Abstract—In this paper, we present a novel analytical framework for the calculation of the level crossing rate (LCR) and the average fade duration (AFD) of fading channels sampled at a certain sampling period $T_s$. These expressions are valid for arbitrary fading distributions with arbitrary correlation, and can be easily computed in terms of the cumulative distribution function (CDF) of the fading envelope and its bivariate CDF. This approach yields interesting insights about the effect of finite sampling in the higher order statistics of fading processes. We also demonstrate that the proposed expressions for sampled fading process converge with the existing expressions for continuous fading processes as the sampling period tends to zero. As a direct application, exact closed-form expressions are given for the LCR and AFD of sampled Rayleigh fading processes, which are suitable to characterize the higher order statistics of the equivalent frequency-domain fading process in multipath Rayleigh fading.

Index Terms—Average fade duration, fading channels, level crossing rate, performance analysis, statistics.

I. INTRODUCTION

Random processes associated with fading channels are usually characterized by their probability density function (PDF) or cumulative distribution function (CDF). These functions allow for the computation of performance metrics such as the symbol error rate (SER), the outage probability (OP) or the system capacity [1].

Second-order statistics of fading channels incorporate information related not only with the scattering environment but with the dynamics of the system. Two valid examples are the level crossing rate (LCR), which provides information about how often the envelope fading crosses a certain threshold, and the average fade duration (AFD), which is related with the amount of time that the envelope remains below this threshold [2, 3].

Since the original work by Rice [4], who proposed a framework for the calculation of the LCR (and consequently the AFD) of a fading process using the joint PDF of the fading envelope and its time derivative, an intense research has been conducted by many authors in order to determine the LCR of different single fading processes such as Nakagami-m [5], Hoyt [6], Weibull [7], $\eta - \mu$ [8] and other distributions [9–11]. Expressions for these statistics have also been obtained for multi-branch reception using different combining strategies [12–14], and Rice’s original framework has also been extended to cope with combinations of multiple independent random processes [15]. It is interesting to note that all these previous works follow the original Rice’s approach for calculating the LCR of a continuous random process and usually are restricted to a particular correlation model, whereas alternative approaches are rarely found in the literature; e.g. using a characteristic function method [16].

Most modern digital communication systems are inherently discrete due to sampling and hence it is usual to use equivalent discrete models for the analysis of such systems. Besides, we observe that in the literature [3, 5, 6, 8, 13] theoretical expressions for the LCR did not exactly match with simulated values (where included) for low values of the threshold level. As the simulation models are discrete by nature, a mismatch between the LCR of continuous and discrete random process can be inferred.

In the literature, analyses devoted to characterize the higher order statistics of sampled fading processes are scarce. In [17], the possibility of missing a level crossing of a continuous signal in a sampling interval is observed for finite values of sampling frequency. Therefore, the LCR of a continuous random process results in an upper bound of the LCR of the associated sampled random process. In [18], it was pointed out that the LCR of orthogonal frequency division multiplexing (OFDM) systems calculated using the continuous approach lead to an overestimation of the LCR. To alleviate this issue, it was suggested to perform a discrete probability calculation instead. However, this approach was not formalized and a closed-form expression was not provided.

In this paper, we derive a general framework for the calculation of higher order statistics of sampled fading processes with arbitrary correlation. Compact expressions in terms of the CDF of the fading envelope and its bivariate CDF are given; hence, the knowledge of the statistics of the derivative of the continuous random process is not needed to calculate the LCR (or the AFD) of the sampled process.

As a direct application, the analysis of sampled Rayleigh fading processes is illustrated, providing exact closed-form expressions for the LCR and AFD of the fading envelope which are proved to converge with the continuous approach as the sampling period tends to zero. This allows to characterize the LCR and AFD of the equivalent frequency domain fading model used in OFDM systems with a 2-D time-frequency correlation model. Results show that the use of a continuous approach for estimating the LCR of sample fading processes leads to an overestimation of the LCR for realistic values of delay spread and Doppler frequency. The proposed framework can be used to analyze the higher order statistics of relevant...
sampled fading processes using existing results in the literature for the bivariate CDF \cite{19-22}.

The remainder of the paper is structured as follows: in Section II, the proposed approach for the calculation of higher-order statistics of sampled random processes is introduced, whereas some applications are presented in Section III. Numerical results are given in Section IV, and the main conclusions are outlined in Section V.

II. PROBLEM FORMULATION

A. Higher-order statistics using Rice’s approach

The LCR of a continuous random process is defined as the average rate at which the envelope \( r \) crosses a certain threshold \( u \) in the positive (or equivalently in the negative) direction \cite{2}, and can be calculated using the original integral expression given by Rice \cite{4}

\[
N_c(u) = \int_0^\infty \tilde{r} p(u, \tilde{r}) \, d\tilde{r}
\]

(1)

where \( \tilde{r} \) denotes the time derivative of the envelope \( r \), and \( p(r, \tilde{r}) \) denotes the joint PDF of \( r \) and \( \tilde{r} \). The subindex \( c \) is included to remark the continuous nature of the random process. Therefore, the calculation of the LCR of a continuous random process is given in a single integral form in terms of \( p(r, \tilde{r}) \).

The AFD of a sampled random process, defined as the average duration of the envelope \( Z \) remaining below a specified threshold level \( u \), can be calculated using (1) as

\[
T_c(u) = \frac{F_r(u)}{N_c(u)}
\]

(2)

where \( F_r(u) \) is the CDF of the fading envelope.

B. Proposed approach

The LCR of a sampled random process, defined as the average rate at which the envelope \( Z \) crosses a certain threshold \( u \) in the positive (or equivalently in the negative) direction can be expressed as

\[
N_Z(u) = \frac{\Pr\{Z_1 < u, Z_2 > u\}}{T_S}
\]

(3)

where \( Z_1 \triangleq Z(t) \), \( Z_2 \triangleq Z(t + T_S) \), and \( T_S \) denotes the sampling period. It can be seen that \( Z_1 \) and \( Z_2 \) are correlated and identically distributed random variables. Thus, their cumulative distribution function (CDF) can be denoted as \( F_Z(x) \triangleq F_{Z_1}(x) = F_{Z_2}(x) \), whereas their joint CDF is defined as \( F_{Z_1, Z_2}(x, y) = \Pr\{Z_1 \leq x, Z_2 \leq y\} \).

Noting that \( \Pr\{Z_1 < u, Z_2 > u\} = \Pr\{Z_1 < u\} - \Pr\{Z_1 < u, Z_2 < u\} \), we can express the LCR of a sampled random process in compact form as

\[
N_Z(u) = \frac{F_Z(u) - F_{Z_1, Z_2}(u, u)}{T_S}.
\]

(4)

Therefore, the LCR of a sampled random process can be expressed in terms of the difference of its CDF and the bivariate CDF of the sampled envelope evaluated at the threshold level \( u \). Note that this approach is general, and valid for arbitrary distributions of the envelope with arbitrary correlation.

It is worth mentioning that the full characterization of the bivariate CDF of the fading envelope is not necessary. In fact, \( F_{Z_1, Z_2}(u, u) \) can be identified with the CDF of the random variable \( Z_M \triangleq \max\{Z_1, Z_2\} \), as

\[
F_{Z_1, Z_2}(u, u) = F_{Z_M}(u).
\]

(5)

Expression (5) corresponds to the CDF of the decision metric at the output of a dual-branch selection combiner (i.e., the outage probability), and hence it is possible to use existing results in the literature for the direct calculation of the LCR \cite{23}.

The AFD of a continuous random process, defined as the average duration of the envelope \( Z \) remaining below a specified threshold level \( u \), can be calculated using (4) as

\[
A_Z(u) = \frac{\Pr\{Z \leq u\}}{N_Z(u)} = T_S \left(1 - \frac{F_{Z_1, Z_2}(u, u)}{F_Z(u)}\right)^{-1}.
\]

(6)

III. APPLICATIONS

A. Sampled Rayleigh fading

Let us consider a the fading process \( g = g_1 + j g_Q \) as a complex circularly symmetric Gaussian RV with zero mean and \( \sigma_g^2 = \sigma_{g_1}^2 + \sigma_{g_Q}^2 \) variance; hence, its envelope \( r = |g| \) follows a Rayleigh distribution whose CDF is given by

\[
F_r(x) = 1 - e^{-x^2/\sigma_g^2}.
\]

(7)

We can use the closed-form expression for the Rayleigh bivariate CDF given in \cite{23} specialized at \( x \)

\[
F_{r_1, r_2}(x, x) = 1 - e^{-x^2/\sigma_g^2} Q_1(k_r x/\sigma_g, k_r x/\sigma_g) - e^{-x^2/\sigma_g^2} (1 - Q_1(k_r x/\sigma_g, k_r x/\sigma_g)),
\]

(8)

where \( Q_1(a,b) \) is the Marcum-Q function, \( r_1 = r(t_0) \) and \( r_2 = r(t_0 + T_S) \) according to the notation introduced in Section II, the correlation coefficient \( \rho = \text{cov}(g_1, g_Q)/\sqrt{\sigma_{g_1}^2 \sigma_{g_Q}^2} \) and \( k_r = \sqrt{2/\left(1-\rho^2\right)} \). It is clear that \( \rho = \rho(T_S) \), although for the sake of compactness we will omit this dependence unless otherwise stated.

Using (4) and (6), the LCR and AFD of a sampled Rayleigh fading process are given in closed-form by

\[
N_r(u) = \frac{e^{-u^2}}{T_S} \left( Q_1(k_r u, k_r u|\rho) - Q_1(k_r u|\rho, k_r u) \right);
\]

(10)

\[
A_r(u) = T_S Q_1(k_r u, k_r u|\rho) - Q_1(k_r u|\rho, k_r u)
\]

(11)

where \( u \) is the threshold level normalized to \( \sigma_g \). Expressions (10) and (11) are novel to the best of the authors’ knowledge, and are valid for an arbitrary correlation of the fading envelope.
B. Convergence with the continuous approach

The convergence to the conventional continuous approach can be seen as \( T_S \to 0 \) (i.e., as \( \rho(T_S) \to 1 \)). For convenience of discussion, we will consider the usual approach of an isotropic scattering environment [3]. This implies that the Doppler spectrum is symmetrical, and the correlation coefficient exhibits a negative curvature at the origin; hence taking the first terms of the Taylor expansion of \( \rho(T_S) \) we have

\[
\rho(T_S)|_{T_S \to 0} \approx 1 - 3T_S^2, \quad (12)
\]

Let us denote the symmetric difference of two Marcum-Q functions as

\[
\overline{Q_1}(a, b) \triangleq Q_1(a, b) - Q_1(b, a), \quad a > b
\]

which can be rearranged using the known equivalence given in [23]

\[
Q_1(a, b) + Q_1(b, a) = 1 + e^{-\frac{(a^2 + b^2)}{2}} I_0(ab),
\]

as

\[
\overline{Q_1}(a, b) = 1 + e^{-\frac{(a^2 + b^2)}{2}} I_0(ab) - 2 \cdot Q_1(b, a).
\]

Using the asymptotic relationships

\[
Q_1(b, a) \approx \left(\frac{a}{b}\right) Q(a - b), \quad a \to \infty
\]

\[
e^{-\frac{(a^2 + b^2)}{2}} I_0(ab) \approx \frac{a - b}{\sqrt{ab}} Q(a - b), \quad ab \to \infty
\]

where \( Q(x) \) is the Gaussian Q function, we obtain the following approximation for the symmetric difference of two Marcum-Q functions as\(^1\)

\[
\overline{Q_1}(a, b) \approx 1 - \left(\frac{b}{a} + \sqrt{\frac{a}{b}}\right) Q(a - b). \quad (18)
\]

Taking the first-order term of the Taylor series for the Gaussian Q function,

\[
Q(x) \approx \frac{1}{2} \left(1 - \sqrt{\frac{2}{\pi}} x\right)
\]

the following compact form for the level crossing rate can be obtained

\[
N_r(u) \approx \frac{e^{-u^2}}{T_S} \left(1 - \frac{1}{2 \sqrt{\rho + \frac{u}{\sqrt{\rho}}} \sqrt{1 - \rho}}\right). \quad (20)
\]

Finally, setting \( \rho(T_S) \approx 1 - 3T_S^2 \), we can calculate

\[
\lim_{T_S \to 0} N_r(u) = \frac{\sqrt{3}}{\pi} u e^{-u^2}
\]

which is coincident with the original expression given by Rice for the LCR of a continuous Rayleigh process.

\(^1\)The use of \( Q_1(a, b) \approx \sqrt{b/a} Q(b - a) \) when \( b \to \infty \) offers a very good approximation when approximating a single \( Q_1 \) function. However, the \( Q_1(a, b) - Q_1(b, a) \approx \sqrt{b/a} Q(b - a) - \sqrt{a/b} Q(a - b) \) approximation is degraded as the second argument in one of the Marcum-Q functions is necessarily lower than the first argument.

C. Scenario of interest: OFDM

In this subsection, we use the previous results in a particular scenario of interest: the calculation of higher-order statistics for a frequency domain equivalent system of OFDM in the presence of multipath Rayleigh fading. Higher-order statistics of OFDM were first tackled in [24], using a continuous approach.

We assume a channel impulse response (CIR) with exponential multipath profile, mean delay spread denoted as \( \tau \), normalized gain, and time variation according to Clarke’s correlation model [3], with maximum Doppler shift \( f_D \). The fading experienced at the \( k \)-th subcarrier in the \( n \)-th symbol period denoted as \( H_{n,k} \) is a complex normal random variable with zero mean, \( \sigma^2 \) variance with a two-dimensional correlation coefficient:

\[
\rho(\Delta n, \Delta k) \triangleq \frac{E[H_{n,k}H_{n+\Delta n,k+\Delta k}^*]}{\sigma^2} = \frac{J_0(2\pi f_D T(\Delta n))}{1 - j2\pi\Delta f \tau(\Delta k)}
\]

where \( \Delta f \) is the subcarrier spacing and \( J_0(x) \) is the zero-th order Bessel function of the first kind.

In this scenario, two LCRs can be defined: for a fixed \( n \) symbol, the LCR of the frequency domain envelope across the available transmission bandwidth, and for a fixed \( k \) subcarrier, the LCR of the frequency response at this subcarrier index along the successive OFDM symbols. It is interesting to note that both cases fit within the analytical framework presented in subsection III-A.

The former LCR (i.e., the LCR in the frequency-domain) is given by

\[
N_f(u) = \frac{e^{-u^2}}{\Delta f} \left(Q_1(k_{\rho_f} u, k_{\rho_f} u) - Q_1(k_{\rho_f} u, k_{\rho_f} u)\right)
\]

where \( u \) is the level threshold normalized to \( \sigma^2 \), the correlation coefficient \( \rho_f \) is given by

\[
\rho_f \triangleq \rho(0, 1) = \frac{1}{1 - j2\pi\Delta f \tau}.
\]

and \( k_{\rho_f} \) is calculated using the definition in subsection III-A setting \( \rho = \rho_f \).

The latter LCR (i.e., the LCR in the time-domain) is given by

\[
N_T(u) = \frac{e^{-u^2}}{T} \left(Q_1(k_{\rho_T} u, k_{\rho_T} u) - Q_1(k_{\rho_T} u, k_{\rho_T} u)\right).
\]

In this case, the correlation coefficient \( \rho_T \) is given by

\[
\rho_T \triangleq \rho(1, 0) = J_0(2\pi f_D T),
\]

and \( k_{\rho_T} \) is calculated using the definition in subsection III-A setting \( \rho = \rho_T \).

Directly from (23) and (25), the average duration of fades can be calculated in both situations as
\[ A_f(u) = \Delta f \frac{e^{\alpha^2} - 1}{Q_1(k_{\rho_f} u, k_{\rho_f} |\rho_f|) - Q_1(k_{\rho_f} u|\rho_f|, k_{\rho_f} u)} \]
\[ A_T(u) = T \frac{e^{\alpha^2} - 1}{Q_1(k_{\rho_x} u, k_{\rho_x} |\rho_T|) - Q_1(k_{\rho_x} u|\rho_T|, k_{\rho_x} u)} \]

D. Other fading distributions

The calculation of the bivariate CDF of two correlated random variables is a classical problem in communication theory. Relevant results for the main fading distributions are available in the literature, e.g., [19–22, 25–27] for Nakagami-m, \( \eta - \mu \), Ricean, Weibull and other distributions.

One of the most direct applications of the bivariate CDF is the evaluation of the outage probability of a dual-branch selection combiner in correlated fading channels. Using the proposed approach, it is also possible to use the bivariate CDF to calculate the higher-order statistics associated with these fading distributions considering a finite sampling period.

IV. Numerical Results

In this Section, we use the derived expressions for the higher-order statistics of OFDM systems affected by multi-path Rayleigh fading to evaluate their behaviour in different scenarios.

Figures 1 and 2 illustrate the level crossing rates for a subcarrier \( k \) across the symbol transmission, and for a symbol \( n \) across the available bandwidth respectively. Montecarlo simulations are included to check the validity of the results. It is appreciated that when the channel frequency response is highly oversampled (i.e., low values of \( \Delta f \cdot \tau \) or \( f_{D} \cdot T \)), the LCR has a very good match with the LCR of the continuous fading process.

\[ \text{Fig. 1. Normalized frequency-domain level crossing rate of subcarrier } k \text{ vs threshold level } u \text{ for different values of } f_{D}. \text{ Solid lines correspond to theoretical expressions, and markers correspond to MC simulations.} \]

\[ \text{Fig. 2. Normalized time-domain level crossing rate of symbol } n \text{ vs threshold level } u \text{ for different values of } \tau. \text{ Solid lines correspond to theoretical expressions, and markers correspond to MC simulations.} \]

As the sampling rate is reduced, it is observed that the LCR of the sampled random process differs is lower than the LCR given by the original Rice expression. In fact, this gap tends to grow as the considered threshold level \( u \) is decreased. Therefore, the calculation of level crossing rate using the continuous approach leads to a non-negligible overestimation of the LCR for low values of the threshold level \( u \).

Figures 3 and 4 show the average durations of fades for the considered scenarios. Since LCR and AFD are inversely proportional, there exists an underestimation in the average fade duration calculated using the continuous approach. In fact, the AFD of a sampled fading process exhibits a floor value as the threshold level is reduced, which is clearly appreciated when the random process are not very oversampled.

\[ \text{Fig. 3. Normalized frequency-domain average fade duration of subcarrier } k \text{ vs threshold level } u \text{ for different values of } f_{D}. \text{ Solid lines correspond to theoretical expressions, and markers correspond to MC simulations.} \]

V. Conclusion

An analytical framework for the calculation of higher-order statistics of sampled fading channels with arbitrary correlation has been proposed. Following this approach, the level crossing
rate and the average fade duration of a sampled random process can be calculated in terms of the CDF and its bivariate CDF (or more precisely, the CDF of the maximum of two correlated RVs). Hence, the calculation of higher-order statistics for the most relevant fading processes can be calculated using existing results in the literature. Additionally, the proposed approach can be used to evaluate the higher-order statistics of continuous random processes using $T_S \rightarrow 0$. Since the proposed formulation is valid for an arbitrary random process, it may result applicable for different physical parameters such as the envelope or the phase.

As a direct application, we have calculated exact closed-form expressions for the level crossing rates of the equivalent frequency domain sampled random process associated with a multipath Rayleigh fading channel. It has been demonstrated that as the sampling rate of the random process is reduced, the actual LCR is lower than the LCR of the associated continuous random process (and consequently, the actual AFD is larger).

ACKNOWLEDGEMENTS

The authors would like to thank the anonymous reviewers for their comments and suggestions, which undoubtedly have improved the quality of the paper. This work was supported in part by Junta de Andalucia (predoc grant, and project “Analysis and design of cooperative adaptive MIMO-OFDM systems”), Spanish Government-FEDER (TEC2010-18451, TEC2011-25473), and the company AT4 Wireless.

REFERENCES