ICA-based technique for radiating sources estimation: application to airport surveillance

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Abstract: As aerial traffic becomes greater, it is more difficult to locate and recognise aircraft in the neighbourhood of civil airports. The technique proposed here resorts to a particular device, monopulse radar, and to a recent tool called the independent component analysis (ICA) to separate messages falling in the same radar beam. The algorithms used to compute the ICA use fourth-order cumulants of the observed signals.

1 Introduction

Air traffic control consists mainly of detecting aircraft, identifying them, and estimating their location and speed. This task is made more and more difficult because of greater aerial traffic. In particular, the probability that two aeroplanes fall in the same radar beam of a civil airport is no longer negligible. Secondary radar is very well matched to this task because aircraft play an active role in the sense that they themselves emit the requested information. For instance 'monopulse radar', which is a particular secondary radar, allows angle measurement and is becoming more widely used. It is proposed in this paper to use the channels that are available in current monopulse radar to improve both angle measurements and aircraft recognition, especially for aircraft falling in the same radar beam. Further improvements are also discussed at the end of this paper that would require modifications of the monopulse radar equipment so that more channels would be available.

2 Secondary radar

The aim of a secondary radar is to receive information from aircraft that allows the ground station to locate and recognise them. Contrary to primary radar, secondary radar needs aircraft to participate in the recognition. For this purpose, aircraft are equipped with a device called a 'transponder'. A question is sent to aircraft in the form of three pulses, $P_1$, $P_2$ and $P_3$, $P_1$ is transmitted in an omnidirectional beam $\Omega$, whereas $P_2$ and $P_3$ are transmitted in a narrow beam $\Sigma$. All aircraft in the radar range are thus questioned if they are supplied with transponders. All transponders measure the levels in $\Sigma$ and $\Omega$. They answer only in the case when the level in $\Sigma$ is larger than that in $\Omega$; in other words, they answer only if they are in the main beam of the secondary radar, as shown in Fig. 1.

![Fig. 1 Principal omnidirectional and difference beams in monopulse radar](image)

In secondary radar, the difference beam is not available.

The answer of the transponder consists of 15 pulses, beginning and ending with known pulses $F_1$ and $F_2$ serving to locate the answer. Between those two pulses, the remaining 13 pulses carry the information (altitude, speed, etc.). The reader is invited to consult References 16 and 10 for complementary information on secondary radar, and Reference 14 for historical remarks. The features of interrogations sent by the ground station (pulses $P_1$, $P_2$ and $P_3$) and corresponding transponder responses obey international norms that have been defined by the International Civil Aviation Organisation (ICAO). Transponders responses are thus realised by a mere amplitude modulation of a carrier whose frequency must be located in the interval $1093 \pm 3$ MHz.

Here we shall consider the particular secondary radar called monopulse radar. This radar involves a third beam $\Lambda$, corresponding to the difference channel (Fig. 1). Among others, this beam is used to locate the target in the main beam by measuring the ratio $\Lambda/\Sigma$. The data to be used in this work contain the two channels $\Lambda$ and $\Sigma$.

The secondary radar system has been designed to be able to question distant aircraft. However, because of the strong increase in aerial traffic, especially in the neighbourhood of airports, it often happens that two aircraft find themselves in the main antenna beam. In such a case, they would answer at the same time, and the transponder messages received by the ground station would be mixed, yielding it impossible to process the messages. In fact, the classical processing of secondary radar is based on the analysis of the amplitude of pulses received in a message. The messages mixture with superimposed pulses (called garbling) leads to a distortion of this amplitude (intermodalisation) and yields in most cases a severe loss of
information in angular gap measurement (AGM) and in interfering aircraft codes.

It is envisaged to utilise the independent component analysis (ICA) concept defined in Reference 5 and detailed in Section 4 to separate garbling messages, the goal being first to recover the codes associated with each aircraft and secondly to access information in AGM if possible. Not only has the contrast maximisation (CM) algorithm described in Reference 5 been used, but also the algorithm based on the contracted quadracovariance (CQ) proposed in Reference 3 by Cardoso.

3 Implementation

To process the response present in the main beam only channels Σ and Λ will be retained among the three channels available. In fact, too many responses are received in the quasimidirectional channel Ω. Moreover, the two algorithms mentioned in the previous Section are not able to recover more sources than channels.

We utilise an antenna of D = 10 m with the phase centre at the middle of the antenna. The band is B = 2.5 MHz, and we may assume the narrow-band hypothesis

\[
B \times D \approx \frac{2.5 \text{ MHz} \times (1/2)}{3 \times 10^6} \approx 4.1 \times 10^{-2}
\]

In such a case, the two channels of the observation can be written as

\[
\Xi(t) = M_1 s_1(t) + M_2 s_2(t) + v(t)
\]

\[
\Delta(t) = M_1 s_1(t) + M_2 s_2(t) + v(t)
\]

where \(v_1\) and \(v_2\) are noises and \(M_{ij}\) are unknown coefficients. Since transponders respond generally at different frequencies and by different messages with different initial phases at emission, it can be reasonably assumed that signals \(s_1(t)\) and \(s_2(t)\) are independent in the statistical sense, at least at orders two and four (i.e. the correlation is null and the cross-cumulants of order four are null).

It will be preferred in the remaining to use a compact form of these equations

\[
y(t) = M(t) + v(t)
\]

The noise component \(v(t)\) is mainly thermal, but may also contain parts of other aircraft responses because of the presence of secondary beams in the spatial directivity of the antenna. It is therefore not assumed that \(v(t)\) is Gaussian. Since responses emitted by different aircraft may be assumed to be independent, the two vector processes \(v(t)\) and \(\dot{v}(t)\) can be assumed to be statistically independent (only independence up to order four will be actually required). It is sought to obtain estimates of original signals contained in \(v(t)\), given only observations \(\Xi(t)\) and \(\Delta(t)\) contained in \(y(t)\). In a second stage, these estimates will be analysed to extract the expected codes.

4 Description of algorithms

It is proposed here to use the independent component analysis (ICA) defined in Reference 5. Several algorithms now exist to compute the ICA and all of them necessarily make use of high-order statistics. In this paper two algorithms in References 3 and 5, are utilised. As pointed out in this Section, the index to be maximised in these algorithms is a measure of statistical indepenence between outputs based on high-order statistics.

4.1 Notation

Consider the instantaneous observation model of the form

\[
y = Mx + v
\]

where \(x\) is a \(K \times 1\) random vector expressing the signal transmitted by the transponders and \(v\) is an additive noise. In this model, only the observed signal \(y\) is assumed to be known. The quantities of interest are \(x\) and possibly the mixing matrix \(M\). As in most radar systems, the data are processed in a narrow band, so that the model above is valid for a given frequency band. In addition, the observations are the result of Hilbert transforms, so that all quantities are complex. Superscripts \((\ast)\) and \((\dagger)\) will denote complex conjugation and hermitian transposition, respectively.

For any random vector \(w\), denote the mean of each component \(\mu_{w,i} = \mathbb{E}[w_i]\) and define the centred moments up to order four

\[
\mu_{w,i,j} = \mathbb{E}[(w_i - \mu_i)(w_j - \mu_j)^\ast]
\]

\[
\mu_{w,i,j,k} = \mathbb{E}[(w_i - \mu_i)(w_j - \mu_j)^\ast(w_k - \mu_k)(w_l - \mu_l)^\ast]
\]

and the corresponding fourth-order cumulant

\[
cum(w)_{ijkl} = \mu_{w,i,j,k} - \mu_{w,i} \mu_{w,j,k} - \mu_{w,i} \mu_{w,j,k} - \mu_{w,i} \mu_{w,j,k}
\]

Cumulants of successive orders are in practice estimated by resorting to so-called \(k\)-statistics [12, 17]. More details about cumulants and their use can be found in References 1, 12 and 13.

The goal of ICA is to find a linear transform \(A\) that tries to make the components of

\[
z = Ay
\]

as independent as possible in the statistical sense. It is convenient to decompose the matrix \(A\) into

\[
A = LQ
\]

where \(Q\) is unitary. There are various ways of performing this decomposition. For instance, if \(L\) is chosen to be lower triangular, there is a unique decomposition, usually called the QR decomposition (the transposition of factorisation of eq. 8) [3]. If \(L\) is imposed to be Hermitian, the factorisation is again unique; this is known as the polar factorisation of \(A\). In the current work, \(L\) is imposed to be the product of a unitary matrix and a diagonal matrix with positive real entries (see Section 5). An important remark is that if we are imposing no other condition on vector \(z\), then for any diagonal matrix \(A\) and any permutation \(P\), the solution \(APz\) will be as good as \(z\). This is indeed an indeterminacy inherent in the problem [5]. It will be assumed in the remainder of this paper that the scaling matrix \(A\) is chosen such that the components of \(z\) have a unit variance (Fig. 2).

4.2 Standardisation

Given a \(K \times 1\) random vector \(w\) with finite covariance \(V_w\), standardisation consists of finding a full rank \(r \times K\) matrix \(L_w\) such that the vector \(\hat{w} = L_w w\) has a unit covariance. In this work, the standardisation is based on the eigenvalue decomposition (EVD) of \(V_w\)

\[
V_w = U \Lambda U^\ast
\]

where \(A\) is an invertible \(r \times r\) diagonal matrix whose entries are real positive and sorted in decreasing order, and \(U\) is a full rank \(K \times r\) matrix such that \(U^\ast U = I\).

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Each column of $U$ can be chosen such that the entry of largest vector modulus is positive real. The standardised random vector is constructed with the transform

$$v = L_w w \quad L_w = A^{-1}U$$

(10)

Fig. 2  General processing diagram

The last operation in a normalisation which fixes the variance of the outputs. With the constraints assumed in the present paper, it is the identity matrix

If the covariance $V_w$ is singular, then $r < K$ and the dimension of $\hat{w}$ is smaller than that of $w$. Denote $L_w = U_A$, then $L_w$ is the pseudoinverse of $L_w$ and $L_w L_w^T = V_w$.

Now, a necessary condition for the components $z_1$ to be independent is that they be uncorrelated (at order two), which means that the covariance $V_z$ of $z$ is diagonal. Thus, by just standardising the observed vector $y$, we have a possible candidate for vector $z$

$$\hat{y} = L_w^T y \quad V_z = I$$

(11)

Unfortunately, the components of $\hat{y}$ are only uncorrelated, and the matrix $A$ is defined for the moment only up to a multiplicative unitary matrix (Fig. 2).

The following Sections describe two means for calculating this unitary matrix that transform the standardised observation $\hat{y}$ into a random vector $z = Q \hat{y}$, whose components have minimal crosscorrelations up to the fourth order. For this purpose, the following notations will be assumed:

$$\gamma_{ijkl} = \text{cum}\langle \hat{y}_i \hat{y}_j \hat{y}_k \hat{y}_l \rangle$$

(12)

$$\kappa_{ijkl} = \text{cum}\langle \hat{y}_i \hat{y}_j \hat{y}_k \hat{y}_l \rangle$$

(13)

These cumulants are usually referred to as 'standardised cumulants'.

4.3 Contracted quadricovariance (CQ) algorithm

The purpose of this Section is to briefly summarise the principles utilised in the CQ algorithm. For further details, refer to References 2 and 3 and further references therein. The algorithm uses a contracted version of the cumulants tensor of eqn. 12, and is sometimes also referred to as the FOBI algorithm or Fourth-Order Blind Identification. Define the matrix

$$(R_{ij})_{kl} = \sum_{i,j} \text{cum}\langle \hat{y}_i \hat{y}_j \rangle$$

(14)

The algorithm consists of computing the EVD of the fourth-order cumulant matrix $R_{ij}$ as

$$R_w = \sum_{r} \nu_r \nu_r^T \nu_r^T$$

and identifying the unit vectors $L_r \nu_r$ to be estimates of the columns of $M$ up to a scalar multiplicative factor.

4.4 Contrast maximisation (CM) algorithm

This algorithm is based on principles that are not given in the paper for reasons of space. A detailed justification of these principles may be found in References 5 and 6.

For the sake of convenience, complementary details regarding practical implementation are given in Appendix 8. A real-time implementation can also be thought of, and is described in Reference 4, in the real case only, however.

The unitary matrix is decomposed into a sequence of $n(\pi - 1)/2$ plane rotations having real cosine and complex sine. Successive complex angles are sought to maximise the contrast function

$$\psi(Q) = \sum_{r} |\kappa_{rppp}|^2$$

(16)

$$\kappa_{ijkl} = \sum_{mov} Q_{il} Q_{jl}^* Q_{kn} Q_{ln}$$

(17)

Clearly, the function to be maximised is thus a polynomial function in the entries of $Q$. The explicit form of the solution is given in Appendix 8.

The performances of the method proposed are first shown with real data, recorded at the airport of Orly, in the Paris suburb (bandwidth < 3 MHz, radar operating in L-band). The results show that the signal-to-noise ratio reached is sufficient to decode the messages correctly (and also localise the two aircraft).

The two-channel approach addresses most situations (two aircraft in the same two degree-wide beam, and at a distance of less than 2 km). As explained in Section 2, the actual current equipment cannot provide more than two channels. However, the procedure could still be improved if more than two aircraft fall in this volume, provided the equipment is revised. It is shown in Section 5.2 how the same principle would perform if the received signal $x$ had four or five channels, which can be the case in the near future.

5 Computer results

Both algorithms described have been extensively tested on simulated data, but also experimented on real data recorded at Orly airport near Paris.

5.1 Conditions of experiments and results obtained with real data

In this experiment, a classical monopulse secondary radar was receiving responses of two transponders located on the ground. An operator could thus vary the delay between the two messages, the angular separation (from 1/2 to 1/4 of beam, the beam being here approximately 3° wide), as well as the emitted codes themselves. The records have been constituted with the help of an 8-bits encoder (Lecroy 9450) at a sampling rate of 48 MHz, the intermediate frequency being around 60 MHz. The delay measured between channels $\Sigma$ and $\Delta$ at the digit to analogue convertor level were of the order of 0.5 nanoseconds.

The beam width and the sweeping speed permitted the recording of a dozen trials for each scenario. Signals were recorded during about 30 microseconds (for somewhat more than twice the response length) and consisted of about 2000 samples. It should be noticed that the encoder was not sensitive to the thermal noise, so that the only noise source is the quantisation noise (which is non-Gaussian). Most of the data have been collected with a signal-to-noise ratio of at least 20 dB, which is a very realistic situation especially in secondary radar where the data transmission phenomenon is observed.

Fifty useful scenarios have been recorded, providing a total of 225 garbling trials. These data have been then formatted in 64-bits complex numbers (32-bits for real
and imaginary parts) and processed by the two blind separation algorithms mentioned in Section 4.

Fig. 3 shows the data before and after separation by algorithm CM. The intermediate data obtained after (second-order) orthogonalisation and before separation by the CM algorithm confirm that the separation cannot be performed without the help of statistics of order larger than two.

After the separation is completed by the CM algorithm, postprocessing is performed on each channel output to recover the code. For this purpose, a search is carried out in a similar manner as in classical secondary radar. The processing of the 225 trials has enabled us to evaluate performances of the separation algorithms in terms of probability of correct estimation. Here, an estimation is considered to be correct if all its bits are correct. The results are summarised in Table 1.

**Table 1: Probabilities of correct code estimation**

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Success rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>Contrast maximisation (CM)</td>
<td>0.71</td>
</tr>
<tr>
<td>Contracted quadricovariance (CQ)</td>
<td>0.47</td>
</tr>
</tbody>
</table>

As far as the secondary radar is concerned, it seems that the measure of independence by a statistical contrast, utilised in the CM method, is better matched to the nature of the present signals. This may be due to the presence of non-Gaussian (quantisation) noise, which is not taken into account in the CQ method. This conclusion should not, however, be extended to other problems since it is only of an experimental nature. Moreover, recent improvements to the CQ method have been proposed in Reference 15. It has been reported there that both methods have the same asymptotic performances.

Nevertheless, the result of 71% success is very encouraging, if we take into account the fact that most records contained responses delayed by less than 100 nanoseconds. This fairly small delay would have almost systematically led to erroneous codes in classical processing.

With only two channels, it is quite evident that no more than two sources can be exactly separated. In the best case, one can indeed only estimate the best filters spatially matched to each emitter direction, in a similar fashion as in the Capon adaptive beamformer [9]. To be able to solve the garbling problem with an instantaneous mixture of more than two sources it will be necessary to think of adding supplementary channels to secondary radars. Sufficiency is argued by simulations in the following Section. Unfortunately, it is not presently envisaged in the operational equipment to add supplementary channels.

### 5.2 Additional results obtained by simulations

In this Section, it is demonstrated that the ICA separation principles also work properly for extracting more than two secondary radar responses, provided there are sufficient channels available. In the following example, a linear array of five uniformly distributed sensors has been chosen. Five transponder responses impinging on this array and are consequently delayed and mixed by a linear transform. Fig. 4 sketches the signals (in normalised amplitude) emitted by each transponder, and Table 2 gives various important features (such as departure from central frequency, distance, bearing, Döppler effect, etc.). The signals observed on the array of sensors is then represented in Fig. 5.

**Table 2: Features of the simulated messages: the IF was 60 MHz**

<table>
<thead>
<tr>
<th>Source</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bearing</td>
<td>0</td>
<td>90</td>
<td>170</td>
<td>220</td>
<td>320</td>
</tr>
<tr>
<td>Site</td>
<td>5</td>
<td>15</td>
<td>60</td>
<td>35</td>
<td>47</td>
</tr>
<tr>
<td>Distance</td>
<td>3.1</td>
<td>3.4</td>
<td>4.1</td>
<td>4.9</td>
<td>5.7</td>
</tr>
<tr>
<td>Frequency departure</td>
<td>0.25</td>
<td>-0.05</td>
<td>-0.5</td>
<td>0.15</td>
<td>0.3</td>
</tr>
<tr>
<td>Speed</td>
<td>200</td>
<td>225</td>
<td>210</td>
<td>215</td>
<td>245</td>
</tr>
</tbody>
</table>
Fig. 6 shows the output signals obtained after the ICA is completed. It is clear that the codes extraction has succeeded in this (typical) very realistic example. Numerous presented seem very promising. In fact a success rate of 70% has been reached without trying to adapt ICA algorithms to the real-world problem addressed. Complementarily work is now in progress which aims to identify the factors that are limiting performance in this particular problem (e.g. linearity of sensors, sampling frequency, coding dynamic, frequency gaps in transponders, etc.). Also work has been recently initiated to design a novel real-time hardware implementation of the monopulse radar.

Fig. 4  Emitted codes in the simulated data
Scale in normalised amplitude

other simulations have been performed and have led to the same kind of results.

6  Concluding remarks

Two algorithms performing ICA have been applied to real data recorded at Orly airport in Paris. The results

Fig. 5  Observations in the proposed simulation scenario

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Fig. 8 Outputs obtained after applying CM to the simulated observations, and before performing post-processing.

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8 Appendix

8.1 Determination of plane rotations

Here, further details concerning the calculation of the optimal plane rotations are given. Each plane rotation is performed similarly, so let us consider the calculation of the first one in plane \[ \{\mathbf{y}_1, \mathbf{y}_2\} \]. Write the rotation as a function of the tangent of the complex angle, denoted \( \theta \)

\[
\mathbf{Q} = \frac{1}{\sqrt{1 + \theta^*\theta}} \begin{pmatrix} 1 & \theta \\ -\theta^* & 1 \end{pmatrix}
\] (18)

The contrast function reduces to

\[
\psi(\mathbf{Q}) = |\kappa_{111}(\theta)|^2 + |\kappa_{2222}(\theta)|^2 + \text{constant}
\] (19)

where \( \kappa_{j_1j_2}(\theta) \) are given by the expressions

\[
\kappa_{1111}(\theta) = \frac{1}{\left[ 1 + \theta^*\theta \right] \left[ 1 + \theta^*\theta \right]} \left( \gamma_{2222} - \theta^*\theta \right)
\] (20)

\[
\kappa_{2222}(\theta) = \kappa_{1111} \left( -\frac{1}{\theta^*\theta} \right)
\] (21)

The contrast may be shown to be a real function of

\[
\zeta = \theta - 1/\theta^*\theta
\]

Since it is always possible to choose \( \theta \) in the unit disc, there is a bijective relation between \( \zeta \) and \( \theta \) defined by the equation \( \theta\theta^* = \zeta\theta^* + 1 \). The problem is thus reduced to finding the maximum of a function \( \Psi(\zeta) \) with respect to \( \zeta \). After simplifications, it can be shown that

\[
\Psi(\zeta) = 2 \sum_{j=2}^{3} \left| \frac{\zeta^j}{\left( \zeta^* + 1 \right)^2} \right|
\] (22)

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the summation being to take for \(0 \leq i + j \leq 4\), with coefficients defined below. Coefficients \(b_{ij}\) are defined as the complex conjugates of \(a_{ij}\):

\[
b_{33} = a_3^* + a_1^*
\]

(23)

\[
b_{23} = a_4 a_3^* - a_0 a_1
\]

(24)

\[
b_{13} = 4(a_0 a_3^* + a_1 a_3^* + a_2 a_3^* + 2a_3 a_0 + a_2 a_1)
\]

(25)

\[
b_{03} = 2(a_1 a_2^* + a_1 a_3^*) + 2a_3 a_2 (a_0 + a_2)
\]

(26)

\[
b_{20} = a_2 (a_2^* - a_1) + 2 a_3 (a_2^* - a_0)
\]

(27)

\[
b_{21} = 4 (a_0 a_2^* - a_0 a_1) + a_2 a_3^* + a_1 a_3^* + 2a_3 a_2 (a_0 + a_2)
\]

(28)

\[
b_{12} = 4 (a_0 a_2^* + a_1 a_3^*) + a_2 a_3^* + a_1 a_3^* + 2a_3 a_2 (a_0 + a_2)
\]

(29)

\[
b_{22} = 4 (a_0 a_2^* + a_1 a_3^*) + 4a_0 a_2 + a_2 a_3^* + 2a_3 a_2 (a_0 + a_2)
\]

(30)

\[
b_{11} = 4 (a_0 a_2^* + a_1 a_3^*) + 4a_0 a_2 + a_2 a_3^* + 2a_3 a_2 (a_0 + a_2)
\]

(31)

with

\[
a_2 = r_1 \bar{a}_2, \quad a_3 = r_1 \bar{a}_3
\]

(32)

The easiest way of finding the maximum of \(\psi(\xi)\) in the unit disc is to compute its value on a fixed spiral grid. This procedure is rather fast since there exist special computer routines dedicated to the calculation of polynomials at prescribed points [11]. But there are also other solutions, as is now commented.

In the real case, the stationary points of \(\psi(\xi)\) must just satisfy a polynomial equation, and the best value of \(\xi\) has just to be chosen among the roots of this polynomial, which turns out to be of degree four and can therefore be solved analytically. In other words, we have here a method that provides us with a set of potential stationary points. The value of the contrast has to be calculated at all the points obtained that fall in the unit disc to be able to pick up the solution \(\rho e^{i\theta}\) that corresponds to the absolute maximum.

We see that this second solution is not necessarily faster than the first. It will be more interesting regarding computational load if a high accuracy is required (in such a case, the first method may be too costly because the spiral grid would need to be very large). The second method may be seen as a means to compute a sort of 'pertinent grid' of values in the unit disc.

8.2 Complete CM algorithm

The CM algorithm consists of standardisation, followed by a series of sweeps (typically four or five, and in any case bounded by \(O(\sqrt{r})\)), in which \(r(\tau - 1)/2\) plane rotations are determined as described in the previous Section, and applied to the transformed data. This leads eventually to the following algorithm.

The CM algorithm receives a \(K \times T\) data matrix, \(Y\), and returns an output data matrix \(Z\), and an estimate of the mixing matrix \(M\), called \(F\).

1. Compute the SVD of \(Y\) as \(Y = PE\Omega^t\), where \(E\) is \(r \times r\) full rank.
2. Standardise the data matrix by retaining the first \(r\) rows of the SVD: \(E = \sqrt{\tau}(T)^{-1}\).
3. Initialise the matrix \(F = P\sqrt{(T)}\), and \(Z = E\).
4. Begin a loop on the sweeps: \(k = 1, \ldots, k \leq \sqrt{r} + 1\). Sweep the \(r(\tau - 1)/2\) pairs \((i, j)\) according to a fixed ordering, cyclic by rows for instance. For each pair, do:
   (i) Estimate the required cumulants of rows \(Z_i\) and \(Z_j\) of matrix \(Z\). Resort to \(\kappa\)-statistics for this purpose [17].
   (ii) Find the variable \(\xi\) that maximises the contrast in eqn. 22 and the corresponding tangent \(\theta\) in the unit disc. From eqn. 18 calculate the corresponding \(r \times r\) rotation matrix \(Q\) that acts in the plane \([i, j]\).
   (iii) Accumulate \(F = FQ^t\).
   (iv) Update the data matrix as \(Z = QZ\). Only rows \(i\) and \(j\) need to be modified.
5. End the loop if all estimated tangents have been very small (compared to \(1/T\)) within a sweep. Otherwise, stop anyway if the number of sweeps reaches \(1 + \sqrt{r}\).
6. Compute the norm of the columns of \(F\): \(||F||_\infty\).
7. Sort the entries of \(A\) in decreasing order, and permute the columns of \(F\) correspondingly.
8. Determine the phase of each column of \(F\) so that the entry of largest modulus is positive real.

The three last steps of the algorithm are not absolutely necessary, but permit the matrix \(F\) to be defined uniquely [5].