Abstract— The generalized monopulse procedure is a tool to estimate parameters by maximizing a beamforming-type objective function. It can be applied to angle estimation (classical monopulse), but also to space-time array processing of any dimension. The performance of this procedure is characterized in this paper by conditional means and covariances for all models of target fluctuation (Swerling 0 to 4). From these the performance of all kinds of parameter estimates with the generalized monopulse formula can be calculated. Applications are presented for a planar array with adaptive beamforming and for space-time adaptive processing for broadband interference suppression leading to some interesting conclusions.

Keywords- Direction finding, monopulse estimation, distribution of monopulse ratio

I. INTRODUCTION

Monopulse estimation has been generalized to arbitrary planar or volume arrays and multiple parameter estimation based on arbitrary beamforming, [1]. Measures of the performance of parameter estimates are often given by the Cramér-Rao bound (CRB). However, this is an asymptotic bound for high signal-to-noise ratio (SNR) or large antenna aperture for any asymptotically unbiased estimator. The challenging and interesting cases like low SNR and with jamming scenarios (leading to biased estimates) are not satisfactorily represented by this bound. The distribution of the monopulse estimates for deterministic and \( \chi^2 \)-distributed target amplitude including a detection threshold has been partly characterized in [2], [3]. A complete characterization in terms of bias and variance for all Swerling target models has been given recently in [6]. These results can be exploited in various ways: for designing arrays and subarrays, for system studies, to predict the effects of adaptive beamforming (ABF) and space-time adaptive processing (STAP), e.g. for airborne radar in air-to-ground mode.

In this paper we apply these results to investigate the bias correction performance and the sensitivity of the orientation and size of the error ellipsoids of parameter estimates with the generalized monopulse procedure for different target fluctuation models. For STAP applications we use the statistical characterization to check the trade-off between coherent and incoherent processing in the case of broadband jammer suppression by fast time STAP. The statistics may also be applied as a priori information in target tracking algorithms, [4], [5].

II. BEAMFORMING AND THE GENERALIZED MONOPULSE RATIO

The generalized monopulse procedure is used to determine the maximum of a function of the “beamforming type” \( \hat{\theta} = \arg \max_{\theta} |S(\theta)|^2 \) with \( S(\theta) = a(\theta)^H z \), with a complex data vector \( z \) and a vector \( a \) for beamforming, which may depend on unknown parameters \( \theta = (\theta_0, \theta_1, \ldots) \). Typically these are the target direction parameters (azimuth/ elevation). The parameter estimate with the generalized monopulse has the form [1]

\[
\hat{\theta} = \theta_0 - C \cdot (\text{Re} [R_x^{-1}] - \mu)
\]

with an initial estimate \( \theta_0 \), and with real correction quantities \( C, \mu, R_x \). \( R_x \) is the averaged monopulse ratio formed from a series of beam outputs \( B_k = (D_k^T, S_k) \) with \( D_k = D(t_k), S_k = S(t_k) \), measured at times \( t_k, k=1,\ldots,K \)

\[
R_x = \sum_{k=1}^K D_k^T S_k - \sum_{k=1}^K |S_k|^2
\]

where the overbar denotes complex-conjugate. Each beam is formed into direction \( \theta_0 \). The difference beams \( D_k = D_k(\theta_0) = (D(\theta_0, t_k), D(\theta_0, t_k), \ldots) \) represent estimates of the derivatives \( \partial S / \partial \theta_0, \partial S / \partial \theta_1, \ldots \). The key point is that this procedure can be applied to nearly any kind of beamforming (deterministic or adaptive).

If spatial beams are formed, a weighting \( a(u) = \left[ \exp \left( j 2 \pi f u / c \right) \right]_{u=1,N} \) is used, for antenna elements at positions \( r_n = (x_n, y_n, z_n)^T \), receiver centre frequency \( f \), and the velocity of light \( c \). The unit vector \( u = \left[ |u| = 1 \right] \) characterizes the direction of a plane wave impinging on the array.

If the data snapshot consists of time samples, the Fourier transform is a beam forming procedure with weighting \( a(f) = \left[ \exp \left( j 2 \pi f t_i \right) \right]_{t=1,N} \). Correspondingly, a frequency
monopulse estimator may be defined with corresponding difference beams.

For a time series of spatial snapshots the monopulse procedure can be applied by writing the data matrix $Z = \{z_1, \ldots, z_N\} \in \mathbb{C}^{N \times K}$ as a column vector $z_n = \text{vec}(Z) \in \mathbb{C}^{NK}$. Combined beamforming in space and time can then be written with a space-time beamforming vector $a_n(\theta) = a(u, f) = a(f) \otimes a(u)$. Time sampling can be here at Nyquist rate (samples in range or “fast time”) or from pulse to pulse (“slow time”); or the data may contain both times, then we have a data cube. Correspondingly we have to use difference beams in space and time.

For arrays with a large number of elements a reduction of the numerical expense is achieved by summing up subarrays. This can be written as $S(u) = m^H T u z$ with the digital weighting $m \in \mathbb{C}^L$ for the subarray outputs and a subarray forming matrix $T \in \mathbb{C}^{N \times L}$ for $L$ subarrays. The classical sidelobe canceller (SLC) for AFB is nothing else than a special subarray configuration. The difference beams are in this configuration formed digitally at subarray level. Subaraying is of particular interest for STAP. Several STAP configurations have been suggested to reduce the high dimensionality of the problem which can be interpreted as forming subarrays in space with a matrix $T_s$ and in time with $T_t$. The whole beamforming procedure with a data matrix $Z$ can then be written as $S = (m^H T^H) Z (T_T m_T)$, or using the rules for the Kronecker product as

$$S = \left( (T_T m_T)^H \otimes (m^H T^H) \right) \cdot \text{vec}(Z) \cdot z_n.$$

This shows that we have the same beamforming structure with a data snapshot. In this way the monopulse procedure can be applied for STAP GMTI angle and Doppler estimation. This has been done in [5].

If the mean and covariance of the monopulse ratio is known, the mean and covariance of the parameter estimates can be calculated by

$$E\{\theta\} = \mu_n - C \cdot E\{\text{Re}\{R\}\} - \mu$$

and

$$\text{cov}\{\theta\} = C \cdot \text{cov}\{\text{Re}\{R\}\} \cdot C^T.$$

Using $\text{Re}\{M\} \cdot \text{Re}\{M\}^T = \frac{1}{2} \left[ \text{Re}\{MM^H\} + \text{Re}\{MM^T\} \right]$ for any matrix $M$ we obtain

$$\text{cov}\{\theta\} = \frac{\mu}{2} \cdot C \cdot \text{Re}\{RR^H\} + \text{Re}\{RR^T\} \cdot C^T,$$

$$- C \cdot \text{Re}\{R\} \cdot E\{\text{Re}\{R\}\} \cdot C^T.$$

### III. CONDITIONAL MEAN AND VARIANCE OF THE MONOPULSE RATIO

In [6] the mean and covariance of the $K$ snapshot complex monopulse ratio $R$ based on $M$ difference beams and one sum beam was calculated. For the calculation we need only the first two moments of the generalized beam outputs $B_k = (D_k, S_k)^T$. For a target with parameters $\theta$ the (space/ time/ subarrayed) snapshot has the structure $z_n = b_n a(\theta) + n_n$, with $b_n$ the complex target amplitude, a plane wave model $a(\theta)$ and a noise/ interference component $n_n$. The vector of beam outputs has the structure $B_k = b_n + v_k$, with $\alpha = (\alpha_0, \alpha_1)^T$ and $\alpha_0 = (d_0^H a(\theta), \ldots, d_M^H a(\theta))$, $\alpha_1 = w^H a(\theta)$ and $d_0, \ldots, d_M$ the weight vectors for difference and sum beam forming and noise contribution $v_k = (d_1, \ldots, d_M, w)^H n_k$. Denote the mean of this beam output vector by

$$E\{B_k\} = t_n = \left( \begin{array}{c} t_0 \\ t_1 \\ \vdots \\ t_M \end{array} \right) = \left( \begin{array}{c} E\{D_k\} \\ E\{S_k\} \end{array} \right)$$

and the covariance matrix by

$$\text{cov}\{B\} = G = \left( \begin{array}{cc} G_0 & G_{0D} \\ G_{D0} & G_S \end{array} \right).$$

Covariance $G$ decomposes into $G = G_{\text{signal}} + G_s$. The noise/ interference contribution given by $v$ is assumed complex Gaussian independent identical distributed (i.i.d.) or short $v \sim CN\{0, G_s\}$. For the complex signal amplitude we assume distributions according to the Swerling cases. Detection is achieved if the average sum beam power exceeds a threshold, i.e. for the event $\Sigma = \{S = (S_1, \ldots, S_k) \mid \eta < \|S\|\}$. Denote the probability of this event (Detection Probability) by $P_\Sigma$. Using (4), (5) the knowledge of $E\{R \mid \Sigma\}$, $E\{R R^T \mid \Sigma\}$ and $E\{R R^H \mid \Sigma\}$ allows now a statistical description of the corresponding parameter estimates.

**Deterministic targets (SW0).** In this case the source amplitudes $b = (b_0, \ldots, b_k)^T$ are considered as a deterministic sequence. The beam output covariance matrix $G$ is then only determined by the noise (interference). Let $p = \frac{G_{0D}}{G_S}$, $\xi_0 = \alpha_0 / \alpha_1$ and $\xi_0^2 = 1 - b^H b$, then we have [6]

$$E\{R \mid \Sigma\} = p + \xi P_{11},$$

$$E\{R R^T \mid \Sigma\} = p p^T + \xi^2 P_{22} + \left[ \xi p^H + \xi p^T \right] P_{11},$$

$$E\{R R^H \mid \Sigma\} = p p^H + \xi^2 P_{22} + \left[ \xi p^H + \xi p^T \right] P_{11} + \xi^2 P_{11} + \left( G_0 / G_S - p p^H \right) P_{00}.$$
with the constants $P_0$, $P_{01}$, $P_{11}$, $P_{12}$, $P_{22}$ defined in [6] by integrals over modified Bessel functions which can be solved numerically.

Rayleigh-distributed targets (SW1 and SW2). Here we assume the source signal i.i.d. with $h \sim CN\left(0,\sigma_e^2\right)$. The beam output covariance matrix $G$ is then determined by both, the signal and the noise/interference contribution. Let $\rho = \frac{G_{00}}{G_s}$, $\zeta = a_0/\alpha_s$, then

$$E\left[R \mid \Sigma\right] = \rho$$
$$E\left[RR^T \mid \Sigma\right] = \rho \rho^T$$

with $G = \begin{bmatrix} \sigma_e^2 & \zeta \bar{z} + G_{s,0} & \sigma_e^2 & \zeta + G_{s,0} \\ \bar{z}^T & \sigma_e^2 & \zeta + G_{s,0} & \sigma_e^2 \end{bmatrix}$ and $P_{01}$ defined as in [6].

Targets with $\chi^2$-distribution (SW3 or SW4). We assume that the signal source power $|h|^2$ is central Chi-square-distributed with four degrees of freedom and $E\left|h^2\right| = \sigma_e^2$. Then [6]

$$E\left[R \mid \Sigma\right] = E_{\psi,z} + \gamma$$
$$E\left[RR^T \mid \Sigma\right] = E_{\psi,z} + \gamma^T + E_{\psi,z}^T + \gamma E_{\psi,z}$$

with $E_{\psi,z} = \varphi K \frac{A_1}{A_0} \tau$, $E_{\psi,z}^T = \varphi^* K (K-1) \frac{A_1}{A_0} \tau^T$

$$E_{\psi,z} = \left(\varphi^* K (K-1) \frac{A_1}{A_0} \varphi F_s \frac{A_1}{A_0} \right) \tau$$
$$+ \frac{A_1}{A_0} \left(\varphi G_{s,0} \frac{v_0}{F_s} \gamma^T + \frac{G_{s,0}}{F_s} \gamma^T\right)$$

where $G_{s,0} = E\left\{v_0 v_0^T\right\}$, $G_{s,s} = E\left\{v_s v_s^T\right\}$, $F_s = \frac{1}{2} \sigma_e^2 |\alpha_s|^2 + G_{s,s}$, $\varphi = \frac{1}{G_{s,s}} - \frac{1}{F_s}$, $\zeta = a_0/\alpha_s$

$\tau = \frac{1}{F_s} \left(G_{s,s} \varphi - E\left\{v_0 v_0^T\right\}\right)$, $\gamma = \zeta - \tau$. The constants $A_0$, $A_1$, $A_4$ can be determined from recursively defined polynomials given in [6].

IV. NUMERICAL EXAMPLES

We choose a planar array which exhibits correlations between the beams which is a key property of the derived new formulas. We consider a thinned array with 192 elements on a dish with radius $32\lambda$ grouped into 24 subarrays, each consisting of 8 elements. The elements of this thinned array are randomly distributed with parabolic density taper without amplitude tapering (thinning is down to 6% of the fully filled array). Correspondingly, we have a pattern with a narrow beamwidth (BW) of about $2^\circ$ and with sidelobes that are pseudo-random uniformly distributed at a level around -23 dB as seen in Figure 1. The difference beams are formed digitally with the subarray outputs by weighting with the $x$- and $y$-coordinates of the subarray centers.

Figure 2 shows the azimuth bias for a single target with SW0, SW1 and SW4 fluctuation for a single snapshot $(K = 1)$ over the true target direction varying in azimuth from -0.6 BW to 0.6 BW. The signal-to-noise ratio (SNR) is measured at the output of the sum beam and varies between 7 to 25 dB. The detection threshold $\eta$ is set at an SNR of 6 dB.

One can see that the bias correction inherent in the generalized monopulse formula is only effective for high SNR. This is consistent with the derivation of this formula where exactly this approximation was used. However, the sensitivity...
to this high SNR assumption is model dependent: With no randomness (SW0) the approximation is quite good; with higher randomness (SW4 to SW1) it degrades. Above 13 dB SNR the approximation is in all cases good and this is consistent with the usual rule of thumb for monopulse.

**Figure 2:** Azimuth monopulse characteristic for different target models and SNR, \( K=1 \)

Figure 3 shows the corresponding standard deviation (std) averaged over the azimuth and elevation components, \( \sigma_{\text{mean}} = \frac{1}{2}(\sigma_u + \sigma_v) \). At low SNR we find a non-uniform sensitivity against the target direction for the different target fluctuation models, for higher SNR the variance increases uniformly with higher randomness (SW4 to SW1).

**Figure 3:** Mean standard deviation of azimuth and elevation monopulse estimates over true target direction, \( K=1 \)

Figure 4 shows the dependence on the number of snapshots \( K \). The detection threshold is adjusted for different values of \( K \) to deliver the same probability of detection. The target position is assumed at an off-axis value of +0.5BW to obtain a situation with meaningful bias. We have plotted the length of the bias and the mean std over the number of snapshots \( K \). In most cases the bias does not vary with the number of snapshots (except for SW0 at low SNR). Note that a remarkable reduction of the variance can be achieved by fairly little averaging over 2 or 3 snapshots.

**Figure 4:** Length of bias and mean std over number of snapshots for target at -BW/2 off look direction.

**Application to ABF.** Figure 5 shows the performance with distorted beams due to adaptive beamforming (ABF). We selected a scenario with a jammer at \((u, v)=(-0.6\text{BW}, 0)\) with 27 dB jammer-to-noise ratio (JNR) and a target of the SW0 type. ABF has to be performed here for the sum and the azimuth and elevation difference beams. We have plotted the bias and std ellipses for target positions on a grid of 1/5 BW in the azimuth and elevation plane. We used the averaged adaptive monopulse ratio with averaging number \( K=3 \). Comparing these plots for different target fluctuation models one can observe that there is practically no dependence of the direction of the bias and the orientation of the std ellipses with respect to the signal model. The plots look nearly the same for the other models. The orientation of the bias and std ellipses depends primarily on the jammer position. This means that in tracking applications, where the shape of the error ellipses is used as a priori information, [4], no serious mismatch is introduced by using the wrong signal model.
Application to STAP. The generalized monopulse procedure can be applied to space-time data snapshots and STAP as explained in (3). Below we show an example of broadband jammer suppression by fast-time STAP with the same 192 element array, but with 10% relative bandwidth of the receiver. The antenna look direction is set to \((u_0, v_0) = (-0.68, 0)\), a broadband jammer is located at \(u_J = (u_0 - 0.8BW, v_0)\). For this jammer direction the dispersion between the elements due to the delays is quite relevant. With spatial-only adaptive processing no angle estimation is possible in this case, because the adaptive null becomes so broad that the beams virtually collapse. However, it is possible to achieve adaptive interference suppression already with one fast time tap. The different target fluctuation models are here assumed to be valid from space-time snapshot to space-time snapshot (typically from pulse to pulse / slow time). No target fluctuations are assumed in fast time. Note that STAP with \(K_t\) taps means \(K_t + 1\) fast-time samples, resulting in a length of the data snapshot vector of \((K_t + 1) \times 24\) and an interference covariance matrix of dimension \([(K_t + 1) \times 24] \times [(K_t + 1) \times 24]\). This is even for a small number of taps a considerable numerical effort. In Figure 6 we have plotted the azimuth bias and mean std for the SW0 and SW4 cases for 4 fast-time samples \((K_t = 3)\). The performance is similar to the narrowband case.

The variance can be reduced as well by using the averaged monopulse ratio. Averaging is here done over the slow time (averaging over \(K\)). This case is shown in Figure 7 for fast-time STAP with 1 tap. Here the variance is even lower than with 3-tap STAP, but this is achieved at a lower numerical expense. Slow-time averaged monopulse could therefore be an alternative to STAP with many taps. Of course, slow time averaging reduces only the variance.
CONCLUSIONS

We have demonstrated the usefulness of the derived mean and covariance of the averaged complex monopulse ratio with given detection threshold by examples with a thinned untapered array. These indicate the following properties of the generalized monopulse procedure:

- Bias correction is only effective for high SNR. The sensitivity against this effect is signal model dependent: With no randomness (SW0) bias correction is quite good; it degrades with higher randomness (SW4 to SW1). Similarly, the variance increases. Above an SNR value of 13 dB a low bias is obtained for all models. This is consistent with the usual rule of thumb for the required SNR for monopulse.

- For the multi-snapshot (averaged) monopulse the bias depends only little on the number of snapshots. A significant reduction of the variance can already be achieved by averaging over two or three snapshots.

- There is practically no dependence of the orientation of the standard deviation ellipses and the bias of the angle estimates with respect to the signal models. The orientation depends primarily of the jammer position in ABF applications. This means that in tracking applications, where the shape of the error ellipses is used as a priori information, no serious mismatch is introduced by using the wrong signal model.

- For STAP broadband jammer suppression applications slow-time averaged monopulse with few taps can be an alternative to STAP with many taps. This may result in a reduction of the dimension in STAP processing.

REFERENCES