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Global stability of an SIR epidemic model with time delays

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Abstract. An SIR disease transmission model is formulated under the assumption that the force of infection at the present time depends on the number of infectives at the past. It is shown that a disease free equilibrium point is globally stable if no endemic equilibrium point exists. Further the endemic point (if it exists) is globally stable with respect to the whole state space except the neighborhood of the disease free state.

Key words: SIR epidemic models – global stability – Time delay – Liapunov functional

1 Introduction

An SIR model was proposed by Cooke (1979) for epidemics which are spread in a human population via a vector (such as a mosquito); i.e. susceptible individuals receive the infection from infectious vectors, and susceptible vectors receive the infection from infectious individuals. It is assumed that when a susceptible vector is infected by a person, there is a time $\tau > 0$ during which the infectious agents develop in the vector and it is only after that time that the infected vector becomes itself infectious. It is also assumed that the vector population is very large and at any time t the infectious vector population is simply proportional to the infectious human population at time $t - \tau$. Thus, if we denote by $S(t)$ the human susceptible population and by $I(t)$ the human infective population, the force of infection at time t is assumed to be given by

$$\beta S(t) I(t - \tau) . \tag{1}$$

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However, it may be more realistic to assume that τ is a distributed parameter and the force of infection (1) has to be substituted by

$$\beta S(t) \int_0^{+\infty} f(\tau) I(t-\tau)\, d\tau \,, \tag{2}$$

where $f(\tau)$ represents the fraction of vector population in which the time taken to become infectious is τ. Further, $f(\tau)$ is assumed to be non-negative, square integrable on $R_{+0} = [0, \infty)$ and satisfies

$$\int_0^{+\infty} f(\tau)\, d\tau = 1, \qquad \int_0^{+\infty} \tau f(\tau)\, d\tau < +\infty \,. \tag{3}$$

Beretta et al. (1988) assumed the delay kernel f to be a γ-distribution (see, Sect. 3) and considered the system of ordinary differential equations obtained from the original set of delay-differential equations by using so called linear chain trick (see, McDonald, 1978). They gave a sufficient condition for a positive equilibrium state of the system to be globally asymptotically stable and also proved that the system with a "weak delay" is always globally asymptotically stable.

In this paper, we consider the stability properties of SIR-models expressed by delay-differential equations with distributed delays for which we do not assume the concrete form. The next section gives the model equations and introduce also the nomenclature. In Sects. 3 and 4, we consider the local and global stability respectively of the equilibria which express a disease free state and an endemic state. In Remark 3 of Sect. 4, it is underlined how the method of Liapunov functionals presented for the model with distributed delays can be applied to the case of discrete delays.

2 Model equations and nomenclature

The SIR model with vital dynamics (Hethcote 1976) is given by

$$\begin{aligned} \dot{S}(t) &= -\beta S(t) I(t) - \mu S(t) + \mu \,, \\ \dot{I}(t) &= \beta S(t) I(t) - \mu I(t) - \lambda I(t) \,, \\ \dot{R}(t) &= \lambda I(t) - \mu R(t) \,, \end{aligned} \tag{4}$$

where a population is divided into three classes denoted by S, I, R: susceptibles, infectives and recovered. The assumptions on the model are

a) The population considered has a constant size N and the variables are normalized to $N = 1$, that is, $S(t) + I(t) + R(t) = 1$ for all t;

b) Births and deaths occur at equal rates μ in N. All the newborns are susceptible. μ is called a *daily death removal rate*;

c) β is the *daily contact rate*, i.e., the average number of contacts per infective per day. A contact of an infective is an interaction which results in infection of the other individual if it is susceptible;

d) λ is the *daily recovery removal rate* of the infectives. Of course, $\beta, \mu, \lambda \in R_+$.

If the force of infection (2) is inserted into the SIR model (4), we obtain

$$\begin{aligned} \dot{S}(t) &= -\beta S(t) \int_0^\infty f(s) I(t-s)\, ds - \mu S(t) + \mu\,, \\ \dot{I}(t) &= \beta S(t) \int_0^\infty f(s) I(t-s)\, ds - \mu I(t) - \lambda I(t)\,, \\ \dot{R}(t) &= \lambda I(t) - \mu R(t)\,. \end{aligned} \tag{5}$$

Because of the conservation law $S(t) + R(t) + I(t) = 1$ for any $t \in R$, we can consider any pair of variables within three variables S, I, R. For example, let us consider $\boldsymbol{x}(t) = (S(t), I(t)) \in \Omega$, where

$$\Omega = \{(S, I) \in R^2_{+0} \,|\, S + I \leqq 1\}\,. \tag{6}$$

According to Kuang (1993), we denote $C((-\infty, 0], R^2)$, the Banach space of continuous functions mapping the interval $(-\infty, 0]$ into R^2 with the topology of uniform convergence; i.e., for $\phi \in C((-\infty, 0], R^2)$, the norm of ϕ is defined as $\|\phi\| = \sup_{\theta \in (-\infty, 0]} |\Phi(\theta)|$, where $|\cdot|$ is a norm in R^2. Furthermore, for $\delta \geqq 0$, $\boldsymbol{x} \in C((-\infty, \delta], R^2)$ and $t \in [0, \delta]$, we define $\boldsymbol{x}_t \in C$ as $\boldsymbol{x}_t(\theta) = \boldsymbol{x}(t + \theta) = (S(t + \theta), I(t + \theta))$ for $\theta \in (-\infty, 0]$. Then, system (5) can be written as

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}_t)\,, \tag{7}$$

i.e., as an autonomous system of delay differential equations, where $R(t)$ is given by $R(t) = 1 - (S(t) + I(t))$ for any $t \in [0, \delta)$.

Denote by Q_H the set of non-negative functions $\phi(\theta) = (\phi_1(\theta), \phi_2(\theta)) \in C((-\infty, 0], R^2)$ such that $\|\phi\| < H$, $H \in R_+$ and $\phi(0) > 0$.

In this paper we will consider stability properties of system (7) with an initial values $\phi \in Q_H$ at time $t = 0$. The following properties of system (7) are easy to check:

(*i*) The function $\boldsymbol{f}$ is Lipschitzian. This implies the local existence, uniqueness and continuous dependence on the i.c. $\phi \in Q_H$ of the solution of (7) for all $t \in [t_0, t_0 + \delta)$.

(*ii*) If there exists $t^* \in [t_0, t_0 + \delta)$ such that $x_i(t^*) = 0$, then $f_i(\boldsymbol{x}_{t^*}) \geqq 0$, $i = 1, 2$. This property implies that, if $\phi \in Q_H$ is a vector function with positive components, then the solution $\boldsymbol{x}(t)$ will remain positive for all $t \in (0, \delta)$. The proof of this property can be performed similarly to the proof given in Beretta and Takeuchi (1993). In particular, if $\phi(0) = (S(0), I(0)) \in \Omega$, then $\boldsymbol{x}(t) = (S(t), I(t)) \in \Omega$ for all $t \in (0, \delta)$.

(*iii*) Positive invariance of Ω implies boundedness of the solutions of (7) for all $t \in [0, \delta)$. This in turn implies continuability of the solutions of (7), together with their properties of uniqueness and continuous dependence on i.c. up to $\delta = +\infty$, i.e., for all $t \in [0, +\infty)$.

(*iv*) The equilibria of (5) satisfying $S(t) = S^*$, $I(t) = I^*$, $R(t) = R^*$ for all $t \in R$ are as follows:

(*iv-a*) The endemic equilibrium point is given by $E_+ = (S^*, I^*) = ((\mu + \lambda)/\beta, \mu(1 - S^*)/\beta S^*)$ provided that $\beta > \mu + \lambda$;

(*iv-b*) The disease free equilibrium point is given by $E_0 = (S^* = 1, I^* = 0)$ which exists for all parameter values.

Let us remark that the equilibrium component for R of the endemic state is simply given by $R^* = 1 - (S^* + I^*)$ or equivalently, by $R^* = \lambda I^*/\mu$.

In the following, we study the stability properties of the equilibria of (5) by using the method of Liapunov functionals. Since (5) can be set in the form (7), if we centre the variables on the equilibrium (E_+ or E_0), we obtain an autonomous system of delay differential equations

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}_t) , \tag{8}$$

where the equilibrium is $\boldsymbol{x} = 0$. Then, we will use the following result (Kuang 1993; Corollary 5.2, p. 30):

Theorem 1 *Assume that* $\omega_1(\cdot)$ *and* $\omega_2(\cdot)$ *are nonnegative continuous scalar functions*: $R_{+0} \to R_{+0}$ *such that* $\omega_1(0) = \omega_2(0) = 0$, $lim_{r \to +\infty} \omega_1(r) = +\infty$ *and that* $V: C \to R$ *is continuous and satisfies*

$$V(\phi) \geqq \omega_1(|\phi(0)|), \qquad \dot{V}(\phi)|_{(8)} \leqq -\omega_2(|\phi(0)|) . \tag{9}$$

Then the solution $\boldsymbol{x} = \boldsymbol{0}$ *of equation* (8) *is uniformly stable and every solution is bounded. If in addition,* $\omega_2(r) > 0$ *for* $r > 0$, *then* $\boldsymbol{x} = \boldsymbol{0}$ *is globally asymptotically stable.*

Concerning the various definitions of stability, we refer to the recent book by Kuang (1993).

3 Local stability

We consider the local asymptotic stability of the equilibria.

Theorem 2 *Whenever* E_+ *of* (5) *exists, it is locally asymptotically stable.*

Proof. System (5) is centred on E_+ by introducing $u_1 = S - S^*$, $u_2 = I - I^*$ and its linear part becomes

$$\dot{u}_1(t) = -(\beta I^* + \mu)u_1 - \beta S^* \int_0^{+\infty} f(s) u_2(t - s)\, ds ,$$

$$\dot{u}_2(t) = \beta I^* u_1 - (\lambda + \mu)u_2 + \beta S^* \int_0^{+\infty} f(s) u_2(t - s)\, ds . \tag{10}$$

Let us consider the Liapunov functional

$$V(\boldsymbol{u}_t) = \frac{1}{2} u_2^2(t) + \frac{1}{2} w(u_1(t) + u_2(t))^2 + \frac{1}{2} \beta S^* \int_0^{+\infty} f(s) \int_{t-s}^{+\infty} u_2^2(v)\, dv\, ds \tag{11}$$

where $w > 0$ is a constant. Let us observe that

$$V(\boldsymbol{u}_t) \geqq \omega_1(|\boldsymbol{u}(t)|) = \tfrac{1}{2}u_2^2(t) + \tfrac{1}{2}w(u_1(t) + u_2(t))^2 , \tag{12}$$

where ω_1 is a positive definite quadratic form of u_1 and u_2, since $w > 0$. Hence, $\omega_1 \geqq 0$, $\omega_1 = 0$ iff $|\boldsymbol{u}(t)| = 0$ and $\lim_{|\boldsymbol{u}(t)| \to +\infty} \omega_1(|\boldsymbol{u}(t)|) = +\infty$.

Furthermore, the time derivative of $V(\boldsymbol{u}_t)$ along the solution of system (10) becomes

$$\begin{aligned}\dot{V}(\boldsymbol{u}_t)|_{(10)} = &-\mu w u_1^2 - (\lambda + \mu)(1 + w)u_2^2 + \tfrac{1}{2}\beta S^* u_2^2 \\ &+ \beta S^* u_2 \int_0^{+\infty} f(s) u_2(t-s)\, ds + [\beta I^* - w(2\mu + \lambda)] u_1 u_2 \\ &- \frac{1}{2}\beta S^* \int_0^{+\infty} f(s) u_2^2(t-s)\, ds \\ \leqq &-\mu w u_1^2 - (\lambda + \mu)(1 + w)u_2^2 + \beta S^* u_2^2 \\ \leqq &-\mu w(u_1^2 + u_2^2) = -\omega_2(|\boldsymbol{u}(t)|)\end{aligned} \tag{13}$$

where the first inequality of (13) is obtained by choosing w as $\beta I^* = w(2\mu + \lambda)$ and $\omega_2(|\boldsymbol{u}(t)|) = \mu w(u_1^2 + u_2^2) = \mu w|\boldsymbol{u}(t)|^2$. Note that $\beta S^* = \mu + \lambda$ and

$$\beta S^* u_2 \int_0^{+\infty} f(s) u_2(t-s)\, ds \leqq \frac{1}{2}\beta S^* u_2^2 + \frac{1}{2}\beta S^* \int_0^{+\infty} f(s) u_2^2(t-s)\, ds .$$

This and Theorem 1 complete the proof. □

Now let us consider the local asymptotic stability of E_0 of (5). It is convenient to choose the variables (I, R) instead of (S, I) and to consider the linearized system of (5) around $E_0 = (I = 0,\ R = 0)$ as follows:

$$\begin{aligned}\dot{I}(t) &= -(\mu + \lambda)I(t) + \beta \int_0^{+\infty} f(s) I(t-s)\, ds \\ \dot{R}(t) &= \lambda I(t) - \mu R(t) .\end{aligned} \tag{14}$$

Since the equation for $I(t)$ is decoupled from that for $R(t)$, the characteristic equations for (14) becomes

$$(\Lambda + \mu)(\Lambda + \mu + \lambda - \beta F(\Lambda)) = 0 , \tag{15}$$

where $F(\Lambda)$ is the Laplace transform of $f(s)$. Of course, one characteristic root is $\Lambda = -\mu < 0$. Therefore, to derive a necessary and sufficient condition for the asymptotic stability of E_0, we need to consider the equation

$$\Lambda + \mu + \lambda - \beta F(\Lambda) = 0 , \tag{16}$$

associated with the stability problem of the trivial solution $I = 0$ of the first equation in (14):

$$\dot{I}(t) = -(\mu + \lambda)I(t) + \beta \int_0^{+\infty} f(s) I(t-s)\, ds . \tag{17}$$

We can prove the following:

Theorem 3 *A necessary and sufficient condition for* E_0 *of* (14) *to be asymptotically stable is*

$$\beta < \mu + \lambda . \tag{18}$$

Proof. Since $\phi(\theta) \geqq 0$ for any $\theta \in (-\infty, 0]$ and $\phi(0) > 0$, $I(t) > 0$ for all $t \geqq 0$, where $I(t)$ is any solution of (17). Therefore, it is enough to consider the Liapunov functional

$$V(I_t) = I(t) + \beta \int_0^{+\infty} f(s) \int_{t-s}^{t} I(v)\, dv\, ds \tag{19}$$

which satisfies that $V(I_t) \geqq I(t) = |I(t)|$ for any $t \geqq 0$. Furthermore,

$$\dot{V}(I_t)|_{(17)} = (\beta - (\mu + \lambda)) I(t) . \tag{20}$$

Therefore, if (18) holds true, E_0 is asymptotically stable.

Next, we will prove the necessity part. First let us consider the case $\beta > \mu + \lambda$. Assume that $I = 0$ of (17) is stable. This means that for any $\varepsilon > 0$ there exists a $\delta(\varepsilon)$, $\delta \leqq \varepsilon$ such that $I(t) < \varepsilon$ for any $t > 0$ and for any $\phi \in Q_\delta$. Since $\phi(\theta)$ is continuous, nonnegative and not identically vanishing for any $\theta \in (-\infty, 0]$, $V(I_0) = \alpha > 0$. Furthermore, for $t > 0$, $V(I_t) \leqq I(t) + \beta T \max\{\varepsilon, \delta\} = I(t) + \beta T\varepsilon$, where $0 < T = \int_0^{+\infty} f(s) s\, ds < +\infty$ by (3). Let us consider now (20) where $I(t) \geqq V(I_t) - \beta T\varepsilon$. We obtain

$$\dot{V}(I_t) \geqq (\beta - (\mu + \lambda)) V(I_t) - (\beta - (\mu + \lambda)) \beta T\varepsilon, \quad V(I_0) = \alpha ,$$

which gives rise to the following inequality:

$$V(I_t) \geqq \beta T\varepsilon + (\alpha - \beta T\varepsilon) \exp\{(\beta - (\mu + \lambda)) t\}$$

for all $t \geqq 0$. Hence

$$I(t) \geqq V(I_t) - \beta T\varepsilon \geqq (\alpha - \beta T\varepsilon) \exp\{(\beta - (\mu + \lambda)) t\} \tag{21}$$

for all $t \geqq 0$. Therefore, if we choose $0 < \varepsilon < \alpha/(\beta T)$, we obtain that $\lim_{t \to +\infty} I(t) = +\infty$, giving rise to a contradiction with stability of $I = 0$.

For the case $\beta = \mu + \lambda$, we observe that $\Lambda = 0$ is a characteristic root of (16), since $F(0) = 1$. Therefore, E_0 is not asymptotically stable for this case. This completes the proof. □

Remark 1. To appreciate the usefulness of Liapunov functionals, let us try to prove the necessity part of Theorem 3 by using only the characteristic equation (16). We know that $\Lambda = 0$ is a characteristic root of (16) when $\beta = \mu + \lambda$. Furthermore, from (16), $\mathrm{Re}\,\Lambda = -\lambda - \mu + \beta\, \mathrm{Re} F(\Lambda)$. Therefore,

$$\left. \frac{\partial\, \mathrm{Re}\,\Lambda}{\partial \beta} \right|_{\Lambda = 0} = \mathrm{Re} F(0) = 1 > 0 , \tag{22}$$

i.e., for $\beta = \mu + \lambda$ one characteristic root of (16) crosses the imaginary axis at the origin and (for $\beta > \mu + \lambda$) enters in the right hand side of the complex plane. In general, (since we do not suppose the concrete form of $F(\Lambda)$), only by using the characteristic equation (16), we cannot exclude the possibility that for some $\beta > \mu + \lambda$ the characteristic roots of (16) again cross the imaginary axis entering in the left hand side of the complex plane and restabilizing E_0. Of course, because of (22), if such a cross occurs, this must be for some $\Lambda = \pm i\omega$ for $\omega > 0$.

We can exclude the possibility that such a cross with the imaginary axis occurs if we specify the analytical form of the delay kernel (in agreement with the result of Theorem 3). For example, let us assume that $f(s)$ is a γ-distribution, i.e.,

$$f(s) = \frac{s^{p-1}}{(p-1)!}\alpha^p e^{-\alpha s}, \tag{23}$$

where $\alpha > 0$, $s \geqq 0$ and $p \in \{1, 2, \ldots, \}$. Its Laplace transform is

$$F(\Lambda) = \left(\frac{\alpha}{\alpha + \Lambda}\right)^p. \tag{24}$$

Then, we can prove

Corollary *If the kernel f is a γ-function* (23), *then* $\Lambda = \pm i\omega$, $\omega > 0$ *cannot be characteristic roots of* (16).

Proof. Consider the characteristic equation (16) for $\Lambda = i\omega$, $\omega > 0$ and with $F(\Lambda)$ given by (24). We obtain

$$(\alpha + i\omega)^p i\omega + (\mu + \lambda)(\alpha + i\omega)^p - \beta\alpha^p = 0. \tag{25}$$

Let us introduce the auxiliary angle variable θ with $\tan\theta = \omega/\alpha$, for $0 < \theta < \pi/2$. From (25), we obtain

$$e^{ip\theta}(i\omega + \mu + \lambda) = \beta(\cos\theta)^p, \quad p = 1, 2, \ldots \tag{26}$$

The imaginary part of (26) is expressed as $\omega\cos(p\theta) + (\mu + \lambda)\sin(p\theta) = 0$ and we have $p\theta = -\arctan\{\omega/(\lambda + \mu)\}$. From the definition of θ, we have $p\arctan\{\omega/\alpha)) = -\arctan\{\omega/(\mu + \lambda))$, which has no solution for any $p = 1, 2, \ldots$ and $\omega > 0$. The same proof can be performed for $\Lambda = -i\omega$ and $\omega > 0$. This completes the proof. □

4 Global stability

Now let us consider the global asymptotic stability of the equilibria.

Theorem 4 *The positive equilibrium point E_+ of* (5) *is globally asymptotically stable with respect to the set* $\tilde{\Omega} = \{(S, I) \in \Omega \mid S < S^* + I^*\}$.

Proof. For the proof it is convenient to choose the variables (I, R) instead of (S, I) in order to have only one integro-differential equation. Let us consider

$$\begin{aligned} \dot{I}(t) &= \beta(1 - I - R)\int_0^{+\infty} f(s)I(t-s)\,ds - \beta S^* I \\ \dot{R}(t) &= \lambda I - \mu R, \end{aligned} \tag{27}$$

where $(R,I) \in \Omega_1 = \{(R,I) \in R^2_+ \,|\, R + I \leqq 1\}$. We centre (27) on $E_+ = (I^*, R^*)$ by introducing $u_1 = I - I^*$ and $u_2 = R - R^*$ with $(u_1, u_2) \in \tilde{\Omega} = \{(u_1, u_2) \in R^2 | -I^* \leqq u_1, -{}^{\varphi *} \leqq u_2, u_1 + u_2 \leqq S^*\}$. We obtain

$$\begin{aligned}\dot{u}_1(t) &= \beta(S^* - u_1 - u_2)\int_0^{+\infty} f(s)u_1(t-s)\,ds - \beta(I^* + S^*)u_1 - \beta I^* u_2 \,,\\ \dot{u}_2(t) &= \lambda u_1 - \mu u_2 \,,\end{aligned} \tag{28}$$

for which we consider the Liapunov functional such as

$$V(u_t) = \frac{w_1}{2}u_1^2 + \frac{w_2}{2}u_2^2 + \frac{1}{2}w_1\beta(I^* + S^*)\int_0^{+\infty} f(s)\int_{t-s}^{t} u_1^2(u)\,du\,ds \,, \tag{29}$$

where $w_i > 0$ for $i = 1, 2$ are positive constants. Then $V(u_t) \geqq k(u_1^2(t) + u_2^2(t))/2$, where $k = \min\{w_1, w_2\} > 0$. If we choose $-w_1\beta I^* + w_2\lambda = 0$, then the time derivative of $V(\boldsymbol{u}_t)$ along the solution of system (26) becomes

$$\begin{aligned}\dot{V}(u_t)|_{(26)} = &-w_2\mu u_2^2 - \frac{1}{2}w_1\beta(I^* + S^*)u_1^2\\ &+ w_1\beta(S^* - u_1 - u_2)u_1\int_0^{+\infty} f(s)u_1(t-s)\,ds\\ &- \frac{1}{2}w_1\beta(I^* + S^*)\int_0^{+\infty} f(s)u_1^2(t-s)\,ds\\ = &-w_2\mu u_2^2 - \frac{1}{2}w_1\beta\int_0^{+\infty} f(s)[\boldsymbol{v}(t,s)\cdot\boldsymbol{A}(t)\boldsymbol{v}(t,s)]\,ds\end{aligned} \tag{30}$$

where $\boldsymbol{v}(t,s) = \mathrm{col}(u_1(t), u_1(t-s))$ and

$$\boldsymbol{A}(t) = \begin{pmatrix} I^* + S^* & -(S^* - (u_1 + u_2)) \\ -(S^* - (u_1 + u_2)) & I^* + S^* \end{pmatrix}.$$

the matrix $\boldsymbol{A}$ is positive definite if

$$-I^* < u_1 + u_2 < I^* + 2S^* \,, \tag{31}$$

where the inequality of the right hand side in (31) is trivially true in $\tilde{\Omega}_1$, since $u_1 + u_2 \leqq S^*$. Now, for any $\varepsilon > 0$, consider $\tilde{\Omega}_{1,\varepsilon} = \{\boldsymbol{u} \in \tilde{\Omega}_1 | u_1 + u_2 > -I^* + \varepsilon\}$. Then for any $u \in \tilde{\Omega}_{1,\varepsilon}$, there exists a minimum eigenvalue of $\boldsymbol{A}$ which is strictly positive. Let us denote such an eigenvalue as $\bar{\lambda}_\varepsilon$. We have

$$[\boldsymbol{v}(t,s)\cdot\boldsymbol{A}(t)\boldsymbol{v}(t,s)] \geqq \bar{\lambda}_\varepsilon(u_1^2(t) + u_1^2(t-s)) \,. \tag{32}$$

By substituting (32) in (30), we obtain

$$\begin{aligned}\dot{V}(\boldsymbol{u}_t)|_{(28)} &\leqq -w_2\mu u_2^2 - \frac{1}{2}w_1\beta\bar{\lambda}_\varepsilon\left[u_1^2 + \int_0^{+\infty} f(s)u_1^2(t-s)\,ds\right]\\ &\leqq -\delta(u_1^2 + u_2^2) \,,\end{aligned} \tag{33}$$

where $\delta = \min[w_2\mu, w_1\beta\bar{\lambda}_\varepsilon/2\}$.

Now note that the set $u_1 + u_2 > -I^*$ corresponds with $I + R > R^*$, which is identical with $\tilde{\Omega}$. Since $S(t) = 1 - (I(t) + R(t))$ for all $t > 0$, this completes the proof. □

Remark 2. For the linearized system, it has been proved that the endemic state E_+ is always locally asymptotically stable without any restriction on the variables space, if E_+ exists (Theorem 2). Further, Theorem 3 implies that the disease free state E_0 is not locally asymptotically stable when E_+ exists. On the other hand, Theorem 4 ensures the global asymptotic stability of E_+ only for the restricted variables space $\tilde{\Omega}$. This suggests that the global asymptotic stability of E_+ is true for whole space Ω defined by (6). This is an open problem.

Let us consider now the global asymptotic stability of E_0. For this case, we can prove it without any restriction on the variable space:

Theorem 5 *Whenever* (18) *is true, equilibrium* E_0 *of* (5) *is globally asymptotically stable with respect to* Ω.

Proof. We choose again the variables (I, R) and consider the space $\tilde{\Omega} = \{(I, R) \in R^2_{+0} \mid R + I \leqq 1\}$ which corresponds to Ω. The equations for (I, R) are

$$\begin{aligned} \dot{I}(t) &= -(\mu + \lambda)I(t) + \beta S(t)\int_0^{+\infty} f(s)I(t-s)\,ds \\ \dot{R}(t) &= \lambda I(t) - \mu R(t)\,, \end{aligned} \tag{34}$$

where $0 \leqq S \leqq 1$ and $(I, R) \in \tilde{\Omega}$. Whenever (18) is true, the positive equilibrium E_+ is not feasible and equilibrium $E_0 = (S^* = 1, R^* = 0, I^* = 0)$ in $\tilde{\Omega}$ simply becomes $E_0 = (0, 0)$ for (34).

Let us consider the following Liapunov functional

$$V(\boldsymbol{x}_t) = I(t) + wR(t) + \beta\int_0^{+\infty} f(s)\int_{t-s}^{t} I(u)\,du\,ds\,, \tag{35}$$

where $w > 0$. Then $V(\boldsymbol{x}_t) \geqq \min\{1, w\}(I(t) + R(t))$ for any $t \geqq 0$. Further,

$$\begin{aligned} \dot{V}(\boldsymbol{x}_t)|_{(34)} &= -(\mu + \lambda)I(t) + \beta I(t) + w\lambda I(t) - w\mu R(t) \\ &\quad + \beta S(t)\int_0^{+\infty} f(s)I(t-s)\,ds - \beta\int_0^{+\infty} f(s)I(t-s)\,ds \\ &\leqq -[(\mu + \lambda) - \beta]I(t) + w\lambda I(t) - w\mu R(t)\,. \end{aligned} \tag{36}$$

Here the last inequality is true because that $0 \leqq S \leqq 1$. Choose $w = [(\mu + \lambda) - \beta]/(2\lambda)$ which is positive, since (18) is true. Then we have

$$\begin{aligned} \dot{V}(\boldsymbol{x}_t)|_{(34)} &\leqq -\tfrac{1}{2}[(\mu + \lambda) - \beta]I(t) - w\mu R(t) \\ &\leqq -k(I(t) + R(t)) = -k|\boldsymbol{x}(t)|_1\,, \end{aligned} \tag{37}$$

for any $t \geqq 0$, where $k = \min\{(\mu + \lambda - \beta)/2,\ w\mu\}$. Hence, the global stability of E_0 follows.

Remark 3. If we insert the force of infection (1) into the SIR model with vital dynamics (4), we obtain

$$\begin{aligned} \dot{S}(t) &= -\beta S(t)I(t-\tau) - \mu S(t) + \mu , \\ \dot{I}(t) &= \beta S(t)I(t-\tau) - \mu I(t) - \lambda I(t) , \\ \dot{R}(t) &= \lambda I(t) - \mu R(t) . \end{aligned} \tag{38}$$

We claim that all the stability properties of the endemic equilibrium point E_+ for $\beta > \mu + \lambda$ and of the disease free equilibrium point E_0 proved for the model with distributed delays still hold true for the discrete SIR delay model. This claim can be checked by the Liapunov functionals obtained from ones for the distributed delay model simply by substituting in them $f(s)$ with $\delta(s-\tau)$, i.e., with the delta Dirac function, and applying Theorem 1. We will check this claim just for Theorem 2, the other results can be proved similarly.

By using the change of variables $u_1 = S - S^*$, $u_2 = I - I^*$ in (38) and considering their linear parts, we obtain

$$\begin{aligned} \dot{u}_1(t) &= -(\beta I^* + \mu)u_1 - \beta S^* u_2(t-\tau) , \\ \dot{u}_2(t) &= \beta I^* u_1 - \beta S^* u_2 + \beta S^* u_2(t-\tau) , \end{aligned} \tag{39}$$

which are to be compared with (10), where E_+ has been transformed into the trivial solution $u_1 = u_2 = 0$ of (39). Then we can prove:

Theorem 6. *Whenever it exists, the positive equilibrium point* E_+ *of* (38) *is locally asymptotically stable.*

Proof. (11) suggests to consider the Liapunov functional

$$\dot{V}(\boldsymbol{u}_t) = \frac{1}{2}u_2^2(t) + \frac{1}{2}w(u_1(t) + u_2(t))^2 + \frac{1}{2}\beta S^* \int_{t-\tau}^{t} u_2^2(v)\,dv \tag{40}$$

where $w > 0$ is a constant.

According to Theorem 1, let us remark that

$$V(\boldsymbol{u}_t) \geqq \omega_1(|\boldsymbol{u}(t)|) = \tfrac{1}{2}u_2^2(t) + \tfrac{1}{2}w(u_1(t) + u_2(t))^2 ,$$

where ω_1 is a positive definite quadratic form, if $w > 0$. Furthermore,

$$\begin{aligned} \dot{V}(\boldsymbol{u}_t)|_{(39)} = &-\mu w u_1^2(t) + [\beta I^* - w(\beta S^* + \mu)]u_1(t)u_2(t) \\ &- \beta S^*(1+w)u_2^2(t) + \beta S^* u_2(t)u_2(t-\tau) \\ &+ \tfrac{1}{2}\beta S^* u_2^2(t) - \tfrac{1}{2}\beta S^* u_2^2(t-\tau) . \end{aligned} \tag{41}$$

If we choose $w > 0$ satisfying $\beta I^* - w(\beta S^* + \mu) = 0$ and we observe that $u_2(t)u_2(t-\tau) \leqq u_2^2(t)/2 + u_2^2(t-\tau)/2$, then from (41) we obtain

$$\dot{V}(\boldsymbol{u}_t)|_{(39)} \leqq -\mu w u_1^2(t) - \beta S^* w u_2^2(t) \leqq -k|\boldsymbol{u}(t)|^2 , \tag{42}$$

where $k = w \min\{\mu, \beta S^*\}$. Then, Theorem 1 assures the asymptotic stability of $u_1 = u_2 = 0$:

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