A summability factor theorem involving an almost increasing sequence for generalized absolute summability

Ekrem Savaş

Istanbul Ticaret University, Department of Mathematics, Uskudar, Istanbul, Turkey

Received 17 July 2006; received in revised form 28 October 2006; accepted 12 December 2006

Abstract

In this paper we prove a general theorem on \(|A; \delta|_k\)-summability factors of infinite series under suitable conditions by using an almost increasing sequence, where \(A\) is a lower triangular matrix with non-negative entries. Also, we deduce a similar result for the weighted mean method.

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Keywords: Absolute summability; Almost increasing summability factors

Let \(A\) be a lower triangular matrix, \(\{s_n\}\) a sequence. Then

\[ A_n := \sum_{v=0}^{n} a_{nv} s_v. \]

A series \(\sum a_n\), with partial sums \(\{s_n\}\), is said to be summable \(|A|_k, k \geq 1\) if

\[ \sum_{n=1}^{\infty} n^{k-1} |A_n - A_{n-1}|^k < \infty, \]  

and it is said to be summable \(|A, \delta|_k, k \geq 1\) and \(\delta \geq 0\) if [2]

\[ \sum_{n=1}^{\infty} n^{\delta k + k-1} |A_n - A_{n-1}|^k < \infty. \]  

We may associate with \(A\) two lower triangular matrices \(\bar{A}\) and \(\hat{A}\) defined as follows:

\[ \bar{a}_{nv} = \sum_{r=v}^{n} a_{nr}, \quad v = 0, 1, 2, \ldots, n \text{ and } n = 0, 1, 2, \ldots. \]

E-mail address: ekremsavas@yahoo.com.

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and

$$\hat{a}_{nv} = \bar{a}_{nv} - \bar{a}_{n-1,v}, \quad n = 1, 2, 3, \ldots.$$  

A positive sequence \( \{b_n\} \) is said to be almost increasing if there exists an increasing sequence \( \{c_n\} \) and two positive constants \( A \) and \( B \) such that \( Ac_n \leq b_n \leq Bc_n \) for each \( n \).

Given any sequence \( \{x_n\} \), the notation \( x_n \propto O(1) \) means \( x_n = O(1) \) and \( 1/x_n = O(1) \). For any matrix entry \( a_{nv} \), \( \Delta v a_{nv} := a_{nv} - a_{nv+1} \).

Quite recently, Rhoades and Savas [5] proved the following theorem for \( |A; \delta|_k \)-summability factors of infinite series.

**Theorem 1.** Let \( \{X_n\} \) be an almost increasing sequence and let \( \{\beta_n\} \) and \( \{\lambda_n\} \) be sequences such that

(i) \( |\Delta \lambda_n| \leq \beta_n \),

(ii) \( \lim \beta_n = 0 \),

(iii) \( \sum_{n=1}^{\infty} n|\Delta \beta_n|X_n < \infty \), and

(iv) \( |\lambda_n|X_n = O(1) \).

Let \( A \) be a lower triangular matrix with non-negative entries satisfying

(v) \( n a_{nn} \propto O(1) \),

(vi) \( a_{n-1,v} \geq a_{nv} \) for \( n \geq v + 1 \),

(vii) \( \bar{a}_{n0} = 1 \) for all \( n \),

(viii) \( \sum_{v=1}^{n-1} a_{nv} \hat{a}_{nv+1} = O(a_{nn}) \),

(ix) \( \sum_{n=v+1}^{m+1} n^{\delta k} |\Delta v \hat{a}_{nv}| = O\left(v^{\delta k} a_{vv}\right) \) and

(x) \( \sum_{n=v+1}^{m+1} n^{\delta k} \hat{a}_{nv+1} = O\left(v^{\delta k}\right) \).

If

(xi) \( \sum_{n=1}^{m} n^{\delta k-1}|t_n|^k = O(X_m) \), where \( t_n := \frac{1}{n+1} \sum_{k=1}^{n} k a_k \),

then the series \( \sum a_n \lambda_n \) is summable \( |A, \delta|_k \), \( k \geq 1, 0 \leq \delta < 1/k \).

In this paper, we shall prove the above theorem in a more general form as follows. Now we have

**Theorem 2.** Let \( \{X_n\} \) be an almost increasing sequence and \( \{\beta_n\} \) and \( \{\lambda_n\} \) be sequences such that the conditions (i)–(iv) of Theorem 1 and the following condition are satisfied,

(viii) \( \sum_{n=1}^{\infty} \frac{\lambda_n}{n} < \infty \).

Let \( A \) be a triangle satisfying conditions (v)–(vii) and (ix)–(xi) of Theorem 1, then the series \( \sum a_n \lambda_n \) is summable \( |A, \delta|_k \), \( k \geq 1, 0 \leq \delta < 1/k \).

The following lemma is pertinent for the proof of Theorem 2.

**Lemma 1 ([3]).** Under the conditions (ii) and (iii) on the sequences \( \{X_n\}, \{\beta_n\} \) and \( \{\lambda_n\} \) as stated in Theorem 2, the following conditions hold,

(1) \( n \beta_n X_n = O(1) \), and

(2) \( \sum_{n=1}^{\infty} \beta_n X_n < \infty \).

**Proof.** Let \( (y_n) \) be the \( n \)th term of the \( A \)-transform of the partial sums of \( \sum_{i=0}^{n} \lambda_i a_i \). Then,

\[
y_n := \sum_{i=0}^{n} \lambda_i a_i = \sum_{i=0}^{n} a_{ni} \sum_{v=0}^{i} \lambda_v a_v
\]

\[
= \sum_{v=0}^{n} \lambda_v a_v \sum_{i=v}^{n} a_{ni} = \sum_{v=0}^{n} \bar{a}_{nv} \lambda_v a_v
\]
and

\[ Y_n := y_n - y_{n-1} = \sum_{v=0}^{n} (\hat{\alpha}_{n,v} - \tilde{\alpha}_{n-1,v}) \lambda_v a_v = \sum_{v=0}^{n} \hat{\alpha}_{n,v} \lambda_v a_v. \]  

(3)

We may write (note that (vii) implies that \( \hat{a}_{n0} = 0 \)):

\[
Y_n = \sum_{v=1}^{n} \left( \frac{\hat{\alpha}_{n,v} \lambda_v}{v} \right) v a_v \\
= \sum_{v=1}^{n-1} \Delta_v \left( \frac{\hat{\alpha}_{n,v} \lambda_v}{v} \right) \sum_{r=1}^{v} r a_r + \hat{\alpha}_{n,n} \frac{\lambda_n}{n} \sum_{r=1}^{n} r a_r \\
= \sum_{v=1}^{n-1} (\Delta_v \hat{\alpha}_{n,v}) \lambda_v \left( \frac{v + 1}{v} \right) t_v + \sum_{v=1}^{n-1} \hat{\alpha}_{n,v+1} (\Delta \lambda_v) \left( \frac{v + 1}{v} \right) t_v + \sum_{v=1}^{n-1} \hat{\alpha}_{n,v+1} \lambda_{v+1} \left( \frac{1}{v} \right) t_v + \frac{(n + 1) \lambda_n}{n} t_n \\
= T_{n1} + T_{n2} + T_{n3} + T_{n4}, \quad \text{say.}
\]

To complete the proof, it is sufficient, by Minkowski’s inequality, to show that

\[ \sum_{n=1}^{\infty} n^{\delta k + k - 1} |T_{nr}|^k < \infty, \quad \text{for } r = 1, 2, 3, 4. \]

From the definition of \( \hat{A} \) and using (vi) and (vii),

\[
\hat{a}_{n,v+1} = \tilde{\alpha}_{n,v} - \tilde{\alpha}_{n-1,v+1} \\
= \sum_{i=v+1}^{n} a_{ni} - \sum_{i=v+1}^{n-1} a_{n-1,i} \\
= 1 - \sum_{i=0}^{v} a_{ni} - 1 + \sum_{i=0}^{n} a_{n-1,i} \\
= \sum_{i=0}^{n} (a_{n-1,i} - a_{n,i}) \geq 0. \]

(4)

Using Hölder’s inequality,

\[
I_1 := \sum_{n=1}^{m} n^{\delta k + k - 1} |T_{n1}|^k = \sum_{n=1}^{m} n^{\delta k + k - 1} \left( \sum_{v=1}^{n-1} \Delta_v \hat{\alpha}_{n,v} \lambda_v \frac{v + 1}{v} t_v \right)^k \\
= O(1) \sum_{n=1}^{m+1} n^{\delta k + k - 1} \left( \sum_{v=1}^{n-1} |\Delta_v \hat{\alpha}_{n,v}||\lambda_v||t_v| \right)^k \\
= O(1) \sum_{n=1}^{m+1} n^{\delta k + k - 1} \left( \sum_{v=1}^{n-1} |\Delta_v \hat{\alpha}_{n,v}||\lambda_v||t_v| \right)^k \left( \sum_{v=1}^{n-1} |\Delta_v \hat{\alpha}_{n,v}| \right)^{k-1}.
\]

Also,

\[
\Delta_v \hat{\alpha}_{n,v} = \hat{\alpha}_{n,v} - \tilde{\alpha}_{n,v+1} \\
= \hat{\alpha}_{n,v} - \tilde{\alpha}_{n-1,v} - \tilde{\alpha}_{n,v+1} + \tilde{\alpha}_{n-1,v+1} \\
= a_{n,v} - a_{n-1,v} \leq 0.
\]
Thus, using (vii),
\[
\sum_{v=0}^{n-1} |\Delta_v \hat{a}_{nv}| = \sum_{v=0}^{n-1} (a_{n-1,v} - a_{nv}) = 1 - 1 + a_{nn} = a_{nn}.
\]
Since \(\{X_n\}\) is an almost increasing sequence, condition (iv) implies that \(\{\lambda_n\}\) is bounded. Then, using (v), (ix),(xi), and (i) and condition (2) of Lemma 1,
\[
I_1 = O(1) \sum_{n=1}^{m+1} n^{\delta k} (na_{nn})^{k-1} \sum_{v=1}^{n-1} |\lambda_v|^k |t_v|^k |\Delta_v \hat{a}_{nv}|
\]
\[
= O(1) \sum_{n=1}^{m+1} n^{\delta k} \left( \sum_{v=1}^{n-1} |\lambda_v|^{k-1} |\lambda_v||\Delta_v \hat{a}_{nv}||t_v|^k \right)
\]
\[
= O(1) \sum_{v=1}^{m} |\lambda_v|^k |t_v|^k \sum_{n=v+1}^{m} n^{\delta k} |\Delta_v \hat{a}_{nv}|
\]
\[
= O(1) \sum_{v=1}^{m} |\lambda_v|^k |t_v|^k \sum_{n=v+1}^{m+1} n^{\delta k} |\Delta_v \hat{a}_{nv}|
\]
\[
= O(1) \sum_{v=1}^{m} |\lambda_v|^k |t_v|^k \left( \sum_{r=1}^{v-1} a_{rr} |t_r|^k r^{\delta k} - \sum_{r=1}^{v} \frac{\lambda_v+1}{\lambda_v} |t_r|^k r^{\delta k} \right)
\]
\[
= O(1) \sum_{v=1}^{m} |\lambda_v|^k |t_v|^k \left( \sum_{r=1}^{m-1} a_{rr} |t_r|^k r^{\delta k} - \sum_{r=1}^{m} |\lambda_m| \frac{\lambda_m}{\lambda_v} |t_r|^k r^{\delta k} \right)
\]
\[
= O(1) \sum_{v=1}^{m} \Delta(\lambda_v) \sum_{r=1}^{m} a_{rr} |t_r|^k r^{\delta k} + |\lambda_m| \sum_{r=1}^{m} |\Delta \lambda_v| X_v + O(1) |\lambda_m| X_m
\]
\[
= O(1) \sum_{v=1}^{m} \beta_v X_v + O(1) |\lambda_m| X_m
\]
\[
= O(1).
\]
Using (i) and Hölder’s inequality
\[
I_2 := \sum_{n=2}^{m+1} n^{\delta k+k-1} |T_{n2}|^k = \sum_{n=2}^{m+1} n^{\delta k+k-1} \sum_{v=1}^{n-1} \hat{a}_{n,v+1} (\Delta \lambda_v)^{v+1} |t_v|^k
\]
\[
= O(1) \sum_{n=2}^{m+1} n^{\delta k+k-1} \left( \sum_{v=1}^{n-1} \hat{a}_{n,v+1} \beta_v |t_v|^k \right)^k
\]
\[
= O(1) \sum_{n=2}^{m+1} n^{\delta k+k-1} \left( \sum_{v=1}^{n-1} \beta_v |t_v|^k \hat{a}_{n,v+1} \right)^{k-1} \left( \sum_{v=1}^{n-1} \hat{a}_{n,v+1} \beta_v \right).
\]
It is easy to see that
\[
\sum_{v=1}^{n-1} \hat{a}_{n,v+1} \beta_v \leq Ma_{nn}.
\]
We have, using (v) and (x),

\[ I_2 = O(1) \sum_{n=2}^{m+1} n^{\delta k} (na_{mn})^{k-1} \sum_{v=1}^{n-1} \hat{a}_{n,v+1} \beta_v |t_v|^k \]

\[ = O(1) \sum_{v=1}^{m} \beta_v |t_v|^k \sum_{n=v+1}^{m+1} n^{\delta k} \hat{a}_{n,v+1}. \]

Therefore,

\[ I_2 = O(1) \sum_{v=1}^{m} \beta_v |t_v|^k \sum_{n=v+1}^{m+1} n^{\delta k} \hat{a}_{n,v+1}. \]

Using summation by parts, (iii), (xi) and condition (2) of Lemma 1,

\[ I_2 := O(1) \sum_{v=1}^{m} v \beta_v \left[ \sum_{r=1}^{v} r^{\delta k-1} |t_r|^k - \sum_{r=1}^{v-1} r^{\delta k-1} |t_r|^k \right] \]

\[ = O(1) \left[ \sum_{v=1}^{m} v \beta_v \sum_{r=1}^{v} r^{\delta k-1} |t_r|^k - \sum_{v=1}^{m-1} (v+1) \beta_{v+1} \sum_{r=1}^{v} r^{\delta k-1} |t_r|^k \right] \]

\[ = O(1) \sum_{v=1}^{m-1} \Delta(v \beta_v) \sum_{r=1}^{v} r^{\delta k-1} |t_r|^k + O(1) m \beta_m \sum_{r=1}^{m} r^{\delta k-1} |t_r|^k \]

\[ = O(1) \sum_{v=1}^{m-1} |\Delta(v \beta_v)| X_v + O(1) m \beta_m X_m \]

\[ = O(1) \sum_{v=1}^{m-1} v |\Delta(\beta_v)| X_v + O(1) m \beta_{v+1} X_v + O(1) \]

\[ = O(1). \]

Using (v), (x), (xi) and Hölder’s inequality, summation by parts, condition (2) of Lemma 1, and (viii), we have

\[ \sum_{n=2}^{m+1} n^{\delta k+k-1} |T_{n3}|^k = \sum_{n=2}^{m+1} n^{\delta k+k-1} \sum_{v=1}^{n-1} \hat{a}_{n,v+1} \lambda_{v+1} \frac{|t_v|^k}{v} \]

\[ \leq \sum_{n=2}^{m+1} n^{\delta k+k-1} \left[ \sum_{v=1}^{n-1} |\lambda_{v+1}| \frac{|t_v|^k}{v} \hat{a}_{n,v+1} \right] \times \left[ \sum_{v=1}^{n-1} \frac{|\lambda_{v+1}|}{v} \right] \]

\[ = O(1) \sum_{n=2}^{m+1} n^{\delta k+k-1} \left[ \sum_{v=1}^{n-1} |\lambda_{v+1}| |t_v|^k \hat{a}_{n,v+1} \frac{1}{v} \right] \times \left[ \sum_{v=1}^{n-1} |\lambda_{v+1}| \frac{1}{v} \right] \]

\[ = O(1) \sum_{n=2}^{m+1} n^{\delta k} (na_{mn})^{k-1} \left[ \sum_{v=1}^{n-1} |\lambda_{v+1}| |t_v|^k \hat{a}_{n,v+1} \frac{1}{v} \right] \times \left[ \sum_{v=1}^{n-1} |\lambda_{v+1}| \frac{1}{v} \right] \]

\[ = O(1) \sum_{n=2}^{m+1} n^{\delta k} (na_{mn})^{k-1} \left[ \sum_{v=1}^{n-1} |\lambda_{v+1}| |t_v|^k \hat{a}_{n,v+1} \right] \times \left[ \sum_{v=1}^{n-1} |\lambda_{v+1}| \frac{1}{v} \right] \]

\[ = O(1) \sum_{n=2}^{m+1} \frac{|\lambda_{v+1}|}{v} |t_v|^k \sum_{n=v+1}^{m+1} n^{\delta k} (na_{mn})^{k-1} \hat{a}_{n,v+1} \]
\[= O(1) \sum_{v=1}^{m} \frac{\lambda_{v+1}}{v} |t_v| k \sum_{n=v+1}^{m+1} n^{\delta k} \hat{a}_{n,v+1} \]

\[= O(1) \sum_{v=1}^{m} \frac{\lambda_{v+1}}{v} \nu^{\delta k} |t_v|^k \]

\[= O(1) \sum_{v=1}^{m-1} (|\Delta \lambda_{v+1}|) \nu^{\delta k} |t_v|^k + O(1) |\lambda_{m+1}| \sum_{r=1}^{m} \nu^{\delta k} |t_r|^k \]

\[= O(1) \sum_{v=1}^{m-1} (|\Delta \lambda_{v+1}|) X_v + O(1) |\lambda_{m+1}| X_m \]

\[= O(1) \sum_{v=1}^{m-1} \beta_{v+1} X_v + O(1) |\lambda_{m+1}| X_m \]

\[= O(1).\]

Finally, using (iv) and (v), we have

\[\sum_{n=1}^{m} n^{\delta k + 1 - j} |T_{n4}|^k = \sum_{n=1}^{m} n^{\delta k + 1 - j} \left| \frac{(n + 1)a_{nn} \lambda_n t_n}{n} \right|^k \]

\[= O(1) \sum_{n=1}^{m} n^{\delta k + 1 - j} |a_{nn}|^k |\lambda_n|^k |t_n|^k \]

\[= O(1) \sum_{n=1}^{m} n^{\delta k} (na_{nn})^{j-1} a_{nn} |\lambda_n|^{j-1} |\lambda_n||t_n|^k \]

\[= O(1) \sum_{n=1}^{m} n^{\delta k} a_{nn} |\lambda_n||t_n|^k \]

\[= O(1),\]

as in the proof of \(I_1\).

Setting \(\delta = 0\) in the theorem yields the following corollary.

**Corollary 1.** Let \(\{X_n\}\) be an almost increasing sequence and \(\{\beta_n\}\) and \(\{\lambda_n\}\) sequences satisfying conditions (i)–(iv) and (viii) of Theorem 2. Let \(A\) be a triangle satisfying conditions (v)–(vii) of Theorem 2. If

(i) \(\sum_{n=1}^{m} \frac{1}{n} |T_{n4}|^k = O(X_m)\), where \(t_n := \frac{1}{n+1} \sum_{k=1}^{n} ka_k\).

then the series \(\sum a_n \lambda_n\) is summable \(|A|_k, k \geq 1.\)

**Corollary 2.** Let \(\{X_n\}\) be an almost increasing sequence, and \(\{\beta_n\}\) and \(\{\lambda_n\}\) be sequences satisfying conditions (i)–(iv) and (viii) of Theorem 2. Also, let \(\{p_n\}\) be a positive sequence such that \(P_n := \sum_{k=0}^{n} p_k \to \infty\), and satisfies

(v) \(np_n \asymp O(P_n)\),

(vi) \(\sum_{n=v+1}^{m+1} n^{\delta k} \left| \frac{p_n}{P_k} \right| T_{n4} t_{n-1} = O \left( \nu^{\delta k} \right) \).

If

(vii) \(\sum_{n=1}^{m} n^{\delta k - 1} |t_n|^k = O(X_m)\), where \(t_n := \frac{1}{n+1} \sum_{k=1}^{n} ka_k\).

then the series \(\sum a_n \lambda_n\) is summable \(|\tilde{N}, p, \delta|_k, k \geq 1\) for \(0 \leq \delta < 1/k\).

**Proof.** Conditions (i)–(iv), (vii) and (viii) of Corollary 2 are, respectively, conditions (i)–(iv), (xi) and (viii) of Theorem 2.
Conditions (vi) and (vii) of Theorem 2 are automatically satisfied for any weighted mean method. Condition (v) of Theorem 2 becomes condition (v) of Corollary 2, and conditions (ix) and (x) of Theorem 2 become condition (vi) of Corollary 2. □

It should be noted that, in [1], an incorrect definition of absolute summability was used (see, [4]). Corollary 2 gives the correct version of Bor and Özaslan’s theorem.

Acknowledgements

The author wish to thank the referees for their constructive comments and suggestions.

References