Class 1 bounds for planar graphs

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Abstract

There are planar class 2 graphs with maximum vertex-degree \( k \), for each \( k \in \{2, 3, 4, 5\} \).

In 1965, Vizing proved that every planar graph with \( \Delta \geq 8 \) is class 1. He conjectured that every planar graph with \( \Delta \geq 6 \) is a class 1 graph. This conjecture is proved for \( \Delta = 7 \), and still open for \( \Delta = 6 \).

Let \( k \geq 2 \) and \( G \) be a \( k \)-critical planar graph. The average face-degree \( F(G) \) of \( G \) is \( \frac{2}{|F(G)|}|E(G)| \). Let \( \Sigma \) be an embedding of \( G \) into the Euclidean plane, and \( v \) be a vertex of the boundaries of the faces \( f_1, \ldots, f_n \). Let \( d_{(G,\Sigma)}(f_i) \) be the degree of \( f_i \), \( F_{(G,\Sigma)}(v) = \frac{1}{n}(d_{(G,\Sigma)}(f_1) + \cdots + d_{(G,\Sigma)}(f_n)) \) and \( F((G, \Sigma)) = \min\{F_{(G,\Sigma)}(v) : v \in V(G)\} \). The local average face-degree of \( G \) is \( F^*(G) = \max\{F((G, \Sigma)) : (G, \Sigma) \text{ is a plane graph}\} \). Clearly, \( F^*(G) \geq 3 \).

The paper studies the question whether there are bounds \( \overline{b}_k, b_k^* \) such that if \( G \) is a \( k \)-critical graph, then \( F(G) \leq \overline{b}_k \) and \( F^*(G) \leq b_k^* \). For \( k \leq 5 \) our results give insights into the structure of planar \( k \)-critical graphs, and the results for \( k = 6 \) give support for the truth of Vizing’s planar graph conjecture for \( \Delta = 6 \).

1 Introduction

We consider finite simple graphs \( G \) with vertex set \( V(G) \) and edge set \( E(G) \). The vertex-degree of \( v \in V(G) \) is denoted by \( d_G(v) \), and \( \Delta(G) \) denotes the maximum vertex-degree of \( G \). If it is clear from the context, then \( \Delta \) is frequently used. The edge-chromatic-number of \( G \) is denoted by \( \chi'(G) \). Vizing \([7]\) proved that \( \chi'(G) \in \{\Delta(G), \Delta(G) + 1\} \). If

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\( \chi'(G) = \Delta(G) \), then \( G \) is a class 1 graph, otherwise it is a class 2 graph. A class 2 graph \( G \) is critical, if \( \chi'(H) < \chi'(G) \) for every proper subgraph \( H \) of \( G \). Critical graphs with maximum vertex-degree \( \Delta \) are also called \( \Delta \)-critical. It is easy to see that critical graphs are 2-connected. A graph is planar if it embeddable into the Euclidean plane. A plane graph \((G, \Sigma)\) is a planar graph \( G \) together with an embedding \( \Sigma \) of \( G \) into the Euclidean plane. That is, \((G, \Sigma)\) is a particular drawing of \( G \) in the Euclidean plane.

In 1964, Vizing\[7\] showed for each \( k \in \{2, 3, 4, 5\} \) that there is a planar class 2 graph \( G \) with \( \Delta(G) = k \). He proved that every planar graph with maximum vertex-degree at least 8 is a class 1 graph, and conjectured that every planar graph \( H \) with \( \Delta(H) \in \{6, 7\} \) is a class 1 graph. Vizing’s conjecture is proved for planar graph with maximum vertex-degree 7 by Grunewald\[2\], Sanders, Zhao\[5\], and Zhang\[12\] independently.

Zhou\[13\] proved for each \( k \in \{3, 4, 5\} \) that if \( G \) is a planar graph with \( \Delta(G) = 6 \) and \( G \) does not contain a circuit of length \( k \), then \( G \) is a class 1 graph. Vizing’s conjecture is confirmed for some other classes of planar graphs which do not contain some specific (chordal) circuits\[1\,9\,10\].

Let \( G \) be a 2-connected planar graph, \( \Sigma \) be an embedding of \( G \) in the Euclidean plane and \( F(G) \) be the set of faces of \((G, \Sigma)\). The degree \( d_{(G, \Sigma)}(f) \) of \( f \) is the length of its facial circuit. If there is no harm of confusion we also write \( d_G(f) \) instead of \( d_{(G, \Sigma)}(f) \). Let \( \overline{F}(G) = \frac{1}{|F(G)|} \sum_{f \in F(G)} d_G(f) \) be the average face-degree of \( G \). Euler’s formula \(|V(G)| - |E(G)| + |F(G)| = 2 \) implies that \( \overline{F}(G) = \frac{2|E(G)|}{|E(G)| - |V(G)| + 2} \).

Let \( v \in V(G) \). If \( d_G(v) = k \), then \( v \) is incident to \( k \) pairwise different faces \( f_1, \ldots, f_k \). Let \( F_{(G, \Sigma)}(v) = \frac{1}{k}(d_{(G, \Sigma)}(f_1) + \cdots + d_{(G, \Sigma)}(f_k)) \) and \( F((G, \Sigma)) = \min\{F_{(G, \Sigma)}(v) : v \in V(G)\} \). Clearly \( F((G, \Sigma)) \geq 3 \). As Figure 1 shows, \( F((G, \Sigma)) \) depends on the embedding.

**Figure 1:** Graph \( G \) has two embeddings \( \Sigma, \Sigma' \) such that \( F((G, \Sigma)) \neq F((G, \Sigma')) \).
The local average face-degree of a 2-connected planar graph $G$ is

$$F^*(G) = \max\{ F((G, \Sigma)) : (G, \Sigma) \text{ is a plane graph} \}.$$ 

This parameter is independent from the embeddings of $G$, and $F^*(G) \geq 3$ for all planar graphs. Let $k$ be a positive integer and $\bar{b}_k = \sup \{ F(G) : G \text{ is a } k\text{-critical planar graph} \}$, and $b^*_k = \sup \{ F^*(G) : G \text{ is a } k\text{-critical planar graph} \}$. If $\bar{b}_k \neq \infty$ and $b^*_k \neq \infty$, then we say that $\bar{b}_k$ is a class 1 bound with respect to the average face-degree for $k$-critical planar graphs, and that $b^*$ is a class 1 bound w. r. t. the local average face-degree for $k$-critical planar graphs, respectively. If $k = 1$ or $k \geq 7$, then every planar graph with maximum vertex-degree $k$ is a class 1 graph, and therefore $\{ F(G) : G \text{ is a } k\text{-critical planar graph} \} = \{ F^*(G) : G \text{ is a } k\text{-critical planar graph} \} = \emptyset$. Hence, $\bar{b}_k$ and $b^*_k$ do not exist in these cases. Therefore, we focus on $k \in \{2, 3, 4, 5, 6\}$ in this paper. The main results are the following two theorems.

**Theorem 1.1.** Let $k \geq 2$ be an integer.

- If $k = 2$, then $\bar{b}_k = \infty$.
- If $k = 3$, then $6 \leq \bar{b}_k \leq 8$.
- If $k = 4$, then $4 \leq \bar{b}_k \leq 4 + \frac{4}{5}$.
- If $k = 5$, then $3 + \frac{1}{5} \leq \bar{b}_k \leq 3 + \frac{3}{4}$.
- If $k = 6$ and $\bar{b}_k$ exists, then $\bar{b}_k \leq 3 + \frac{1}{3}$.

**Theorem 1.2.** Let $k \geq 2$ be an integer.

- If $k \in \{2, 3, 4\}$, then $b^*_k = \infty$.
- If $k = 5$, then $3 + \frac{1}{5} \leq b^*_k \leq 7 + \frac{1}{2}$.
- If $k = 6$ and $b^*_k$ exists, then $b^*_k \leq 3 + \frac{2}{5}$.

The next section states some properties of critical and of planar graphs. These results are used for the proofs of Theorems 1.1 and 1.2 which are given in Section 3.

## 2 Preliminaries

Let $G$ be a 2-connected graph. A vertex $v$ is called a $k$-vertex, or a $k^+$-vertex, or a $k^-$-vertex if $d_G(v) = k$, or $d_G(v) \geq k$, or $d_G(v) \leq k$, respectively. Let $N(v)$ be a set of vertices
which are adjacent to \( v \), \( N(u, v) = N(u) \cup N(v) \), \( N(N(u)) = \{ w | vw \in E(G), v \in N(u) \} \), and \( N(N(u, v)) = N(N(u)) \cup N(N(v)) \).

Let \((G, \Sigma)\) be a plane graph. A face \( f \) is called \( k \)-face, or a \( k^+ \)-face, or a \( k^- \)-face, if \( d_{(G, \Sigma)}(f) = k \), or \( d_{(G, \Sigma)}(f) \geq k \), or \( d_{(G, \Sigma)}(f) \leq k \), respectively. We will use the following well-known results on critical graphs.

**Lemma 2.1.** Let \( G \) be a critical graph and \( e \in E(G) \). If \( e = xy \), then \( d_G(x) \geq 2 \), and\( d_G(x) + d_G(y) \geq \Delta(G) + 2 \).

**Lemma 2.2** (Vizing’s Adjacency Lemma \([7]\)). Let \( G \) be a critical graph and \( e \in E(G) \). If \( e = xy \), then \( x \) is adjacent to at least \((\Delta(G) - d_G(y) + 1)\) \( \Delta(G) \)-vertices other than \( y \).

**Lemma 2.3** (\([12]\)). Let \( G \) be a critical graph and \( xy \in E(G) \). If \( d(x) + d(y) = \Delta(G) + 2 \), then

1. every vertex of \( N(x, y) \setminus \{x, y\} \) is a \( \Delta(G) \)-vertex,
2. every vertex in \( N((N(x, y)) \setminus \{x, y\} \) has degree at least \( \Delta(G) - 1 \),
3. if \( d(x) < \Delta(G) \) and \( d(y) < \Delta(G) \), then every vertex in \( N((N(x, y)) \setminus \{x, y\} \) has degree \( \Delta(G) \).

**Lemma 2.4** (\([5]\)). No critical graph has pairwise distinct vertices \( x, y, z \), such that \( x \) is adjacent to \( y \) and \( z \), \( d(z) < 2\Delta(G) - d(x) - d(y) + 2 \), and \( xz \) is in at least \( d(x) + d(y) - \Delta(G) - 2 \) triangles not containing \( y \).

We will use the following results on lower bounds for the number of edges in critical graphs.

**Theorem 2.5** (\([9]\)). If \( G \) is a 3-critical graph, then \( |E(G)| \geq \frac{4}{3} |V(G)| \).

**Theorem 2.6** (\([11]\)). Let \( G \) be a \( k \)-critical graph. If \( k = 4 \), then \( |E(G)| \geq \frac{12}{7} |V(G)| \), and if \( k = 5 \), then \( |E(G)| \geq \frac{15}{7} |V(G)| \).

**Theorem 2.7** (\([4]\)). If \( G \) is a 6-critical graph, then \( |E(G)| \geq \frac{1}{2} (5|V(G)| + 3) \).

**Lemma 2.8.** Let \( t \) be a positive integer and \( \epsilon > 0 \).

1. For \( k \in \{2, 3, 4\} \) there is a \( k \)-critical planar graph \( G \) and \( F^*(G) > t \).
2. There is a 2-critical planar graph \( G \) with \( F(G) > t \).
3. There is a 3-critical planar graph \( G \) such that \( 6 - \epsilon < F(G) < 6 \).
4. There is a 4-critical planar graph \( G \) such that \( 4 - \epsilon < F(G) < 4 \).
5. There is a 5-critical planar graph $G$, such that $3 + \frac{1}{3} - \varepsilon < \mathcal{F}(G) < 3 + \frac{1}{3}$ and $F^*(G) = 3 + \frac{1}{5}$.

Proof. The odd circuits are the only 2-critical graphs. Hence, the second statement and the first statement for $k = 2$ are proved. Let $X$ and $Y$ be two circuits of length $n \geq 4$, with $V(X) = \{x_i : i \in \{0, \ldots, n\}\}$, $V(Y) = \{y_i : i \in \{0, \ldots, n\}\}$ and edges $x_ix_{i+1}$, $y_iy_{i+1}$, where the indices are added modulo $n$. Consider an embedding, where $Y$ is inside $X$. Add edges $x_iy_i$ to obtain a planar cubic graph $G$ with $F^*(G) = \frac{3}{4}(n+3)$. Add edges $x_iy_{i+1}$ to obtain a 4-regular planar graph $H$ with $F^*(H) = \frac{3}{4}(n+9)$. Subdividing one edge in $G$ and one in $H$ yields a critical planar graph $T$ with $\Delta(T) = 3$, and a critical planar graph $H_n$ with $\Delta(H_n) = 4$. If $n > 4t$, then $F^*(G_n) > t$ and $F^*(H_n) > t$.

Since $|F(G_n)| = n+3$, and $\sum_{f \in F(G_n)} d_{G_n}(f) = 6n+8$, it follows that $\mathcal{F}(G_n) = 6 - \frac{10}{n+3}$. Furthermore, $|F(H_n)| = 2n + 4$, $\sum_{f \in F(H_n)} d_{H_n}(f) = 8n + 10$ and therefore, $\mathcal{F}(H_n) = 4 - \frac{3}{2n+2}$. Now, the statements for 3-critical and 4-critical graphs follow. Examples of these graphs are given in Figure 2.

![Figure 2: Examples for $k \in \{2, 3, 4\}$](image)

Let $m \geq 4$ be an integer. Let $C_i = [c_{i,1}c_{i,2} \cdots c_{i,4}c_{i,1}]$ be a circuit of length 4 for $i \in \{1, m\}$, and $C_i = [c_{i,1}c_{i,2} \cdots c_{i,8}c_{i,1}]$ be a circuit of length 8 for $i \in \{2, \ldots, m-1\}$. Consider an embedding, where $C_i$ is inside $C_{i+1}$ for $i \in \{1, \ldots, m-1\}$. Add edges $c_{1,j}c_{2,j-1}$, $c_{1,j}c_{2,j}$, $c_{1,j}c_{2,j+1}$ for $j \in \{1, \ldots, 4\}$, edges $c_{i,j}c_{i+1,j}$ for $i \in \{2, \ldots, m-2\}$ and $j \in \{1, \ldots, 8\}$, edges $c_{i,j}c_{i+1,j+1}$ for $i \in \{2, \ldots, m-2\}$ and $j \in \{2, 4, 6, 8\}$ and edges $c_{m-1,2j-2c_{m,j}}, c_{m-1,2j-1c_{m,j}}$ and $c_{m-1,2jc_{m,j}}$ for $j \in \{1, \ldots, 4\}$ to obtain a 5-regular planar graph (the indices are added modulo 8). Subdividing one edge in this 5-regular planar graph yields a critical planar graph $T_m$ with $\Delta(T_m) = 5$ (Figure 3 illustrates $T_6$).

Since $|F(T_m)| = 12m - 22$ and $\sum_{f \in F(T_m)} d_{T_m}(f) = 40m - 80$, it follows that $\mathcal{F}(T_m) = \frac{10}{3} - \frac{10}{15m-33}$. Furthermore, $F^*(T_m) = 3 + \frac{1}{5}$.

The following lemma is implied by Euler’s formula directly.
Lemma 2.9. If $G$ is a planar graph, then $|E(G)| = \frac{F(G)}{F(G) - 2}(|V(G)| - 2)$.

3 Proofs

3.1 Theorem 1.1

The statement for $k = 2$ and the lower bounds for $b_k$ if $k \in \{3, 4, 5\}$ follow from Lemma 2.8. The other statements of Theorem 1.1 are implied by the following proposition.

Proposition 3.1. Let $G$ be a $k$-critical planar graph.

1. If $k = 3$, then $\overline{F}(G) < 8$.
2. If $k = 4$, then $\overline{F}(G) < 4 + \frac{4}{5}$.
3. If $k = 5$, then $\overline{F}(G) < 3 + \frac{3}{4}$.
4. If $k = 6$, then $\overline{F}(G) < 3 + \frac{1}{3}$.

Proof. Let $k = 3$ and suppose to the contrary that $\overline{F}(G) \geq 8$. With Lemma 2.9 and Theorem 2.5 we deduce $\frac{4}{7}|V(G)| \leq |E(G)| \leq \frac{4}{3}(|V(G)| - 2)$, a contradiction.

The other statements follow analogously with the Lemma 2.9 and Theorem 2.6 ($k \in \{4, 5\}$) and Theorem 2.7 ($k = 6$).

3.2 Theorem 1.2

The statement for $k \in \{2, 3, 4\}$ and for the lower bound for $b_5$ follow from Lemma 2.8. It remains to prove the upper bounds for $b_5^*$ and $b_6^*$. The result for $b_5^*$ is implied by the following theorem.
Theorem 3.2. If $G$ is a planar 5-critical graph, then $F^*(G) \leq 7 + \frac{1}{2}$.

Proof. Suppose to the contrary that $F^*(G) = r > 7 + \frac{1}{2}$. Let $\Sigma$ be an embedding of $G$ into the Euclidean plane and $F^*(G) = F((G, \Sigma))$. Let $V = V(G)$, $E = E(G)$, and $F$ be the set of faces of $(G, \Sigma)$. We are going to proceed a discharging procedure in $G$, by which we eventually deduce a contradiction. Define the initial charge $ch$ in $G$ as $ch(x) = d_G(x) - 4$ for $x \in V \cup F$. Euler’s formula $|V| - |E| + |F| = 2$ can be rewritten as:

$$\sum_{x \in V \cup F} ch(x) = \sum_{x \in V \cup F} (d_G(x) - 4) = -8.$$ 

We define suitable discharging rules to change the initial charge function $ch$ to the final charge function $ch^*$ on $V \cup F$ such that $\sum_{x \in V \cup F} ch^*(x) \geq 0$ for all $x \in V \cup F$. Thus,

$$-8 = \sum_{x \in V \cup F} ch(x) = \sum_{x \in V \cup F} ch^*(x) \geq 0,$$

which is the desired contradiction.

Note that if a face $f$ sends charge $-\frac{1}{3}$ to a vertex $v$, then this can also be considered as $f$ receives charge $\frac{1}{3}$ from $v$. The discharging rules are defined as follows.

R1: Every $3^+$-face sends $\frac{d_G(f) - 4}{d_G(f)}$ to each incident vertex.

R2: Let $v$ be a 5-vertex of $G$.

R2.1: If $u$ is a 2-neighbor of $v$, then $v$ sends $\frac{2}{3} + \frac{2}{|2r| - 3}$ to $u$.

R2.2: If $u$ is a 3-neighbor of $v$, then $v$ sends charge to $u$ as follows:

R2.2.1: if $u$ has a 4-neighbor, then $v$ sends $\frac{1}{3} + \frac{2}{|3r| - 6}$ to $u$;

R2.2.2: if $u$ has no 4-neighbor, then $v$ sends $\frac{2}{3} + \frac{4}{3(|3r| - 6)}$ to $u$.

R2.3: If $u$ is a 4-neighbor of $v$ and $u$ is adjacent to $n$ 5-vertices ($2 \leq n \leq 4$), then $v$ sends $\frac{4}{n(|4r| - 9)}$ to $u$.

R2.4: If $v$ is adjacent to five $4^+$-vertices, then $v$ sends $\frac{1}{3} (\frac{4}{|5r| - 12} + \frac{2}{|2r| - 3})$ to each 5-neighbor which is adjacent to a 2-vertex.

Claim 3.2.1. If $u$ is an $k$-vertex, then $u$ receives at least $\frac{4 - k}{3} - \frac{4}{|rk| - 3k + 3}$ in total from its incident faces by R1. In particular, if $u$ is incident with at most two triangles, then $u$ receives at least $\frac{1}{3} - \frac{4}{|rk| - 4k + 6}$ in total from its incident faces.

Proof. Note that if $a$ and $b$ are integers and $2 \leq a \leq b$, then $\frac{1}{a} + \frac{1}{b+1} \geq \frac{1}{a} + \frac{1}{b}$ ($\otimes$).

Let $u$ be a $k$-vertex which is incident with faces $f_1, f_2, \cdots, f_k$. According to rule R1, $u$ totally receives charge $S = \sum_{i=1}^{k} \frac{d_G(f_i) - 4}{d_G(f_i)} = k - 4 \sum_{i=1}^{k} \frac{1}{d_G(f_i)}$ from its incident faces. Since $\sum_{i=1}^{k} d_G(f_i) \geq |rk|$, it follows with ($\otimes$) that $S \geq k - 4(\frac{1}{3}(k - 1) + \frac{1}{|rk| - 3(k - 1)}) = \frac{4 - k}{3} - \frac{4}{|rk| - 3k + 3}$. 

\[
\Rightarrow \quad S \geq \frac{4 - k}{3} - \frac{4}{|rk| - 3k + 3}.
\]
\[ \frac{4-k}{3} - \frac{4}{|rk|-3k+3} \]. In particular, if \( u \) is incident with at most two triangles, then we deduce with (\( \bowtie \)) that \( S \geq k - 4\left(\frac{2}{3} + \frac{1}{4}(k-3) + \frac{1}{|rk|-6-4(k-3)}\right) = \frac{1}{3} - \frac{4}{|rk|-4k+6}. \)

**Claim 3.2.2.** The charge that a 5-vertex sends to a 4-neighbor by R2.3 is smaller than or equal to the charge that a 5-vertex sends to a 5-neighbor which is adjacent to a 2-vertex by R2.4, that is, \( \frac{4}{n(4r-9)} \leq \frac{2}{3} - \frac{4}{|2r|-3} \).

**Proof.** Since \( \frac{4}{n(4r-9)} \leq \frac{2}{3} - \frac{4}{|2r|-3} \), and \( \frac{1}{3}(\frac{4}{5r+1-12} + \frac{2}{|2r+1-3|}) \leq \frac{2}{3}(\frac{4}{|5r-12|} + \frac{2}{|2r-3|}) \), we only need to prove that \( \frac{2}{3} - \frac{4}{|2r|-3} \leq \frac{1}{3}(\frac{4}{5r+1-12} + \frac{2}{|2r+1-3|}) \), which is equivalent to \( 2r^2 - 15r + 23 \geq 0 \) by simplification. Clearly, this inequality is true for every \( r \geq 5 + \frac{2}{5} \).

It remains to check the final charge for all \( x \in V \cup F \).

Let \( f \in F \), then \( ch^*(f) \geq d_G(f) - 4 - d_G(f) \frac{dg(f)-4}{dg(f)} = 0 \) by R1.

Let \( v \in V \). If \( d_G(v) = 2 \), then \( v \) receives at least \( \frac{2}{3} - \frac{4}{|2r|-3} \) in total from its incident faces by Claim 3.2.1. By Lemma 2.1, \( v \) has two 5-neighbors. Thus, \( v \) receives \( \frac{2}{3} + \frac{2}{|2r|-3} \) from each of them by R2.1. So we have \( ch^*(v) \geq d_G(v) - 4 + (\frac{2}{3} - \frac{4}{|2r|-3}) = 0 \).

If \( d_G(v) = 3 \), then \( v \) receives at least \( \frac{1}{3} - \frac{4}{|3r|-6} \) in total from its incident faces by Claim 3.2.1. By Lemmas 2.1 and 2.2, \( v \) has three 4°-neighbors, and two of them have degree 5.

If \( v \) has a 4-neighbor, then by R2.2.1, \( ch^*(v) \geq d_G(v) - 4 + (\frac{1}{3} - \frac{4}{|3r|-6}) = 0 \).

If \( v \) has at least three 5-neighbors, then by R2.1 and R2.4, \( ch^*(v) \geq d_G(v) - 4 - (\frac{1}{3} - \frac{4}{|5r|-12}) - (\frac{2}{3} + \frac{2}{|2r|-3}) + 3(\frac{4}{3(3r-6)} + \frac{2}{3(|3r|-6)}) = 0 \).

Next assume that \( v \) has a 3-neighbor, then by Lemma 2.2, \( v \) has at least three 5-neighbors. In this case, \( v \) sends nothing to each 5-neighbor. Let \( u \) be the remaining neighbor of \( v \). Then \( d_G(u) \in \{3, 4, 5\} \).

If \( d_G(u) = 3 \), then by Lemma 2.4, \( u \) is incident with at most two triangles. Thus, by Claim 3.2.1, \( u \) receives a charge of at least \( \frac{1}{3} - \frac{4}{|5r|-14} \) in total from its incident faces. Moreover, since both \( u \) and \( v \) have no 4°-neighbor it follows by rule R2.2.2 that \( ch^*(v) \geq d_G(v) - 4 - (\frac{1}{3} - \frac{4}{|5r|-14}) - (\frac{2}{3} + \frac{2}{|2r|-3}) + 3(\frac{4}{3(|5r|-12)} + \frac{2}{|2r|-3}) = \frac{8}{3} - \frac{4}{|5r|-14} - \frac{8}{3(|3r|-6)}. \)

If \( d_G(w) = 4 \), and if \( w \) is adjacent to \( u \), then by Lemma 2.3, \( w \) has three 5-neighbors. Hence, by R2.2 and R2.3, \( ch^*(v) \geq d_G(v) - 4 - (\frac{1}{3} + \frac{4}{|5r|-12}) - (\frac{1}{3} + \frac{2}{|3r|-6}) - \frac{4}{3(|4r|-9)} = \frac{8}{3} - \frac{4}{|5r|-14} - \frac{8}{3(|3r|-6)} - \frac{4}{3(|4r|-9)}. \)
5. A vertex \(v\) at most one 4\(^{-}\)-face. We say a light vertex sends charge only to \(u\) and therefore, \(V\) with Claim 3.2.2 that \(\sum v \in G \sum x \geq 8\); good-light otherwise. Since \(r > 7 + \frac{1}{2}\) it follows that \(ch^*(x) \geq 0\) for all \(x \in V \cup F\). \(\square\)

The result for \(k = 6\) in Theorem 1.2 is implied by the following theorem.

**Theorem 3.3.** If \(G\) is a planar 6-critical graph, then \(F^*(G) \leq 3 + \frac{2}{5}\).

**Proof.** Suppose to the contrary that \(F^*(G) > 3 + \frac{2}{5}\). Let \(\Sigma\) be an embedding of \(G\) into the Euclidean plane and \(F^*(G) = F((G, \Sigma))\). We have

\[
\sum_{f \in F(G)} (2d_G(f) - 6) = 4|E(G)| - 6|F(G)|
\]

\[
= 4|E(G)| - 6(|E(G)| + 2 - |V(G)|) \quad \text{(by Euler’s formula)}
\]

\[
= 6|V(G)| - 2|E(G)| - 12
\]

\[
\leq |V(G)| - 15 \quad \text{(by Theorem 2.7)}
\]

and therefore, \(-|V(G)| + \sum_{f \in F(G)} (2d_G(f) - 6) \leq -15\). \((\ast)\)

Define the initial charge \(ch(x)\) for each \(x \in V(G) \cup F(G)\) as follows: \(ch(v) = -1\) for every \(v \in V(G)\) and \(ch(f) = 2d_G(f) - 6\) for every \(f \in F(G)\). It follows from inequality \((\ast)\) that \(\sum_{x \in V(G) \cup F(G)} ch(x) \leq -15\).

A vertex \(v\) is called heavy if \(d_G(v) \in \{5, 6\}\) and \(v\) is incident with a face of length 4 or 5. A vertex \(v\) is called light if \(2 \leq d_G(v) \leq 4\) and \(v\) is incident with no \(6^+\)-face and with at most one \(4^+\)-face. We say a light vertex \(v\) is bad-light if \(v\) has a neighbor \(u\) such that \(d_G(u) + d_G(v) = 8\); good-light otherwise.

Discharge the elements of \(V(G) \cup F(G)\) according to following rules.

**R1:** every \(4^+\)-face \(f\) sends \(\frac{2d_G(f) - 6}{d_G(f)}\) to each incident vertex.

**R2:** every heavy vertex sends \(\frac{3}{10}\) to each bad-light neighbor, and \(\frac{1}{10}\) to each good-light neighbor.

Let \(ch^*(x)\) denote the final charge of each \(x \in V(G) \cup F(G)\) after discharging. On one hand, the sum of charge over all elements of \(V(G) \cup F(G)\) is unchanged. Hence, we
have $\sum_{x \in V(G) \cup F(G)} ch^*(x) \leq -15$. On the other hand, we show that $ch^*(x) \geq 0$ for every $x \in V(G) \cup F(G)$ and hence, this obvious contradiction completes the proof.

It remains to show that $ch^*(x) \geq 0$ for every $x \in V(G) \cup F(G)$.

Let $f \in F(G)$. If $d_G(f) = 3$, then no rule is applied for $f$. Thus, $ch^*(f) = ch(f) = 0$.

If $d_G(f) \geq 4$, then by R1 we have $ch^*(f) = ch(f) - d_G(f) \frac{2d_G(f) - 6}{d_G(f)} = 0$.

Let $v \in V(G)$. First we consider the case when $v$ is heavy. On one hand, since $F((G, \Sigma)) > 3 + \frac{2}{5}$, it follows that either $v$ is incident with a $5^+$-face and another $4^+$-face or $v$ is incident with at least three $4$-faces. In both cases, $v$ receives at least $\frac{13}{10}$ in total from its incident faces by R1. On the other hand, we claim that $v$ sends at most $\frac{3}{10}$ out in total. If $v$ is adjacent to a bad-light vertex $u$, then all other neighbors of $v$ have degree 6 by Lemma 2.3. Hence, $v$ sends $\frac{3}{10}$ to $u$ by R2 and nothing else to its other neighbors.

If $v$ is adjacent to no bad-light vertex, then $v$ has at most three good-light neighbors by Lemma 2.2. Hence, $v$ sends $\frac{1}{10}$ to each good-light neighbor by R2 and nothing else to its other neighbors. Therefore, $ch^*(v) \geq ch(v) + \frac{13}{10} - \frac{3}{10} = 0$.

Second we consider the case when $v$ is not heavy. In this case, $v$ sends no charge out. If $v$ is incident with a $6^+$-face, then $v$ receives at least 1 from this $6^+$-face by R1. This gives $ch^*(v) = ch(v) + 1 = 0$. If $v$ is incident with at least two $4^+$-faces, then $v$ receives at least $\frac{1}{2}$ from each of them by R1. This gives $ch^*(v) = ch(v) + \frac{1}{2} + \frac{1}{2} = 0$. We are done in both cases above. Hence, we can assume that $v$ is incident with no $6^+$-face and with at most one $4^+$-face, that is, $v$ is light. From $F((G, \Sigma)) > 3 + \frac{2}{5}$ it follows that $v$ is incident to a face $f_v$ such that $d_G(f_v) \in \{4, 5\}$.

If $d_G(f_v) = 4$, then $v$ has degree 2. It follows that the two neighbors of $v$ are heavy. Thus, $v$ receives $\frac{1}{2}$ from $f_v$ by R1 and $\frac{3}{10}$ from each neighbor by R2. Hence, $ch^*(v) = ch(v) + \frac{1}{2} + \frac{3}{10} + \frac{3}{10} > 0$.

If $d_G(f_v) = 5$, then $v$ receives $\frac{1}{2}$ from $f_v$. If $v$ is not a bad-light 4-vertex, then by Lemma 2.2 every neighbor of $v$ has degree 5 or 6. Hence, both of the two neighbors of $v$ contained in $f_v$ are heavy. By R2, each of them sends charge at least $\frac{1}{10}$ to $v$, and therefore, $ch^*(v) \geq ch(v) + \frac{1}{2} + \frac{1}{10} + \frac{1}{10} = 0$. If $v$ is a bad-light 4-vertex, then at least one of the two neighbors of $v$ contained in $f_v$ is heavy. Thus, this heavy neighbor sends charge $\frac{3}{10}$ to $v$, and therefore, $ch^*(v) \geq ch(v) + \frac{4}{5} + \frac{3}{10} > 0$. □

4 Concluding remarks

Problem 4.1. What are the precise values of $b_k$ and $b_k^*$?
Seymour’s exact conjecture [6] says that every critical planar graph $G$ is overfull, i.e. $|V(G)|$ is odd and $|E(G)| = \Delta(G) \lfloor \frac{1}{2} |V(G)| \rfloor + 1$. If this conjecture is true for $k \in \{3, 4, 5\}$, then $b_k$ is equal to the lower bound given in Theorem 1.1. It is also not clear whether $\overline{b}_k$ and $b_k^*$ or $\overline{F}(G)$ and $F^*(G)$ are related to each other, respectively.

By Proposition 3.1, $\overline{F}(G)$ has an upper bound for every critical planar graph $G$. However, this is not always true for class 2 planar graphs. Similarly, Theorems 3.2 and 3.3 can not be generalized to class 2 planar graphs.

References


[12] L. Zhang, Every graph with maximum degree 7 is of class 1, Graphs Combin. 16 (2000) 467-495.