

We will show that (34) holds with $\beta = \alpha/2^m$. Suppose that this is not true; then there exist $y \in K(C, \bar{x}), u \in B(0, \beta \|y\|^m)$ and $z \in \bar{R}^p_+$ such that

$$\text{dg}(\bar{x}; y) \in -\text{int } D$$

and

$$\bar{d}^m f(\bar{x}; y) = u - z \leq u.$$  

(Not that $\bar{d}^m f(\bar{x}; y)$ can have some components equal to $-\infty$.) Let $\varepsilon > 0$ be such that

$$u + \varepsilon e \in B(0, \beta \|y\|^m).$$

where $e := (1, \ldots, 1)^T \in \bar{R}^p$. Then condition (38) implies

$$\bar{d}^m f(\bar{x}; y) \not\in u + \varepsilon e.$$  

Since $D$ is a cone and $D \neq Y$, condition (37) implies $\text{dg}(\bar{x}; y) \neq 0$, and so, $y \neq 0$ (it should be noted that $\text{dg}(\bar{x}; 0) = 0$ by definition (28) because $\nu = y$ is an allowable choice for $\nu$). Therefore, by Proposition 8 and condition (37), we can find $\delta > 0$ such that

$$\delta \leq \frac{\|y\|^m}{2},$$

$$\bar{x} + (0, \delta)B(y, \delta) \subset U,$$

$$\frac{f(\bar{x} + tv) - f(\bar{x})}{t^m} < u + \varepsilon e \quad \text{for all } t \in (0, \delta) \text{ and } v \in B(y, \delta),$$

$$\frac{g(\bar{x} + tv) - g(\bar{x})}{t} \in -D \quad \text{for all } t \in (0, \delta) \text{ and } v \in B(y, \delta).$$

Since $y \in K(C, \bar{x})$, we have

$$\bar{x} + (0, \delta)B(y, \delta) \cap C \neq \emptyset.$$  

Moreover, condition (44) and the convexity of $D$ give

$$g(\bar{x} + tv) \in g(\bar{x}) - D \subset -D \quad \text{for all } t \in (0, \delta) \text{ and } v \in B(y, \delta).$$
It follows from (41)–(44) and (46) that there exist \( \lambda \in (0, \delta) \) and \( w \in B(y, \delta) \) such that
\[
\lambda \frac{f(x + \lambda w) - f(x)}{\lambda} < u + \epsilon e,
\]
(47)
\[
g(x + \lambda w) - D. \quad (49)
\]
Conditions (39) and (48) imply that
\[
\frac{f(x + \lambda w) - f(x)}{\lambda} \in B(0, \beta \|y\|^m) - \mathbb{R}^n_+,
\]
or equivalently,
\[
f(x + \lambda w) - f(x) \in B(0, \beta \|\lambda y\|^m) - \mathbb{R}^n_+.
\] (50)
(Note that \( f(x + \lambda w) - f(x) \) cannot have infinite values because \( f \) is finite-valued.) Now, taking \( x := x + \lambda w \), we obtain, by (47), (49) and (50),
\[
x \in C \cap U \setminus \{\bar{x}\}, \ g(x) \in -D \quad \text{and} \quad f(x) - f(x) \in B(0, \beta \|\lambda y\|^m) - \mathbb{R}^n_+.
\] (51)
It follows from (41) and the inequality \( \|w - y\| < \delta \) that
\[
\|y\| - \|w\| \leq \|y - w\| < \delta \leq \|y\|/2,
\]
which gives \( \|y\|/2 < \|w\| \). From this inequality, and from the definition of \( \beta \), we deduce
\[
B(0, \beta \|\lambda y\|^m) \subset B(0, \alpha \|\lambda y\|^m) = B(0, \alpha \|x - \bar{x}\|^m).
\] (52)
Conditions (51) and (52) give contradiction with (36).
(b) The proof is almost the same as for the part (a); there are only a few changes that should be noted:
(i) Instead of (37), we have \( \bar{d}g(x, y) < 0 \), which implies, by Proposition 8, that there exists \( \delta > 0 \) such that
\[
g(x + tv) - g(x) < 0 \quad \text{whenever} \ 0 < t < \delta \quad \text{and} \quad \|v - y\| < \delta.
\] (53)
From this we infer that \( y \neq 0 \) since otherwise we get a contradiction \((0 < \epsilon)\) by inserting \( v = 0 \) in (53).
(ii) Conditions (44) and (46) should be replaced, respectively, by the following ones:
\[
g(x + tv) - g(x) < 0 \quad \text{for all} \ t \in (0, \delta) \quad \text{and} \quad v \in B(y, \delta),
\]
\[
g(x + tv) < g(x) \leq 0 \quad \text{for all} \ t \in (0, \delta) \quad \text{and} \quad v \in B(y, \delta). \quad \square
\]

**Example 12.** Let \( f : \mathbb{R} \to \mathbb{R}^2 \) be defined by
\[
f(x) := (x^3, x^2).
\]
Consider the multiobjective optimization problem (17), where
\[
S := \{x \in \mathbb{R} : -g(x) = x \in \mathbb{R}_+ \} = \mathbb{R}_+.
\]
(Here \( D := \mathbb{R}_+ \), \( C := \mathbb{R} \) and \( g : \mathbb{R} \to \mathbb{R} \) is defined by \( g(x) := -x \)). First, let us show that \( \bar{x} = 0 \) is a strict local Pareto minimizer of order \( m \) for any integer \( m \geq 2 \). Indeed, we can take \( U := B(0, 1) \) and \( \alpha = 1 \), then
\[
S \cap U \setminus \{\bar{x}\} = \mathbb{R}_+ \cap U \setminus \{0\} = \{x : 0 < x < 1\}.
\]
To verify condition (19), observe that, for each \( x \) such that \( 0 < x < 1 \), and for each \((u_1, u_2) \in (x^3, x^2) + \mathbb{R}^2_+ \), we have \( u_1 \geq x^3, u_2 \geq x^2 \), which implies
\[
u_1^2 + u_2^2 \geq x^6 + x^4 > x^4,
\]
and consequently,
\[
\|(u_1, u_2)\| = \sqrt{u_1^2 + u_2^2} > x^2 \geq x^m.
\]
This implies that
\[
((x^3, x^2) + \mathbb{R}^2_+) \cap B((0, 0), \alpha x^m) = \emptyset.
\]
Next, we have that $dg(0; y)$ exists for all $y \in \mathbb{R}$ and $dg(0; y) = -y < 0$ for $y > 0$. Moreover, $K(C, \bar{x}) = \mathbb{R}$, and, using equality (27), we compute, for $y > 0$,

$$d^m f(0; y) = \begin{cases} 
(0, y^2) & \text{if } m = 2, \\
(y^2, \infty) & \text{if } m = 3, \\
(\infty, \infty) & \text{if } m > 3.
\end{cases}$$

Therefore, for each $\beta \in (0, 1)$, condition (35) holds, i.e.

$$d^m f(0; y) \not\in B(0, \beta y^m) - \mathbb{R}_+^3.$$  

for all $m \geq 2$ and $y > 0$. This relation is true for $m = 2$ because $\beta < 1$, and for $m \geq 3$ because $d^m f(0; y)$ contains components equal to $\infty$, while the components of every vector in $B(0, \beta y^m) - \mathbb{R}_+^3$ are either finite or equal to $-\infty$.

**Example 13.** Let $f = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2$ and $g : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x_1, x_2) := (x_1^2 + x_2, x_1^2 - x_2), \quad g(x_1, x_2) := -|x_2|.$$  

Consider problem (17), where

$$S := \{(x_1, x_2) \in \mathbb{R}^2 : -g(x_1, x_2) = |x_2| \in \mathbb{R}_+ \} = \mathbb{R}^2.$$  

(Here $D := \mathbb{R}_+$ and $C := \mathbb{R}^2_+.$) The point $\bar{x} = (0, 0)$ is a strict local Pareto minimizer of order 2. To see this, we can take $\alpha = 1/2$ and $U := B(\bar{x}, 1)$, then

$$S \cap U \setminus \{\bar{x}\} = \mathbb{R}^2 \cap U \setminus \{\bar{x}\} = \left\{(x_1, x_2) : 0 < x_1^2 + x_2^2 < 1 \right\}.$$  

Now, let $(x_1, x_2) \in S \cap U \setminus \{\bar{x}\}$ and $(h_1, h_2) \in \mathbb{R}^2_+$; we consider two cases:

1. $0 \leq x_2 < 1$, then $x_2 \geq x_2^2$, hence

$$\|f(x_1, x_2) + (h_1, h_2)\| \geq |f_1(x_1, x_2) + h_1| = x_1^2 + x_2 + h_1 \geq x_1^2 + x_2^2,$$

$$(x_1^2 + x_2^2) \cdot \left(1 + \frac{1}{2} (x_1^2 + x_2^2) \right) = \left(\alpha \|(x_1, x_2) - \bar{x}\|^2 \right),$$

(54)

2. $-1 < x_2 < 0$, then $x_2 \leq x_2^2$, hence

$$\|f(x_1, x_2) + (h_1, h_2)\| \geq |f_2(x_1, x_2) + h_2| = x_1^2 - x_2 + h_2 \geq x_1^2 + x_2^2,$$

$$(x_1^2 + x_2^2) \cdot \left(1 + \frac{1}{2} (x_1^2 + x_2^2) \right) = \left(\alpha \|(x_1, x_2) - \bar{x}\|^2 \right).$$

(55)

Conditions (54) and (55) imply that (19) holds with $m = 2$.

We shall verify condition (35). We have $K(C, \bar{x}) = \mathbb{R}^2$ and $dg(\bar{x}; (y_1, y_2)) = -|y_2| < 0$ for $y_2 \neq 0$. Take any vector $y = (y_1, y_2)$ with $y \neq 0$. Then

$$\bar{d}^2 f_1(\bar{x}; (y_1, y_2)) = \limsup_{(t, v) \to (0^+, y)} \frac{f_1(tv_1, tv_2)}{t^2} = \limsup_{(t, v) \to (0^+, y)} \frac{tv_1^2 + tv_2}{t^2}$$

$$= \limsup_{(t, v) \to (0^+, y)} \left( \frac{v_1^2 + v_2}{t} \right) = \begin{cases} 
\infty & \text{if } y_2 > 0, \\
-\infty & \text{if } y_2 < 0.
\end{cases}$$

and

$$\bar{d}^2 f_2(\bar{x}; (y_1, y_2)) = \limsup_{(t, v) \to (0^+, y)} \frac{f_2(tv_1, tv_2)}{t^2} = \limsup_{(t, v) \to (0^+, y)} \frac{tv_1^2 - tv_2}{t^2}$$

$$= \limsup_{(t, v) \to (0^+, y)} \left( \frac{v_1^2 - v_2}{t} \right) = \begin{cases} 
-\infty & \text{if } y_2 > 0, \\
\infty & \text{if } y_2 < 0.
\end{cases}$$

Consequently,

$$\bar{d}^2 f(\bar{x}; (y_1, y_2)) = \begin{cases} 
(\infty, -\infty) & \text{if } y_2 > 0, \\
(-\infty, \infty) & \text{if } y_2 < 0.
\end{cases}$$

i.e. (35) holds because no vector in $B(0, \beta \|y\|^m) - \mathbb{R}_+^2$ can have components equal to $\infty$.  

6. Sufficient optimality conditions

The first theorem on sufficient conditions is formulated for an arbitrary constraint set $S$ (we do not assume condition (18)).

**Theorem 14.** Let $\dim X < \infty$, and let $\tilde{x}$ be a feasible point for problem (17).

(a) If $m > 1$ and

$$d^m f^\gamma (\tilde{x}; y) \not\in -\mathbb{R}_+^p \quad \text{for all } y \in K_f (S, \tilde{x}) \setminus \{0\},$$

then $\tilde{x} \in S\text{Str} (m, f, S)$.

(b) If

$$d^m f_i (\tilde{x}; y) \not\in -\mathbb{R}_+^p \quad \text{for all } y \in K (S, \tilde{x}) \setminus \{0\},$$

then $\tilde{x} \in S\text{Str} (1, f, S)$.

**Proof.** (a) By (26) and (33), condition (56) is equivalent to

$$\max_{i \in I} d^m f_i^\gamma (\tilde{x}; y) > 0 \quad \text{for all } y \in K_f (S, \tilde{x}) \setminus \{0\}.$$  

(58)

Now, it follows from the implication (c) $\Rightarrow$ (a) in [3, Theorem 3.1] that $\tilde{x} \in S\text{Str} (m, f, S)$. The proof of part (b) is similar.

In the next theorem we shall use the following notation for the closure of the cone generated by $D + g(\tilde{x})$:

$$D_{g(\tilde{x})} := \text{cl cone}(D + g(\tilde{x})).$$  

(59)

It follows from the convexity of $D$ that $D_{g(\tilde{x})}$ is a closed convex cone.

**Theorem 15.** Let $\dim X < \infty$, and let $\tilde{x}$ be a feasible point for problem (17)-(18). Suppose that $dg(\tilde{x}; y)$ exists for all $y \in X$. If

$$d^m f(\tilde{x}; y) \not\in -\mathbb{R}_+^p, \quad \forall y \in K (C, \tilde{x}) \cap \{ u \in X : dg(\tilde{x}; u) \in -D_{g(\tilde{x})} \} \setminus \{0\},$$

then $\tilde{x} \in \text{Str} (m, f, S)$.

**Proof.** (Using the idea of Theorem 4.1 in [5].) Suppose on the contrary that $\tilde{x} \not\in \text{Str} (m, f, S)$. Then it follows from Definition 9(a) that, for each positive integer $n$, there exist $x_n \in S \cap B(\tilde{x}, 1/n) \setminus \{\tilde{x}\}$ and $d_n = (d_{n,1}, \ldots, d_{n,p}) \in \mathbb{R}_+^p$ such that

$$f(x_n) - f(\tilde{x}) + d_n \in B \left( \frac{1}{n} \|x_n - \tilde{x}\|^m \right),$$

which is equivalent to

$$\frac{f(x_n) - f(\tilde{x})}{\|x_n - \tilde{x}\|^m} + \frac{d_n}{\|x_n - \tilde{x}\|^m} \in B \left( \frac{1}{n} \frac{1}{m} \right).$$

(61)

We may assume, by choosing a subsequence if necessary, that $v_n := (x_n - \tilde{x})/\|x_n - \tilde{x}\|$ converges to some vector $v$ with $\|v\| = 1$. Define $t_n := \|x_n - \tilde{x}\|$, then $t_n \to 0+$ and $x_n = \tilde{x} + t_nv_n$ for all $n$. Since $x_n \in S \subset C$, we have $v \in K (C, \tilde{x})$ by (29). Since $dg(\tilde{x}; v)$ exists, it must satisfy

$$dg(\tilde{x}; v) := \lim_{n \to \infty} \frac{g(\tilde{x} + t_nv_n) - g(\tilde{x})}{t_n}.$$

(62)

Moreover, $g(\tilde{x} + t_nv_n) = g(x_n) \in -D$, and consequently,

$$g(\tilde{x} + t_nv_n) - g(\tilde{x}) \in \text{cone}(-D - g(\tilde{x})) \subset -D_{g(\tilde{x})}, \quad \forall n.$$

(63)

Conditions (62) and (63), $v \in K (C, \tilde{x}), \|v\| = 1$, and the closedness of $D_{g(\tilde{x})}$ imply that

$$v \in K (C, \tilde{x}) \cap \{ u \in X : dg(\tilde{x}; u) \in -D_{g(\tilde{x})} \} \setminus \{0\}.$$  

(64)

It follows from (61) that

$$\lim_{n \to \infty} \left( \frac{f(x_n) - f(\tilde{x})}{\|x_n - \tilde{x}\|^m} + \frac{d_{n,i}}{\|x_n - \tilde{x}\|^m} \right) = 0, \quad \forall i \in I,$$

which can be rewritten as

$$\lim_{n \to \infty} \left( \frac{f(\tilde{x} + t_nv_n) - f(\tilde{x})}{t_n^m} + \frac{d_{n,i}}{t_n^m} \right) = 0, \quad \forall i \in I.$$

(65)
Now, using (22) and (65), we obtain, for each $i$,
\[
\overrightarrow{d^m f_i}(\bar{x}; v) \leq \liminf_{n \to \infty} \frac{f_i(x + t_n v_n) - f_i(\bar{x})}{t_n^m} = \lim_{n \to \infty} \left( \frac{f(\bar{x} + t_n v_n) - f(\bar{x})}{t_n^m} + \frac{d_{n,i}}{t_n^m} \right) + \liminf_{n \to \infty} \left( \frac{d_{n,i}}{t_n^m} \right).
\]
Equation (66).

The condition $d_n \in \mathbb{R}_+^p$, implies that $-d_{n,i}/t_n^m < 0$, and by (66), $\overrightarrow{d^m f_i}(\bar{x}; v) < 0 \ (\forall i)$. Therefore,
\[
\overrightarrow{d^m f}(\bar{x}; v) < -\overrightarrow{\mathbb{R}_+^p},
\]
which, in view of (64), contradicts (60).

**Example 16.** Consider problem (17), where $f = (f_1, f_2) : \mathbb{R} \to \mathbb{R}^2$ is defined by
\[
f(x) := \begin{cases} 
(x \sin \frac{1}{x}, x \left(2 - \sin \frac{1}{x}\right)) & \text{if } x \neq 0, \\
(0, 0) & \text{if } x = 0,
\end{cases}
\]
and the feasible set is given by
\[
S := \{ x \in \mathbb{R} : -g(x) = |x| \in \mathbb{R}_+, x \in \mathbb{R}_+ \} = \mathbb{R}_+.
\]
(Here $D := \mathbb{R}_+, C := \mathbb{R}_+$, and $g : \mathbb{R} \to \mathbb{R}$ is defined by $g(x) := -|x|$.) Let $\bar{x} = 0$. For all $y \in \mathbb{R}$, we have that $dg(0; y) = -|y|$ exists and $dg(0; y) \in -Dg(0) = \mathbb{R}_+$. It is clear that $K(\mathbb{R}_+, 0) = \mathbb{R}_+$, hence
\[
K(\mathbb{R}_+, 0) \cap \{ u \in \mathbb{R} : dg(\bar{x}; u) \in -Dg(0) \} = \mathbb{R}_+.
\]
It is not difficult to see that, for all $y \in \mathbb{R}_+$,
\[
df_1(0; y) = -y, \quad \overleftarrow{df}_1(0; y) = y, \quad df_2(0; y) = y, \quad \overleftarrow{df}_2(0; y) = 3y,
\]
therefore
\[
df(0; y) = (-y, y) \not\in -\overrightarrow{\mathbb{R}_+^2}, \quad \forall y \in \mathbb{R}_+ \setminus \{0\},
\]
and so, condition (60) is satisfied. Moreover, if we take $\alpha = 1/2$, then
\[
\|f(x)\| \geq \|f_2(x)\| = x \left(2 - \sin \frac{1}{x}\right) > \alpha x, \quad \forall x \in S \setminus \{0\} = \mathbb{R}_+ \setminus \{0\},
\]
hence condition (19) holds with $m = 1$ and $U = \mathbb{R}$, which means that $\bar{x} = 0$ is a strict (global) Pareto minimizer of order one.

Note that this example cannot be solved by using Theorem 4.1 of [5] because $df(0; y) \neq \overleftarrow{df}(0; y)$ for all $y \neq 0$, hence $df(0; \cdot)$ does not exist.

### 7. Characterizations of strict local Pareto minima

We next establish a characterization of a strict local Pareto minima of order $m$ for the multiobjective optimization problem (17) without any restrictions on the constraint set $S$ as in (18). This result is similar to [10, Theorem 3.1], but includes strict local minima of arbitrary order.

**Theorem 17.** Suppose that $\dim X < \infty$, $\bar{x} \in S$, and $d^m f(\bar{x}; y)$ exists for all $y \in X$. Then the following conditions are equivalent:

(a) $\bar{x} \in \text{StrL}(m, f, S)$.

(b) There exists $\beta > 0$ such that
\[
d^m f(\bar{x}; y) \not\in B(0, \beta \|y\|^m) - \overrightarrow{\mathbb{R}_+^p} \quad \text{for all } y \in K(S, \bar{x}) \setminus \{0\}.
\]

(c) $d^m f(\bar{x}; y) \not\in -\overrightarrow{\mathbb{R}_+^p}$ for all $y \in K(S, \bar{x}) \setminus \{0\}$.

**Proof.** To prove (a) $\Rightarrow$ (b), one should repeat the proof of Theorem 11(a) with $C = S$ and without $g$. Note that we must add the restriction $y \neq 0$ in (67), which was previously guaranteed by condition (37). Similarly, (c) $\Rightarrow$ (a) is obtained from Theorem 15. (b) $\Rightarrow$ (c) is trivial. \(\square\)
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