Evolution semi-linear hyperbolic equations in a bounded domain

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Abstract. In this article, our goal is to prove the existence and uniqueness of solution for 1D and 2D semi-linear hyperbolic equations in a bounded domain with a monotone nonlinear term. We use elliptic regularization and a finite difference scheme in time to build the approximate solutions for the semi-linear hyperbolic equations, and we utilize the regularization method together with the monotonicity and convexity of the nonlinear term to show the existence of the resulting stationary problems. Finally, the existence of the solution for the evolution problem is done by studying the convergence of the approximate solutions and by using the standard Minty method, and the uniqueness is achieved.

1. Introduction

We aim to study in this article the initial and boundary value problem in dimension one and two for some first-order semi-linear reaction convection equation

\[ \partial_t u + Au + P(u) = f, \]

where \( Au = \partial_x u + \partial_y u \) and \( P : \mathbb{R} \to \mathbb{R} \) is a monotone function growing at the rate of a polynomial of odd order; a related problem was studied in space dimension one in [JP07] in the context of singular perturbation and boundary layer theory. The second-order semi-linear hyperbolic equations, in particular the non-linear wave equation, have been studied by many authors, see [Liu03, Sat68, Gla73] and references therein, where the authors considered the Dirichlet boundary conditions. In this article, we study the first-order semi-linear hyperbolic equation (1.1) and consider the dissipative boundary conditions in the sense of [BS07, Chapter 3] (see Remark 3.1), which is a first attempt in order to further study the first-order semi-linear hyperbolic systems. We are going to use the finite difference method in time to study the problem (1.1).

Finite difference discretization in time for approximating the solutions PDEs has been used since 1970’s to establish existence of their solutions, see e.g. [Tem77] or its latest edition [Tem01] for Navier Stokes equations. This method consists of discretizing the time domain
into subintervals and replacing the time derivative in the partial differential equation by a finite difference approximation. The time-dependent parabolic problem is thus transformed into a sequence of stationary problems which can be solved by Galerkin methods.

The finite difference discretization in time method applied to some hyperbolic PDEs appeared in the late 1970’s, see e.g. [Mar79, Kac84, Mun87]. The evolution problem is then transformed to a stationary problem but is still of hyperbolic type. In our case, the stationary problems are semi-linear hyperbolic with monotonic nonlinear term. To show the existence of the solutions to these nonlinear problems, we use the perturbation method, see [Lio73, Oma74]. To show the convergence of the approximate solutions, we first carry out a priori estimates to obtain some uniform bounds and hence the weak convergence of the approximate solutions, and then pass to the limit. To deal with the nonlinearity, we use fundamental results in the theory of monotone and accretive operators discovered during the latter half of the century by Minty [Min62, Min63], Browder [Bro65], Lions [Lio69], Brezis [Bre73] and many others. We also use a result in convex analysis, see Theorem B.1 from [ET76], to cope with non zero initial data and still be able to keep the monotonicity of the problem. Note that many stationary or time dependant problems of type (1.1) with a monotone nonlinearity are studied in the last references. However, a major difference with our study is that the operator $A + P$ is not coercive unlike in the previous references.

Our work is organized as follows. We first consider a one dimensional problem of the semi-linear hyperbolic equation in Section 2. In Section 3, we then use the result in Section 2 by treating the time variable as a spacial variable to construct the solutions for the stationary problem in dimension two, and then use the similar techniques as in Section 2 to obtain the existence and uniqueness of our problem.

2. A one dimensional problem

In this section, our aim is to prove the existence and uniqueness of solution of a semi-linear hyperbolic equation in space dimension one. We first study the stationary boundary value problem, and then use finite differences in time to study the evolution problem. The semi-linear hyperbolic equation that we consider reads

$$\frac{\partial u(x,t)}{\partial t} + \frac{\partial u(x,t)}{\partial x} + P(u(x,t)) = f(x,t),$$

(2.1)

where $x \in \Omega = (0,1)$, $t \in (0,T)$, $f \in L^{\frac{m+1}{m}}(0,T; L^{\frac{m+1}{m}}(\Omega))$ and $P : \mathbb{R} \to \mathbb{R}$ is monotone and has a polynomial growth at infinity in the following sense:

$$(P(u) - P(v))(u - v) \geq 0,$$

(2.2a)

$$\alpha_1 u^{m+1} - \beta_1 \leq P(u) u \leq \alpha_2 u^{m+1} + \beta_1,$$

(2.2b)

$$|P(u)| \leq \alpha_2 |u|^m + \beta_2,$$

(2.2c)

for some positive constants $\alpha_1, \alpha_2, \beta_1, \beta_2$ and an odd integer $m$.

An example of such a $P$ is an increasing polynomial of odd order $m = 2s-1$ with a positive leading coefficient $a_{2s-1} > 0$ of the form

$$P(u) = \sum_{j=0}^{2s-1} a_j u^j.$$ 

(2.3)
We first see that $P$ satisfies (2.2a) since $P$ is increasing, and the estimates in Lemma 2.1 below assert that $P$ also satisfies (2.2b) and (2.2c).

**Lemma 2.1.** Assume that $P$ is of the form (2.3) with the leading coefficient $a_{2s-1} > 0$. Then

\[
\begin{align*}
\frac{1}{2}a_{2s-1}u^{2s} - b_1 & \leq P(u)u \leq \frac{3}{2}a_{2s-1}u^{2s} + b_1 \quad (2.4a) \\
\frac{1}{2}a_{2s-1}u^{2s} - b_2|u| & \leq P(u)u \leq \frac{3}{2}a_{2s-1}u^{2s} + b_2|u| \quad (2.4b) \\
|P(u)| & \leq \frac{3}{2}a_{2s-1}|u|^{2s-1} + b_2, \quad (2.4c)
\end{align*}
\]

where $b_1$, $b_2$ depend only on the coefficients $a_j$’s and $s$.

**Proof.** Using Young’s inequality, we are able to find $b_1, \tilde{b}_2 > 0$ such that

\[
\begin{align*}
\left|\sum_{j=0}^{2s-2} a_j u^{j+1}\right| & \leq \frac{1}{2}a_{2s-1}u^{2s} + b_1, \\
\left|\sum_{j=0}^{2s-2} a_j u^{j+1}\right| & \leq |u| \sum_{j=0}^{2s-2} |a_j||u|^j \leq |u| \left(\frac{1}{2}a_{2s-1}|u|^{2s-1} + \tilde{b}_2\right).
\end{align*}
\]

These two inequalities imply (2.4a) and (2.4b), and (2.4c) is a consequence of (2.4b). □

We endow (2.1) with the following initial and boundary conditions

\[
\begin{align*}
u(t, 0) = g(t), \\
u(0, x) = u_0(x).
\end{align*}
\]

(2.5)

Our agenda is as follows; we first study in Section 2.1 the stationary problem corresponding to (2.1); see also (2.27). We then use this result to show the existence of solution to a finite difference scheme in time of (2.1), (2.27). The existence of solution of (2.1), (2.27) in $L^\infty(0, T; L^2(\Omega) \cap L^{m+1}(\Omega_T))$ is achieved by studying the convergence property of the finite difference solutions in Section 2.2. Since the solution is in the weak sense, we develop a trace theorem and an integration by parts result as tools to obtain the existence result. These results are presented in the Appendix A. The uniqueness of solution is done by the monotonicity of $P$ and an integration by parts argument.

Throughout this work, we will interchangeably use the notation $\Omega_T = \Omega \times (0, T)$ at convenience. We now study the stationary problem that will help us to build approximate solutions to the time dependent problem (2.1). We also set $V = L^{m+1}(\Omega)$ and $V' = L^{m+1'}(\Omega_T)$ to be dual spaces of each other. In the following, we denote by $\langle \cdot, \cdot \rangle$ the duality pairing between $L^r(\Omega)$ and $L^{r'}(\Omega)$, where $1 < r < +\infty$, and $r'$ is the conjugate exponents of $r$ satisfying that $1/r + 1/r' = 1$. We also use the notations $\| \cdot \|$ for the norm in $L^2(\Omega)$.

**2.1. The stationary boundary value problem.** In this subsection, we use a regularization and variational inequality method to study the following boundary value problem:

\[
\begin{align*}
\frac{\partial u}{\partial t} + P(u) + \lambda u = f, & \quad \text{in } \Omega, \\
u(0) = g_0.
\end{align*}
\]

(2.6)
for some constant $\lambda > 0$ and $g_0 \in \mathbb{R}$. The reason why we introduce an extra term $\lambda u$ in (2.6) is to prepare for using the finite difference scheme for the evolution problem. In order to show the existence, we introduce the following elliptic regularization problem

$$-\epsilon \partial_{xx}u^\epsilon + \partial_x u^\epsilon + P(u^\epsilon) + \lambda u^\epsilon = f, \quad \text{in } \Omega, \quad (2.7a)$$

$$u^\epsilon(0) = g_0, \quad u^\epsilon(1) = 0. \quad (2.7b)$$

We are going to apply Theorem B.2 to obtain a unique solution $u^\epsilon$ of (2.7a)-(2.7b). Notice that by Sobolev embedding, we have $H^1(\Omega) \subset C(\overline{\Omega})$. We set $X = H^1(\Omega)$ with the usual norm $\|u\|_{H^1}^2 = \|u\|^2 + \|\partial_x u\|^2$, and

$$\mathcal{K} = \{ u \in H^1(\Omega) : u(0) = g_0 \}.$$  

We then define the form $a_\epsilon(u,v)$, for all $u, v \in H^1(\Omega)$,

$$a_\epsilon(u,v) = \epsilon \int_\Omega \partial_x u \partial_x v \, dx + \int_\Omega u_x v \, dx + \int_\Omega P(u)v \, dx + \lambda \int_\Omega uv \, dx, \quad (2.8)$$

and the operator $A_\epsilon : X \to X'$, for all $u, v \in H^1(\Omega)$,

$$\langle A_\epsilon u, v \rangle = a_\epsilon(u,v).$$

By definition, it is easy to verify that $\mathcal{K}$ is a non-empty convex closed set $\{ u \equiv g_0 \in \mathcal{K} \}$ and that $A_\epsilon$ is weakly continuous over finite dimensional subspaces of $X$ and bounded. For $u, v \in \mathcal{K}$, we have

$$\langle A_\epsilon u - A_\epsilon v, u - v \rangle = \epsilon \int_\Omega (\partial_x u - \partial_x v)^2 \, dx + \int_\Omega (u - v \partial_x (u - v)) \, dx$$

$$+ \int_\Omega (P(u) - P(v)) (u - v) \, dx + \lambda \int_\Omega (u - v)^2 \, dx$$

$$\geq \int_\Omega (u - v \partial_x (u - v)) \, dx = \frac{1}{2}(u(1) - v(1))^2$$

$$\geq 0,$$

and the equality holds if and only if $u = v$. We thus deduce that $A_\epsilon$ is strictly monotone.

To show that $A_\epsilon$ is coercive, we first note that $u(x) = g_0 + \int_0^x \partial_x u(x') \, dx'$, which implies that

$$\|u\|^2 \leq 2g_0^2 + 2\|\partial_x u\|^2; \quad (2.10)$$

and then for $u \in \mathcal{K}$, integration by parts using (2.2b) and (2.10) yield

$$a_\epsilon(u,u) = \epsilon \int_\Omega (\partial_x u)^2 \, dx + \frac{1}{2}u(1)^2 - \frac{1}{2}u(0)^2 + \int_\Omega P(u)u \, dx + \lambda \int_\Omega u^2 \, dx$$

$$\geq \epsilon \int_\Omega (\partial_x u)^2 \, dx - \frac{1}{2}g_0^2 + \frac{1}{2}u(1)^2 + \alpha_1 \int_\Omega u^{m+1} \, dx - \beta_1 + \lambda \int_\Omega u^2 \, dx$$

$$\geq \frac{\epsilon}{4} \|u\|_{H^1}^2 - \frac{1 + \epsilon}{2}g_0^2 - \beta_1 + \frac{1}{2}u(1)^2 + \alpha_1 \int_\Omega u^{m+1} \, dx + \lambda \int_\Omega u^2 \, dx. \quad (2.11)$$

Hence, for $u \in \mathcal{K}$,

$$\frac{\langle A_\epsilon u, u \rangle}{\|u\|_{H^1}} = \frac{a_\epsilon(u,u)}{\|u\|_{H^1}} \geq \frac{\epsilon}{4} \|u\|_{H^1}^2 - \frac{(1 + \epsilon)g_0^2 + 2\beta_1}{2\|u\|_{H^1}^2} \to +\infty, \quad \text{as } \|u\|_{H^1} \to +\infty. \quad (2.12)$$
Applying Theorem B.2 to \( A_\epsilon \), we find that for any \( f \in L^{(m+1)/m}(\Omega) \subset X' = (H^1(\Omega))' \) there exists a unique \( u^\epsilon \in K \) such that

\[
\langle A_\epsilon u^\epsilon - f, v - u^\epsilon \rangle \geq 0, \quad \forall \ v \in K.
\] (2.13)

**Lemma 2.2.** Assume that \( f \in L^{(m+1)/m}(\Omega) \). Then there exists a unique \( u^\epsilon \) which satisfies (2.7a)-(2.7b), and the following energy estimates independent of \( \epsilon \) hold:

- \( u^\epsilon \) is bounded in \( L^{m+1}(\Omega) \),
- \( \sqrt{\epsilon \partial_x u^\epsilon} \) is bounded in \( L^2(\Omega) \),
- \( u^\epsilon \) is bounded in \( W^{1,(m+1)/m}(\Omega) \).

(2.14)

**Proof.** We already showed the existence of \( u^\epsilon \) satisfying (2.13), and we now prove that \( u^\epsilon \) satisfies (2.7a)-(2.7b). For any \( w \in C^\infty(\Omega) \) with \( w(0) = 0 \), choosing \( v = u^\epsilon \pm w \in K \) in (2.13) gives

\[
\langle A_\epsilon u^\epsilon - f, \pm w \rangle \geq 0,
\] (2.15)

which yields

\[
\langle A_\epsilon u^\epsilon, w \rangle = \langle f, w \rangle.
\] (2.16)

If we further assume that \( w \in C_c^\infty(\Omega) \), then integrating by parts in (2.16) gives

\[
\int_\Omega (-\epsilon \partial_{xx} u^\epsilon + \partial_x u^\epsilon + P(u^\epsilon) + \lambda u^\epsilon) w \, dx = \int_\Omega f w \, dx,
\] (2.17)

which implies that (2.7a) holds in the sense of distributions on \((0,1)\).

We have \( u^\epsilon(0) = g_0 \) since \( u^\epsilon \in K \), and we are left to recover the boundary condition \( u^\epsilon_x(1) = 0 \). From (2.7a), we obtain that \( \partial_{xx} u^\epsilon \) belongs to \( L^{(m+1)/m}(\Omega) \), which shows that \( u \) belongs to \( C^1(\overline{\Omega}) \) by the Sobolev embedding \( W^{2,(m+1)/m}(\Omega) \subset C^1(\overline{\Omega}) \). Hence the trace \( u^\epsilon_x |_{x=1} \) makes sense, and integrating by parts in (2.16) gives

\[
\int_\Omega (-\epsilon \partial_{xx} u^\epsilon + \partial_x u^\epsilon + P(u^\epsilon) + \lambda u^\epsilon) w \, dx + \epsilon u^\epsilon_x(1) w(1) = \int_\Omega f w \, dx.
\] (2.18)

Combining (2.17) and (2.18) leads to \( \epsilon u^\epsilon_x(1) w(1) = 0 \), which implies that \( u^\epsilon_x(1) = 0 \). Therefore, \( u^\epsilon \) satisfies (2.7a)-(2.7b).

We now turn to prove the energy estimates (2.14). Choosing \( v \equiv g_0 \in K \) in (2.13) yields

\[
\langle A_\epsilon u^\epsilon, u^\epsilon \rangle \leq \langle f, u^\epsilon \rangle + \langle A_\epsilon u^\epsilon - f, g_0 \rangle.
\] (2.19)

Using Hölder’s inequality and Young’s inequality, we obtain

\[
\langle f, u^\epsilon \rangle \leq \| f \|_{V'} \| u^\epsilon \|_V \leq c \| f \|_{V'}^{(m+1)/m} + \frac{\alpha_1}{4} \| u^\epsilon \|_V^{m+1};
\] (2.20)
and using (2.2c) for $P(u^\epsilon)$ and Young’s inequality for the term $P(u^\epsilon)g_0$, we find

$$
(A_u u^\epsilon - f, g_0) = \int_\Omega u^\epsilon g_0 \, dx + \int_\Omega P(u^\epsilon)g_0 \, dx + \lambda \int_\Omega u^\epsilon g_0 \, dx - \int_\Omega f g_0 \, dx
$$

$$
= g_0(u^\epsilon(1) - g_0) + \int_\Omega P(u^\epsilon)g_0 \, dx + \lambda \int_\Omega u^\epsilon g_0 \, dx - \int_\Omega f g_0 \, dx
$$

$$
\leq \frac{1}{2}(u^\epsilon(1))^2 - \frac{1}{2}g_0^2 + \int_\Omega \frac{\alpha_2}{4}(u^\epsilon)^{m+1} + c(g_0^{m+1} + g_0^{(m+1)/m}) \, dx
$$

$$
+ \lambda \int_\Omega \frac{1}{2}((u^\epsilon)^2 + g_0^2) \, dx + \int_\Omega \left( \frac{m}{m + 1} f^{(m+1)/m} + \frac{1}{m + 1} g_0^{(m+1)/m} \right) \, dx.
$$

(2.21)

for some constant $c > 0$.

Combining (2.19)-(2.21), we find with (2.11) that

$$
\frac{1}{4} \|\sqrt{\epsilon} \partial_x u^\epsilon\|^2 + \frac{\alpha_1}{2} \|u^\epsilon\|_V^{m+1} \leq c(\|f\|_V^{(m+1)/m} + g_0^{m+1} + g_0^{(m+1)/m}) + \frac{1}{2} \lambda g_0^2.
$$

Hence, we established the first two estimates in (2.14). The last estimate in (2.14) follows by applying Lemma A.1 with $\lambda(x) = 1$, $g^\epsilon = f - P(u^\epsilon) - \lambda u^\epsilon$, (2.7a) and the first estimate. \(\square\)

The first and last estimates in (2.14) show that there exists a subsequence of $u^\epsilon$, still denoted by $u^\epsilon$, which weakly converges to some $u$ in $L^{m+1}(\Omega)$ and in $W^{1,(m+1)/m}(\Omega)$. By the compact Sobolev embedding $W^{1,(m+1)/m} \subset C^{0,\alpha}(\Omega)$ valid for $0 < \alpha < 1/4$, we obtain that $u^\epsilon$ strongly converges to $u$ in $C(\overline{\Omega})$, which, together with the dominated convergence theorem, implies that $P(u^\epsilon)$ weakly converges to $P(u)$ in $L^{(m+1)/m}(\Omega)$. Therefore, passing to the limit in (2.14), we conclude that the limit $u$ solves (2.6) \(1\) (at least in the sense of distributions). It remains to verify that $u$ satisfies the boundary conditions (2.6)\(2\). Using the uniform boundedness of (2.14), (2.7a) gives

$$
- \epsilon \partial_{xx} u^\epsilon + \partial_x u^\epsilon \text{ is uniformly bounded in } L_x^{(m+1)/m}(0,1).
$$

(2.22)

Applying Lemma A.1 to $u^\epsilon$ with $p = (m+1)/m$ and $X = \mathbb{R}$, we see that the $\partial_x u^\epsilon$ are uniformly bounded in $L_x^{(m+1)/m}(0,1)$, and the corresponding traces converge, i.e. $u(0) = g_0$. We thus find a solution $u$ satisfying (2.6).

Therefore, we have the following result.

**Lemma 2.3.** Assume that $f \in L^{(m+1)/m}(\Omega)$ and $\lambda > 0$. Then there exists a unique $u \in L^{m+1}(\Omega) \cap W^{1,(m+1)/m}(\Omega)$ satisfying (2.6).

**Proof.** We only need to prove the uniqueness. Suppose that $u_1, u_2$ both satisfy (2.6) and belong to $L^{m+1}(\Omega) \cap W^{1,(m+1)/m}(\Omega)$, and let $w = u_1 - u_2$. Then $w$ satisfies

$$
w_x + P(u_1) - P(u_2) + \lambda w = 0,
$$

$$
w(0) = 0.
$$

(2.23)

Since $w \in L^{m+1}(\Omega)$ and $P(u_1), P(u_2) \in L^{(m+1)/m}(\Omega)$, $w_x \in L^{(m+1)/m}(\Omega)$, and it follows that $ww_x$ belongs to $L^1(\Omega)$ by Hölder’s inequality, and that

$$
\int_\Omega ww_x \, dx = \frac{1}{2} |w(1)|^2 - \frac{1}{2} |w(0)|^2 = \frac{1}{2} |w(1)|^2.
$$

(2.24)
We now multiply (2.23) by \( w \) and use (2.24); we find
\[
\frac{1}{2}w(1)^2 + \int_{\Omega} (P(u_1) - P(u_2))(u_1 - u_2) \, dx + \lambda \int_{\Omega} w^2 \, dx = 0. 
\tag{2.25}
\]
The left-hand side of (2.25) is always positive thanks to the monotonicity (2.2a) of \( P \). Hence, we must have \( w = 0 \) if (2.25) holds. We thus completed the proof. \( \square \)

**Remark 2.1.**

i) The same result would be true without uniqueness if \( \lambda = 0 \). Uniqueness can be recovered if \( \lambda = 0 \) and \( P \) is strictly monotone.

ii) With the same method we could replace (2.1) by
\[
a(x)\frac{\partial u}{\partial x} + P(u) + \lambda u = f, 
\tag{2.26}
\]
with \( a \in C^1(\Omega) \) such that \( a(0) \neq 0 \). The boundary conditions would be imposed at \( x = 0 \) and/or \( x = 1 \), depending on the sign of \( a(0) \) and \( a(1) \).

**2.2. The evolution problem.** In this section, we show the existence and uniqueness of solution to the following semi-linear hyperbolic equation
\[
\begin{array}{l}
\partial_t u + \partial_x u + P(u) = f, \text{ in } \Omega \times (0, T), \\
u(0, t) = g(t), \\
u(x, 0) = u_0(x), 
\end{array} 
\tag{2.27}
\]
where \( P : \mathbb{R} \to \mathbb{R} \) is as above, that is \( P \) satisfies the assumptions (2.2a),(2.2b) and (2.2c). We suppose that \( f \in L^{\frac{m+1}{m}}(0, T; L^{\frac{m+1}{m}}(\Omega)) \), \( g \in L^2(0, T) \) and \( u_0 \in L^2(\Omega) \). The implicit Euler finite difference scheme for (2.27) is set as follows: we let \( \Delta t = T/N \), \( N \in \mathbb{N} \) and we set \( u^0 = u_0 \), and then construct the \( u^n \) for \( n = 1, \cdots, N \) by induction solving:
\[
\begin{aligned}
\frac{u^n - u^{n-1}}{\Delta t} + \partial_x u^n + P(u^n) &= f^n, \\
u^n(0) &= g^n, 
\end{aligned} 
\tag{2.28a}
\]
where
\[
f^n = \frac{1}{\Delta t} \int_{(n-1)\Delta t}^{n\Delta t} f(x, t) \, dt, \\
g^n = \frac{1}{\Delta t} \int_{(n-1)\Delta t}^{n\Delta t} g(t) \, dt. 
\tag{2.28b}
\]
Given \( u^{n-1} \in L^{\frac{m+1}{m}}(\Omega) \), the existence of a unique solution \( u^n \in W^{1,\frac{m+1}{m}}(\Omega) \) of (2.27) is guaranteed by Lemma 2.3. We now need to prove some estimates independent of \( N \) in order to pass to the limit \( k = 1/N \to 0 \).

**Remark 2.2.** Regarding the boundary value problem (2.26) in Remark 2.1 ii), we could also study the following evolution problem
\[
\begin{array}{l}
\partial_t u + a(x, t)\partial_x u + P(u) + \lambda u = f, \\
u(x, 0) = u_0(x), 
\end{array} 
\tag{2.29}
\]
where \( a \in C^1(\Omega_T) \) with \( a(0, t)a(1, t) \neq 0, \forall t \in [0, T] \). The boundary conditions would be imposed at \( x = 0 \) and/or \( x = 1 \), depending on the sign of \( a(0, t) \) and \( a(1, t) \).
2.2.1. *A priori estimates.* Taking the inner product of each side of (2.28a) with $2u^n \Delta t$, we find that
\[ 2\langle u^n - u^{n-1}, u^n \rangle + 2\Delta t \langle \partial_x u^n, u^n \rangle + 2\Delta t \langle P(u^n), u^n \rangle = 2\Delta t \langle f^n, u^n \rangle. \]
Using the identity
\[ 2\langle u^n - u^{n-1}, u^n \rangle = \|u^n - u^{n-1}\|^2 + \|u^n\|^2 - \|u^{n-1}\|^2, \]
the equation becomes
\[ \|u^n - u^{n-1}\|^2 + \|u^n\|^2 - \|u^{n-1}\|^2 + 2\Delta t \langle \partial_x u^n, u^n \rangle + 2\Delta t \langle P(u^n), u^n \rangle = 2\Delta t \langle f^n, u^n \rangle. \]
Using the Cauchy-Schwarz inequality and the Young inequality for the right hand side, we obtain, recalling that $V = L^{m+1}(\Omega)$:
\[ \|u^n - u^{n-1}\|^2 + \|u^n\|^2 - \|u^{n-1}\|^2 \leq 2\Delta t \|f^n\|_{V'} \|u^n\|_V \leq c_m \Delta t \|f^n\|_{V'}^{m+1} + \alpha_1 \Delta t \|u^n\|_{m+1}', \]
where $c_m := 2 \sqrt{\alpha m/(m+1)} \sqrt{\alpha_1 m}$. From the assumption (2.2b), we find that $\langle P(u^n), u^n \rangle \geq \alpha_1 \|u^n\|_{m+1}' - \beta_1$. This implies
\[ \|u^n - u^{n-1}\|^2 + \|u^n\|^2 - \|u^{n-1}\|^2 + \Delta t ((u^n(1))^2 - (g^n)^2) + \alpha_1 \Delta t \|u^n\|_{m+1}' \leq c_m \Delta t \|f^n\|_{V'}^{m+1} + 2\beta_1 \Delta t. \]
We sum the above inequalities for $n = 1, \ldots, n_0$ and obtain:
\[ \sum_{n=1}^{n_0} \|u^n - u^{n-1}\|^2 + \|u^n\|^2 + \Delta t \sum_{n=1}^{n_0} (u^n(1))^2 + \alpha_1 \Delta t \sum_{n=1}^{n_0} \|u^n\|_{m+1}' \leq c_m \Delta t \sum_{n=1}^{N} \|f^n\|_{V'}^{m+1} + \Delta t \sum_{n=1}^{N} (g^n)^2 + \|u_0\|^2 + 2\beta_1 T. \]
We have by (2.28b)
\[ \Delta t \sum_{n=1}^{N} \|f^n\|_{V'}^{m+1} \leq \|f\|_{L^{m+1}(0,T;V')}^{m+1}, \quad \Delta t \sum_{n=1}^{N} (g^n)^2 \leq \|g\|_{L^2(0,T)}^2, \]
and we conclude that
\[ \sum_{n=1}^{n_0} \|u^n - u^{n-1}\|^2 + \|u^n\|^2 + \Delta t \sum_{n=1}^{n_0} (u^n(1))^2 + \alpha_1 \Delta t \sum_{n=1}^{n_0} \|u^n\|_{m+1}' \leq K_0 = K_0(P,T,f,g) := c_m \|f\|_{L^{m+1}(0,T;V')}^{m+1} + \|g\|_{L^2(0,T)}^2 + \|u_0\|^2 + 2\beta_1 T, \]
for all $1 \leq n_0 \leq N$.
We have proven the following bounds:
Lemma 2.4.

\[
\begin{align*}
\|u^n\|^2 & \leq K_0 \text{ for all } 1 \leq n \leq N, \\
\Delta t \sum_{n=1}^{n_0} \|u^n\|_{V'}^{m+1} & \leq \frac{K_0}{\alpha_1}, \\
\Delta t \sum_{n=1}^{n_0} (u^n(1))^2 & \leq K_0, \\
\sum_{n=1}^{N} \|u^n - u^{n-1}\|^2 & \leq K_0,
\end{align*}
\]

where \(K_0\) is defined in (2.31), and depends only on the data.

2.2.2. Passage to the limit. We now introduce two approximate solutions to (2.1), which are denoted by \(u_k\) and \(\tilde{u}_k\) with \(k = \Delta t\): for each \(t \in I_n : = ((n-1)\Delta t, n\Delta t]\), \(n = 1, \ldots, N\), we set

\[
\begin{align*}
\begin{cases}
\partial_t \tilde{u}_k + \partial_x u_k + P(u_k) = f_k, & \text{in } \Omega \times (0, T), \\
\tilde{u}_k(x, 0) = u_0(x), \\
\tilde{u}_k(0, t) = g_k(t),
\end{cases}
\end{align*}
\]

that is, \(u_k\) is the step function on the interval \((0, T)\) with values taken from the right of each interval \(I_n\), and \(\tilde{u}_k\) is the piecewise linear function that interpolates \(u^{n-1}\) and \(u^n\) on \(I_n\). We first have an equivalent form of the scheme (2.28a) as follows

\[
\begin{align*}
\Delta t \|\partial_t \tilde{u}_k\|_{L^2(0, T; L^2(\Omega))}^2 + \|u_k(\cdot, t)\|_{L^2(0, T)}^2 + \|u^n\|_{x=1}^2_{L^2[0, T]} + \alpha_1\|u_k\|_{L^{m+1}(0, T, V)}^{m+1} \leq K_0(f, g)
\end{align*}
\]

for all \(0 \leq t \leq T\).

Using (2.4c) and \(\|u_k\|_{L^{m+1}(\Omega)}\) being bounded, we find that \(\|P(u_k)\|_{L^{m+1}(\Omega)}\) is also bounded. Therefore, as \(k \to 0\)

\[
\begin{align*}
\begin{cases}
u_k, \tilde{u}_k & \text{is bounded set of } L^\infty(0, T; L^2(\Omega)), \\
u_k & \text{is bounded set of } L^{m+1}(0, T; V) = L^{m+1}(\Omega), \\
P(u_k) & \text{is bounded set of } L^{m+1}(0, T; V'), \\
u_k(1) & \text{is bounded set of } L^2(0, T), \\
\sqrt{\Delta t} \partial_t \tilde{u}_k & \text{is bounded set of } L^2(0, T; L^2(\Omega)).
\end{cases}
\end{align*}
\]

As a consequence of the estimates (2.36), we find from equation (2.34) that

\[
\partial_t \tilde{u}_k + \partial_x u_k \in \text{bounded set of } L^{m+1}(0, T; V').
\]
We now have an estimate of the distance between the two functions \( u_k \) and \( \tilde{u}_k \): direct computations show that
\[
\int_{(n-1)\Delta t}^{n\Delta t} \| \tilde{u}_k - u_k \|^2 dt = \int_{(n-1)\Delta t}^{n\Delta t} \left( \frac{u^n - u^{n-1}}{\Delta t} \right)^2 dt = \frac{1}{3} \frac{u^n - u^{n-1}}{\Delta t}^2 \Delta t^3.
\]
This implies
\[
\| \tilde{u}_k - u_k \|_{L^2(\Omega_T)} = \sqrt{\frac{3}{\Delta t}} \| \partial_t \tilde{u}_k \|_{L^2(\Omega_T)} \Delta t.
\]  
(2.38)

From (2.38) and (2.36)_5, we find
\[
\| \tilde{u}_k - u_k \|_{L^2(0,T;L^2(\Omega_T))} = \mathcal{O}(\Delta t^{\frac{1}{2}}).
\]  
(2.39)

By (2.36) and (2.39), there exist subsequence of \( u_k \) and \( \tilde{u}_k \) which we still denote by \( k \), and \( u \in L^\infty(0,T;L^2(\Omega)) \cap L^{m+1}(\Omega_T) \), \( \chi \in L^{m+1}(0,T;V') \), \( \kappa \in L^2(0,T) \) such that for \( k \to 0 \)
\[
\begin{cases}
  u_k, \tilde{u}_k \rightharpoonup u \text{ weak-* in } L^\infty(0,T;L^2(\Omega)) \text{ and weakly in } L^{m+1}(\Omega_T), \\
  P(u_k) \rightharpoonup \chi \text{ weakly in } L^{\frac{m+1}{m}}(0,T;V'), \\
  u_k(1) \rightharpoonup \kappa \text{ weakly in } L^2(0,T).
\end{cases}
\]  
(2.40)

From (2.37), we are also able to extract a subsequence still denoted by \( k \) such that \( \partial_t \tilde{u}_k + \partial_x u_k \) weakly converges to some \( \xi \) in \( L^{m+1}(0,T;V') \). From (2.40)_1, we infer that \( \partial_t \tilde{u}_k, \partial_x u_k \) converge to \( \partial_t u, \partial_x u \) in the sense of distributions respectively. We thus have that \( \xi = \partial_t u + \partial_x u \), and that
\[
\partial_t \tilde{u}_k + \partial_x u_k \rightharpoonup \partial_t u + \partial_x u \text{ weakly in } L^{m+1}(0,T;V').
\]  
(2.41)

We have \( u \in L^\infty(0,T;L^2(\Omega)) \cap L^{m+1}(\Omega_T), \) \( \partial_t u + \partial_x u \in L^{m+1}(0,T;V') = L^{\frac{m+1}{m}}(\Omega_T) \). Thanks to Lemma A.2 with \( U = \Omega_T, a = b = 1, p = (m+1)/m, \) the traces \( u(0,t) \) and \( u(x,0) \) make sense.

We now denote by \( \tilde{u}_k, \tilde{u}_k, \tilde{f}_k \) the extensions of the functions \( u_k, \tilde{u}_k, f_k \) by zero to the domain \((0,1) \times (-\infty, T)\); \( \tilde{u}_0 \) is the extension of \( u_0 \) to \((0,1) \times (-\infty, T)\) and \( \tilde{g}_k \) is the extension of \( g_k \) to \((0,1) \times (-\infty, T)\). We infer from (2.40)_2, (2.41) and (2.34)_2,3 that
\[
\partial_t \tilde{u}_k(x,t) - \delta_0(t) \tilde{u}_0(x) + \partial_t \tilde{u}_0(x,t) - \delta_0(x) \tilde{g}_k(t) + P(\tilde{u}_k(x,t)) = \tilde{f}_k(x,t)
\]

where \( \delta_0 \) is the dirac delta function having mass at 0. Passing to the limit as \( k \to 0 \), we find in the sense of distributions
\[
\partial_t \tilde{u}(x,t) - \delta_0(t) \tilde{u}_0(x) + \partial_t \tilde{u}(x,t) - \delta_0(x) \tilde{g}(t) + \tilde{\chi}(x,t) = \tilde{f}(x,t)
\]

where \( \tilde{u}, \tilde{\chi}, \tilde{f} \) are the extensions of \( u, \chi, f \) on \((0,1) \times (-\infty, T)\) and \( \tilde{g} \) is the extension of \( g \) to \((-\infty, T)\). This equation and \( \partial_t u + \partial_x u + \chi = f \) on \( \Omega_T \) imply
\[
\begin{cases}
  \partial_t u + \partial_x u + \chi = f, \\
  u(x,0) = u_0(x), \\
  u(0,t) = g(t).
\end{cases}
\]  
(2.42)

**Remark 2.3.** To recover the initial condition and boundary condition for (2.42), we can also take the inner product of (2.34) with \( \varphi(x) \psi(t) \) for \( \varphi(x) \in \mathcal{D}(\Omega), \psi(t) \in \mathcal{D}([0,T]) \), \( \varphi(1) = \psi(T) = 0 \) and carry out the integration by parts to obtain
\[
-\langle \tilde{u}_k, \varphi \psi_k \rangle_{L^2(\Omega)} - \langle u_0, \varphi \psi_0 \rangle_{L^2(\Omega)} - \langle g_k, \varphi(0) \psi \rangle_{L^2(0,T)} + \langle P(u_k), \varphi \psi \rangle = \langle f_k, \varphi \psi \rangle.
\]
We then let \( k \) run to zero and find that
\[
-\langle u, \varphi \psi \rangle - \langle u_0, \varphi \psi(0) \rangle_{L^2_0(\Omega)} - \langle u, \varphi_x \psi \rangle - \langle g, \varphi(0) \psi \rangle_{L^2_T(0,T)} + \langle \chi, \varphi \psi \rangle = \langle f, \varphi \psi \rangle.
\]
This also implies (2.42). \( \square \)

There remains to show that \( \chi = P(u) \). For that purpose, let \( v \in L^{m+1}(0,T;V) \) so that \( P(v) \in L^{\frac{m+1}{m}}(0,T;V') \); we consider
\[
X_k = \int_0^T \langle P(u_k) - P(v), u_k - v \rangle dt.
\]
We first see that \( X_k \geq 0 \) thanks to the monotonicity (2.2a) of \( P \), and then write
\[
X_k = X_k^1 + X_k^2 + X_k^3,
\]
where
\[
X_k^1 = \int_0^T \langle P(u_k), u_k \rangle dt, \quad X_k^2 = \int_0^T \langle P(u_k), v \rangle dt, \quad X_k^3 = \int_0^T \langle P(v), (u_k - v) \rangle dt.
\]
The weak convergence in (2.40) immediately shows that as \( k \to 0 \):
\[
X_k^2 \to \int_0^T \langle \chi, v \rangle dt, \quad \text{and} \quad X_k^3 \to \int_0^T \langle P(v), (u - v) \rangle dt.
\]
For \( X_k^1 \), we use the scheme (2.34) and write
\[
X_k^1 = \int_0^T \langle f_k, u_k \rangle dt - \int_0^T \langle \partial_t \tilde{u}_k, u_k \rangle dt - \int_0^T \langle \partial_x u_k, u_k \rangle dt
\]
\[
= \int_0^T \langle f_k, u_k \rangle dt - \frac{1}{2} \sum_{n=1}^N \| u^n - u^{n-1} \|^2 - \frac{1}{2} \| u^0 \|^2 + \frac{1}{2} \| u_0 \|^2
\]
\[
- \frac{1}{2} \| u_k \|_{x=1}^2 L^2(0,T) + \frac{1}{2} \| g_k \|_{L^2(0,T)}^2
\]
\[
\leq \int_0^T \langle f_k, u_k \rangle dt - \frac{1}{2} \| u_k \|_{x=1}^2 L^2(0,T) + \frac{1}{2} \| u_0 \|^2 - \frac{1}{2} \| u_{x=1}^2 L^2(0,T) + \frac{1}{2} \| g_k \|_{L^2(0,T)}^2.
\]
From Lemma A.3 and since \( u_0 \in L^2(\Omega) \) and \( g \in L^2(0,T) \), we know that the traces of \( u \) at \( x = 0,1 \) and \( t = 0,T \) are \( L^2 \)-functions. We also have that \( \liminf_{k \to 0} \| u_k \|_{x=T} \geq \| u \|_{x=T} \),
\[
\liminf_{k \to 0} \| u_k \|_{x=1}^2 L^2(0,T) \geq \| u_{x=1}^2 L^2(0,T), \quad \int_0^T \langle f_k, u_k \rangle dt \to \int_0^T \langle f, u \rangle dt, \quad \text{and} \quad \| g_k \|_{L^2(0,T)} \to \| g \|_{L^2(0,T),}
\]
which together yield
\[
\limsup_{k \to 0} X_k^1 \leq \int_0^T \langle f, u \rangle dt - \frac{1}{2} \| u \|_{x=T}^2 + \frac{1}{2} \| u_0 \|^2 - \frac{1}{2} \| u_{x=1}^2 L^2(0,T) + \frac{1}{2} \| g \|_{L^2(0,T)}^2.
\]
(2.43)
Since we have \( u \in L^{m+1}(\Omega_T) \) and \( \partial_t u + \partial_x u \in L^{\frac{m+1}{m}}(\Omega_T) \), applying Lemma A.3 to \( u \) with \( y = t \), \( \mathcal{U} = \Omega_T \), \( p = m+1 \) and \( q = \frac{m+1}{m} \) and using (2.42), we obtain that
\[
\frac{1}{2} \left( \|u|_{t=T}\|^2 - \|u_0\|^2 + \|u|_{x=1}\|^2_{L^2(0,T)} - \|g\|^2_{L^2(0,T)} \right) = \int_{\Omega_T} \langle u, u_t + u_x \rangle \, dx \, dt
\]
\[
= \int_{\Omega_T} \langle u, f - \chi \rangle \, dx \, dt. \tag{2.44}
\]
Combining (2.43) and (2.44), we find that
\[
\limsup_{k \to 0} X^1_k \leq \int_0^T \langle \chi, u \rangle \, dt.
\]
We finally conclude that
\[
\limsup_{k \to 0} X_k \leq \int_0^T \langle \chi, u \rangle \, dt - \int_0^T \langle \chi, v \rangle \, dt - \int_0^T \langle P(v), (u - v) \rangle \, dt = \int_0^T \langle \chi - P(v), (u - v) \rangle \, dt,
\]
which, together with \( X_k \geq 0 \), implies that for all \( v \in L^{m+1}(0,T; V) \),
\[
\int_0^T \langle \chi - P(v), (u - v) \rangle \, dt \geq 0.
\]
The standard Minty method (see [Lio69]) then yields \( \chi = P(u) \).

2.2.3. Main result of Section 2.

**Theorem 2.1.** Assume that \( f \in L^{\frac{m+1}{m}}(0,T; L^{\frac{m+1}{m}}(\Omega)), g \in L^2(0,T) \) and \( u_0 \in L^2(\Omega) \) are given. Then there exists a unique solution \( u = u(t,x) \in L^{\infty}(0,T; L^2(\Omega)) \cap L^{m+1}(0,T; L^{m+1}(\Omega)) \) satisfying (2.1) and (2.5).

**Proof.** The existence of a solution \( u \) was proved in Section 2.2.1, and we thus only need to show the uniqueness. Let \( u_1, u_2 \) belong to \( L^{\infty}(0,T; L^2(\Omega)) \cap L^{m+1}(0,T; L^{m+1}(\Omega)) \) and both satisfy (2.1)-(2.5). Set \( w = u_1 - u_2 \); we then see that \( w \) satisfies
\[
\begin{cases}
  w_t + w_x = P(u_2) - P(u_1), \\
  w(t,0) = 0, \\
  w(0,x) = 0.
\end{cases} \tag{2.45}
\]

Since \( u_1, u_2 \in L^{m+1}(\Omega_T) \), we find from (2.45)\_1 that \( w_t + w_x \) belongs to \( L^{(m+1)/m}(\Omega_T) \) by assumption (2.2b). Applying Lemma A.3 to \( w \) with \( y = t \), \( \mathcal{U} = \Omega_t := \Omega \times (0,t) \), \( p = m+1 \) and \( q = (m + 1)/m \) and using (2.45)\_2, we obtain
\[
\int_0^t w(1,t')^2 \, dt' + \int_{\Omega_t} w(x,t)^2 \, dx = \int_{\Omega_t} w(w_t + w_x) \, dx \, dt', \quad \forall t \in [0,T],
\]
which, together with (2.45)\_1, implies that
\[
\int_{\Omega_t} w(x,t)^2 \, dx + \int_{\Omega_t} (u_1 - u_2)(P(u_1) - P(u_2)) \, dx \, dt' \leq 0, \quad \forall t \in [0,T]. \tag{2.46}
\]
Therefore, we can conclude that \( u_1(\cdot,t) = u_2(\cdot,t) \) for all \( t \in [0,T] \) and hence \( u_1 = u_2 \), and we thus completed the proof. □
3. A two dimensional problem

In this section, we consider the rectangular domain \( \Omega = (0,1)^2 \) in \( \mathbb{R}^2 \) and our aim is to show the existence and uniqueness of solution to the following semi-linear hyperbolic equation:

\[
\begin{cases}
\partial_t u + Au + P(u) = f, \text{ in } \Omega \times (0,T), \\
u(0,0,t) = g_1(y,t), u(x,0,t) = g_2(x,t) \\
u(x,y,0) = u_0(x,y),
\end{cases}
\tag{3.1}
\]

where \( Au = \partial_x u + \partial_y u \) and \( P : \mathbb{R} \rightarrow \mathbb{R} \) is as in Section 2 satisfying \((2.2a)-(2.2c)\). We suppose that \( f \in L^{m+1}_m(0,T;L^{m+1}_m(\Omega)), g_1 \in L^2(0,T;L^2_y(0,1)), g_2 \in L^2(0,T;L^2_x(0,1)) \) and \( u_0 \in L^2(\Omega) \).

The standard finite difference scheme of \((3.1)\) is set as follows: Let \( \Delta t = T/N; \) we set \( u^0 = u_0, \) and construct the \( u^n \) for \( n = 1, \cdots, N \) recursively by setting

\[
\begin{cases}
u^n - \frac{u^{n-1}}{\Delta t} + Au^n + P(u^n) = f^n, \\
u^n(0,y) = g^n_1(y), u^n(x,0) = g^n_2(x),
\end{cases}
\tag{3.2a}
\]

where

\[
f^n = \frac{1}{\Delta t} \int_{(n-1)\Delta t}^{n\Delta t} f(x,y,t) dt, \quad g^n_1(y) = \frac{1}{\Delta t} \int_{(n-1)\Delta t}^{n\Delta t} g_1(y,t) dt, \quad g^n_2(x) = \frac{1}{\Delta t} \int_{(n-1)\Delta t}^{n\Delta t} g_2(x,t) dt.
\tag{3.2b}
\]

We rewrite the equation \((3.2a)_1\) as

\[
\begin{cases}
\partial_t u^n + \partial_x u^n + P(u^n) + \frac{1}{\Delta t} u^n = f^n + \frac{1}{\Delta t} u^{n-1}, \\
u^n(0,y) = g^n_1(y), u^n(x,0) = g^n_2(x).
\end{cases}
\tag{3.3}
\]

Given \( u^{n-1} \in L^{m+1}_m(\Omega), \) the existence and uniqueness of \( u^n \) in \( L^{m+1}_m(\Omega) \) is guaranteed by applying Theorem 2.1 with \( t = y \) and \( P(u) = P(u) + \frac{1}{\Delta t} u \) and noting that \( P(u) + \frac{1}{\Delta t} u \) is strictly monotone thanks to \( \lambda = \frac{1}{\Delta t} > 0. \)

3.1. A priori estimates. We still use the previous notations, i.e. \( V = L^{m+1}_m(\Omega) \) and \( V' = L^{m+1}_m(\Omega) \) to be dual spaces of each other; \( \langle \cdot, \cdot \rangle \) is the duality pairing between \( L^r(\Omega) \) and \( L^{r'}(\Omega), \) for \( 1/r + 1/r' = 1 \) and \( \| \cdot \| \) for the norm in \( L^2(\Omega). \)

Taking the inner product of each side of \((3.2a)\) with \( 2u^n \Delta t, \) and using the identity \((2.30),\) we find that

\[
\|u^n - u^{n-1}\|^2 + \|u^n\|^2 - 2 \|u^{n-1}\|^2 + 2 \langle Au^n, u^{n}\rangle \Delta t + 2 \langle P(u^n), u^{n}\rangle \Delta t = 2 \langle f^n, u^{n}\rangle \Delta t.
\]

Using the Cauchy-Schwarz inequality and the Young inequality for the right hand side, we find that

\[
\|u^n - u^{n-1}\|^2 + \|u^n\|^2 - 2 \|u^{n-1}\|^2 + 2 \langle Au^n, u^{n}\rangle \Delta t + 2 \langle P(u^n), u^{n}\rangle \Delta t \\
\leq 2 \|f^n\|_{V'} \|u^n\|_V \Delta t \leq c_m \|f^n\|_{V'} \frac{\delta}{\Delta t} \Delta t + \alpha_1 \|u^n\|_V \frac{\delta}{\Delta t} \Delta t,
\]

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where \( c_m := \frac{2 \sqrt{m}}{m+1} \sqrt{\alpha_1} \). We know from (2.2b) that \( \langle P(u^n), u^n \rangle \geq \alpha_1 \|u^n\|^{m+1} - \beta_1 \) and this implies
\[
\|u^n - u^{n-1}\|^2 + \|u^n\|^2 - \|u^{n-1}\|^2 + 2 \Delta t \langle Au^n, u^n \rangle + \alpha_1 \Delta t \|u^n\|^{m+1} \leq c_m \Delta t \|f^n\|_{V^n}^{m+1} + 2 \beta_1 \Delta t.
\]

Since \( g^n_1 \in L^2_y(0,1) \), \( g^n_2 \in L^2_x(0,1) \), and \( Au^n \in L^\frac{m+1}{m}(\Omega_T) \), we apply Lemma A.4 with \( U = \Omega \) and \( a = b = c = 1 \) to obtain
\[
2 \langle Au^n, u^n \rangle = 2(\partial_x u^n + \partial_y u^n, u^n) = \|u^n\|_{x=1}^2 L^2_{\dot{y}}(0,1) - \|g^n_1(y)^2\|_{L^2_{\dot{y}}(0,1)}^2 + \|u^n\|_{y=1}^2 L^2_x(0,1) - \|g^n_2(x)^2\|_{L^2_x(0,1)}^2.
\]

Therefore
\[
\|u^n - u^{n-1}\|^2 + \|u^n\|^2 - \|u^{n-1}\|^2 + \Delta t \langle \|u^n\|_{x=1}^2 L^2_{\dot{y}}(0,1) + \|u^n\|_{y=1}^2 L^2_x(0,1) \rangle + \alpha_1 \Delta t \|u^n\|^{m+1} \leq c_m \Delta t \|f^n\|_{V^n}^{m+1} + 2 \beta_1 \Delta t + \Delta t \|g^n_1(y)^2\|_{L^2_{\dot{y}}(0,1)} + \Delta t \|g^n_2(x)^2\|_{L^2_x(0,1)}^2.
\]

We sum the above inequalities for \( n = 1, \ldots, n_0 \):
\[
\sum_{n=1}^{n_0} \|u^n - u^{n-1}\|^2 + \|u^n\|^2 + \Delta t \sum_{n=1}^{n_0} \left( \|u^n\|_{x=1}^2 L^2_{\dot{y}}(0,1) + \|u^n\|_{y=1}^2 L^2_x(0,1) \right) + \alpha_1 \Delta t \sum_{n=1}^{n_0} \|u^n\|^{m+1} \leq c_m \Delta t \sum_{n=1}^{n_0} \|f^n\|_{V^n}^{m+1} + 2 \beta_1 T + \Delta t \sum_{n=1}^{n_0} \|g^n_1(y)^2\|_{L^2_{\dot{y}}(0,1)} + \Delta t \sum_{n=1}^{n_0} \|g^n_2(x)^2\|_{L^2_x(0,1)} + \|u_0\|^2.
\]

Since we have
\[
\Delta t \sum_{n=1}^{N} \|f^n\|_{V^n}^{m+1} \leq \|f\|_{L^\frac{m+1}{m}(0,T;V')}^{m+1},
\]
\[
\Delta t \sum_{n=1}^{N} \|g^n_1\|_{L^2_{\dot{y}}(0,1)}^2 \leq \|g_1\|_{L^2(0,T;L^2_{\dot{y}}(0,1))}^2,
\]
\[
\Delta t \sum_{n=1}^{N} \|g^n_2\|_{L^2_x(0,1)}^2 \leq \|g_2\|_{L^2(0,T;L^2_x(0,1))}^2,
\]
we conclude that, for all \( 1 \leq n_0 \leq N \),
\[
\sum_{n=1}^{n_0} \|u^n - u^{n-1}\|^2 + \|u^n\|^2 + \Delta t \sum_{n=1}^{n_0} \|u^n\|_{x=1}^2 L^2_{\dot{y}}(0,1) + \Delta t \sum_{n=1}^{n_0} \|u^n\|_{y=1}^2 L^2_x(0,1) + \alpha_1 \Delta t \sum_{n=1}^{n_0} \|u^n\|^{m+1} \leq \mathcal{K}_0,
\]
(3.4)
where \( \mathcal{K}_0 := c_m \|f\|_{L^\frac{m+1}{m}(0,T;V')}^{m+1} + \|g_1\|_{L^2(0,T;L^2_{\dot{y}}(0,1))}^2 + \|g_2\|_{L^2(0,T;L^2_x(0,1))}^2 + \|u_0\|^2 + 2 \beta_1 T. \)
3.2. Passage to the limit. We introduce two approximate solutions \( u_k \) and \( \tilde{u}_k \) defined on \( t \in I_n := ((n-1)\Delta t, n\Delta t], n = 1 \ldots, N \) as follows: \( u_k \) is the step function on \( I_n \) with values taken from the right of each interval \( I_n \), and \( \tilde{u}_k \) is the piecewise linear function that interpolates \( u^{n-1} \) and \( u^n \) on \( I_n \). We first have an equivalent form of the scheme (3.2a) as follows

\[
\begin{align*}
\partial_t \tilde{u}_k + Au_k + P(u_k) &= f_k, \quad \text{in } \Omega \times (0, T), \\
u_k(0, y, t) &= g_{1k}(y, t), \quad u_k(x, 0, t) = g_{2k}(x, t), \\
\tilde{u}_k(x, y, 0) &= u_0(x, y),
\end{align*}
\]

where \( f_k \), \( g_{1k} \) and \( g_{2k} \) are the step functions on \( (0, T) \), respectively equal to \( f^n \), \( g^n_2 \) and \( g^n_2 \) on the interval \( I_n \), \( n = 1 \ldots, N \). From (3.4), we find

\[
\Delta t \| \partial_t \tilde{u}_k \|^2_{L^2(0; T; L^2(\Omega))} + \| u_k(t) \|^2 + \| u^n \|_{x=1}^2 \leq 2 \nabla^2 \| u|_{I_n}\|^m_{1,1(0, T; V)} \leq \mathcal{K}_0,
\]

for all \( 0 \leq t \leq T \).

Therefore, as \( k \to 0 \):

\[
\begin{align*}
&u_k, \tilde{u}_k \in \text{bounded set of } L^\infty(0, T; L^2(\Omega)), \\
&u_k \in \text{bounded set of } L^{m+1}(0, T; V) = L^{m+1}(\Omega_T), \\
&P(u_k) \in \text{bounded set of } L^{m+1}_m(0, T; V'), \\
&u_k|_{\partial \Omega} \in \text{bounded set of } L^2(0, T; L^2(\partial \Omega)), \\
&\sqrt{\nabla t} \partial_t \tilde{u}_k \in \text{bounded set of } L^2(0, T; L^2(\Omega)).
\end{align*}
\]

As a consequence of these bounds, from equation (3.5), we find that \( \partial_t \tilde{u}_k + Au_k \) is bounded in \( L^{m+1}_m(\Omega_T) \) independently of \( k \).

From these bounds along with (2.38), we extract a subsequence still denoted by \( k \) such that for \( k \to 0 \)

\[
\begin{align*}
&u_k, \tilde{u}_k \rightharpoonup u \text{ weak-* in } L^\infty(0, T; L^2(\Omega)) \text{ and weakly in } L^{m+1}(\Omega_T), \\
&P(u_k) \rightharpoonup \chi \text{ weakly in } L^{m+1}_m(0, T; V'), \\
&u_k|_{\partial \Omega} \rightharpoonup \kappa \text{ weakly in } L^2(0, T; \partial \Omega), \\
&\partial_t \tilde{u}_k + Au_k \to \xi = \partial_t u + Au \text{ weakly in } L^{m+1}_m(0, T; V').
\end{align*}
\]

We have \( u \in L^\infty(0, T; L^2(\Omega)) \cap L^{m+1}(\Omega_T), \partial_t u + Au \in L^{m+1}_m(\Omega_T) \). Thanks to Lemma A.4 with \( \mathcal{M} = \Omega_T, a = b = c = 1 \) and \( p = (m+1)/m \), the traces \( u|_{t=0} \) and \( u|_{\partial \Omega} \) make sense. As in dimensional one case, by extending all functions by zero to the domain \( (-\infty, 1)^2 \times (-\infty, T) \), letting \( k \to 0 \) and combing with (2.40), (2.41), and (3.5)\_2,3, we find the initial condition and boundary condition for the limit equation reads

\[
\begin{align*}
&\partial_t u + Au + \chi = f, \\
u(0, y, t) = g_1(y, t), \quad u(x, 0, t) = g_2(x, t), \\
u(x, y, 0) = u_0(x, y).
\end{align*}
\]

Thanks to Lemma A.5 with \( a = b = c = 1 \) and \( u|_{x=0} = g_1(y, t) \in L^2(0, T; L^2(y, 0, 1)), u|_{y=0} = g_2(x, t) \in L^2(0, T; L^2(x, 1)), u|_{t=0} = u_0(x, y) \in L^2(\Omega), \) all the traces \( u|_{x=1}, u|_{y=1} \) and \( u|_{t=T} \) are
square integrable functions, and the integration by parts is valid for $\langle \partial_t u + Au, v \rangle$; hence we can similarly (as in Section 2.2.2) show that

$$\int_0^T (\chi - P(v), (u - v)) dt \geq 0,$$

for all $v \in L^{m+1}(0, T; V)$ and thus by the Minty method $\chi = P(u)$. This shows that $u$ is a solution of (3.1).

We now state the result of existence and uniqueness of a solution $u$ for (3.1):

**Theorem 3.1.** Assume that $f \in L^{m+1}(\Omega_T)$, $g_1 \in L^2(0, T; L^2(0, 1))$, $g_2 \in L^2(0, T; L^2(0, 1))$ and $u_0 \in L^2(\Omega)$ are given. Then there exists a unique function $u \in L^{\infty}(0, T; L^2(\Omega)) \cap L^{m+1}(\Omega_T)$ satisfying (3.1).

**Proof.** The existence of a solution $u$ was proved in Section 3.2, and we thus only need to show the uniqueness. Let $u_1, u_2$ belong to $L^{\infty}(0, T; L^2(\Omega)) \cap L^{m+1}(0, T; L^{m+1}(\Omega))$ that both satisfy (3.1). Setting $w = u_1 - u_2$, we then see that $w$ satisfies

$$\begin{cases}
w_t + Au = P(u_2) - P(u_1), \\
w(t, 0, y) = w(t, x, 0) = 0, \\
w(0, x) = 0.
\end{cases}$$

(3.10)

Since $u_1, u_2 \in L^{m+1}(\Omega_T)$, we find from (3.10) that $w_t + Au$ belongs to $L^{(m+1)/m}(\Omega_T)$ by assumption (2.2b). Applying Lemma A.5 to $w$ with $z = t$, $M = \Omega_t := \Omega \times (0, t)$, $p = m + 1$ and $q = (m + 1)/m$ and using (3.10)2,3, we obtain that for all $t \in [0, T]$,

$$\int_0^1 \int_0^t w(x, 1, t')^2 dx dt + \int_0^1 \int_0^t w(1, y, t')^2 dy dt + \int_\Omega w(x, y, t)^2 dxdy = \int_\Omega w(x + w_0) dxdy,$$

which, together with (3.10)1, implies that

$$\int_\Omega w(x, y, t)^2 dxdy + \int_{\Omega_t} (u_1 - u_2)(P(u_1) - P(u_2)) dxdt \leq 0, \quad \forall t \in [0, T].$$

(3.11)

Therefore, we can conclude that $u_1(\cdot, t) = u_2(\cdot, t)$ for all $t \in [0, T]$ and hence $u_1 = u_2$, and we thus completed the proof. 

**Remark 3.1 (The generality of operator $A$ and the boundary conditions).** We could study the following general form of $A = a(x, y, t)\partial_x u + b(x, y, t)\partial_y u$ where $a, b \in C^1(\overline{\Omega_T})$ are never zero on the boundary $\partial \Omega$. The boundary conditions imposed on the boundary $\partial \Omega$ depend on the signs of $a, b$ on the boundary. For example, we could impose the following boundary conditions

$$\begin{align*}
u(0, y, t) &= g_1(y, t) \text{ when } a(0, y, t) > 0, & u(1, y, t) &= g_1(y, t) \text{ when } a(1, y, t) < 0, \\
u(x, 0, t) &= g_2(x, t) \text{ when } b(x, 0, t) > 0, & u(x, 1, t) &= g_2(x, t) \text{ when } b(x, 1, t) < 0.
\end{align*}$$

(3.12)

The boundary conditions in (3.12) are strictly dissipative in the sense of [BS07, Chapter 3]. Our main result in Theorem 3.1 is still valid without much changing of its proof. Indeed, Theorem 2.1 or 2.2 can be applied with $t = y$ to the problem (3.3) induced by the finite difference scheme in time in the general case.
Appendix A. Trace theorem

In this Appendix we give some trace theorems which we used in the article. See Appendix A in [ST10] for some other trace results.

We fix $p$ such that $1 < p < +\infty$, and we also let $q = p'$ the conjugate exponent of $p$ (i.e. $1/p + 1/q = 1$).

**Lemma A.1.** Let $X$ be a reflexive Banach space, and $\lambda = \lambda(x) \in C^0([0, 1])$ such that $\lambda(x) \geq c_0$ for some positive constant $c_0$. Assume that two sequences of function $u^\epsilon, g^\epsilon \in L^p_x(0, 1; X)$ satisfy

\[
\begin{cases}
-\epsilon u^\epsilon_{xx} + \lambda(x)u^\epsilon_x = g^\epsilon, \\
\qquad u^\epsilon(0) = u_0, \quad u^\epsilon_x(1) = 0,
\end{cases}
\]  

(A.1)

with $g^\epsilon$ bounded in $L^p_x(0, 1; X)$ independently of $\epsilon$, $u_0 \in X$ and $p > 1$. Then the $u^\epsilon_x$ are bounded in $L^p_x(0, 1; X)$ independently of $\epsilon$, and, for any subsequence $u^\epsilon \to u, g^\epsilon \to g$ (strongly or weakly) converging in $L^p_x(0, 1; X)$, $u^\epsilon(0)$ converges to $u(0)$ in $X$ (weakly at least), and hence $u(0) = u_0$.

**Proof.** For simplicity, we drop $\epsilon$, and by solving (A.1), we obtain

\[
u_x(x) = \int_x^1 \frac{1}{\epsilon} e^{-\int_x^x \lambda(x)/\epsilon dx} g(x_1) dx_1.
\]  

(A.2)

Taking the $X$-norm of (A.2) and raising to the $p$-th power, and using Hölder’s inequality, we find

\[
\|u_x\|_X^p \leq \left( \int_x^1 \frac{1}{\epsilon} e^{-\int_x^x \lambda(x)/\epsilon dx} dx_1 \right)^{p/q} \int_x^1 \frac{1}{\epsilon} e^{-\int_x^x \lambda(x)/\epsilon dx} \|g(x_1)\|_X^p dx_1.
\]  

(A.3)

Direct computations show that the first integral in (A.3) is less than

\[
\int_x^1 \frac{1}{\epsilon} e^{-\int_x^1 c_0/\epsilon dx} dx_1 = \int_x^1 \frac{1}{\epsilon} e^{-c_0(x_1-x)/\epsilon} dx_1 \leq \frac{1}{c_0}.
\]

Then integrating (A.3) from 0 to 1 with respect to $x$ and switching the order of integration for $x, x_1$, we finally arrive at

\[
\int_0^1 \|u_x\|_X^p dx \leq \left( \frac{1}{c_0} \right)^p \int_0^1 \|g\|_X^p dx.
\]  

(A.4)

It follows that the $u^\epsilon_x$ are uniformly bounded in $L^p_x(0, 1; X)$; hence $u \in C([0, 1]; X)$. The existence of the trace $u(0)$ and its linear continuous dependence on $\{u, g\}$ follow easily. □

**Remark A.1.** If the coefficient function in the convection term is negative, that is, instead of (A.1) we have

\[
\begin{cases}
-\epsilon u^\epsilon_{xx} - \lambda(x)u^\epsilon_x = g^\epsilon, \\
\qquad u^\epsilon(1) = u_1, \quad u^\epsilon_x(0) = 0,
\end{cases}
\]  

(A.5)

for two sequences $u^\epsilon, g^\epsilon \in L^p_x(0, 1; X)$, then the same result as in Lemma A.1 follows; that means the $u^\epsilon_x$ are bounded in $L^p_x(0, 1; X)$ independently of $\epsilon$, and also for any subsequence $u^\epsilon \to u, g^\epsilon \to g$ which converge (strongly or weakly) in $L^p_x(0, 1; X)$, $u^\epsilon(1)$ converges to $u(1)$ in $X$ (weakly at least), and hence $u(1) = u_1$.  

Let \( \mathcal{U} = (0, L_1) \times (0, L_2) \) be a subset of \( \mathbb{R}^2 \), and we consider the space:

\[
\mathcal{X}(\mathcal{U}) = \{ u \in L^p(\mathcal{U}), Tu = au_x + bu_y \in L^p(\mathcal{U}) \},
\]

endowed with its natural norm \( (\| u \|^p_{L^p(\mathcal{U})} + \| Tu \|^p_{L^p(\mathcal{U})})^{1/p} \), where \( L_1 \) and \( L_2 \) are positive constants, and \( a \) and \( b \) are non-zero constants. Then we have the following trace result.

**Lemma A.2.** If \( u \in \mathcal{X}(\mathcal{U}) \), then the traces of \( u \) are defined on all of \( \partial \mathcal{U} \), i.e. the traces of \( u \) are defined at \( x = 0, L_1 \), and \( y = 0, L_2 \), and they belong to the respective spaces \( W_y^{-1,p}(0, L_2) \) and \( W_x^{-1,p}(0, L_1) \). Furthermore the trace operators are linear continuous in the corresponding spaces, e.g., \( u \in \mathcal{X}(\mathcal{U}) \rightarrow u|_{x=0} \) is continuous from \( \mathcal{X}(\mathcal{U}) \) into \( W_y^{-1,p}(0, L_2) \).

**Proof.** Since \( u \in L^p(\mathcal{U}) \), we see that \( u_y \) belongs to \( L_y^p(0, L_1; W_y^{-1,p}(0, L_2)) \), which, together with \( Tu \in L^p(\mathcal{U}) \) and \( a, b \) are non-zeros, implies that \( u_x \in L_x^p(0, L_1; W_y^{-1,p}(0, L_2)) \). In combination with \( u \in L_y^p(0, L_1; L_y^p(0, L_2)) \), we obtain that \( u \in C_x([0, L_1]; W_y^{-1,p}(0, L_2)) \). Hence, the traces of \( u \) are well-defined at \( x = 0 \) and \( L_1 \), and belong to \( W_y^{-1,p}(0, L_2) \). The continuity of the corresponding mappings is easy. The proof for the traces at \( y = 0 \) and \( L_2 \) is similar. \( \square \)

**Lemma A.3.** Assume that \( u \in L^p(\mathcal{U}) \) and \( Tu \in L^q(\mathcal{U}) \) with \( a, b > 0 \). Then by Lemma A.2 with \( p = \min\{p, q\} \), we know that the traces of \( u \) on \( \partial \mathcal{U} \) are well-defined. Furthermore, we assume that the traces \( u(0, y) \in L^2(0, L_2) \) and \( u(x, 0) \in L^2(0, L_1) \). Then the traces \( u(L_1, y), u(x, L_2) \) belong to \( L^2(0, L_2), L^2(0, L_1) \) respectively, and the identity holds

\[
\int_{\mathcal{U}} u(Tu) \, dxdy = \frac{a}{2} (\| u(L_1, y) \|_{L^2(0, L_2)}^2 - \| u(0, y) \|_{L^2(0, L_2)}^2)
+ \frac{b}{2} (\| u(x, L_2) \|_{L^2(0, L_1)}^2 - \| u(x, 0) \|_{L^2(0, L_1)}^2).
\]

(A.6)

**Proof.** Without loss of generality, we may assume that \( b = 1 \). We first denote by \( \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4 \) the boundaries \( x = 0, x = L_1, y = 0, y = L_2 \) respectively, and then introduce a new coordinate system \((x', y')\) such that

\[
\begin{pmatrix}
x' \\
y'
\end{pmatrix}
=
\begin{pmatrix}
a & 1 \\
-1 & a
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix},
\]

which is equivalent to

\[
\begin{pmatrix}
x \\
y
\end{pmatrix}
=
\frac{1}{1 + a^2}
\begin{pmatrix}
a & -1 \\
1 & a
\end{pmatrix}
\begin{pmatrix}
x' \\
y'
\end{pmatrix}.
\]

We denote by \( \Gamma'_i \) the image of \( \Gamma_i \) by this transformation for all \( i \in \{1, 2, 3, 4\} \); we also denote by \( \mathcal{U}' \) the image of \( \mathcal{U} \) and denote by \( u' \) the transform of \( u \). Let \((\alpha_{y'}, y')\) and \((\beta_{y'}, y')\) be the end points of the intersection of \( \mathcal{U}' \) with the line \( y' \) fixed, \((\alpha_{y'}, y') \in \Gamma'_1 \) or \( \Gamma'_3 \), \((\beta_{y'}, y') \in \Gamma'_2 \) or \( \Gamma'_4 \) (see Figure A.1).
Therefore, for $u \in L^p(\mathcal{U})$ and $Tu \in L^q(\mathcal{U})$, we see that $u' \in L^p(\mathcal{U}')$ and $u'_{x'} \in L^q(\mathcal{U}')$, which shows that $u'u'_{x'}$ belongs to $L^1(\mathcal{U}')$ by Hölder’s inequality. Hence for fixed $y'$, we see that

$$u'(\beta'y', y')^2 = u'(\alpha'y', y')^2 + 2 \int_{\alpha'y'}^{\beta'y'} u'(x', y')u'_{x'}(x', y') \, dx'.$$

Integrating (A.7) with respect to $y'$ from $-L_1$ to $aL_2$, we obtain

$$\int_{-L_1}^{aL_2} u'(\beta'y', y')^2 \, dy' = \int_{-L_1}^{aL_2} u'(\alpha'y', y')^2 \, dy' + 2 \int_{-L_1}^{aL_2} \int_{\alpha'y'}^{\beta'y'} u'(x', y')u'_{x'}(x', y') \, dx' \, dy'. \tag{A.8}$$

Since the traces of $u$ at $\Gamma_1 = \{x = 0\}$ and $\Gamma_3 = \{y = 0\}$ are $L^2$ functions, we also have that the traces of $u'$ at $\Gamma_1'$ and $\Gamma_3'$ are $L^2$ functions. Therefore, we infer from (A.8) that the traces of $u'$ at $\Gamma_2'$ and $\Gamma_4'$ are $L^2$ functions, which shows that the traces of $u$ at $\Gamma_2 = \{x = L_1\}$ and $\Gamma_4 = \{y = L_2\}$ are $L^2$ functions. Now, transforming back into the original system $(x, y)$, (A.8) is equivalent to

$$a \int_0^{L_2} u(L_1, y)^2 \, dy + \int_0^{L_1} u(x, L_2)^2 \, dx = a \int_0^{L_2} u(0, y)^2 \, dy + \int_0^{L_1} u(x, 0)^2 \, dx + 2 \int_0^{L_1} \int_0^{L_2} u(Tu) \, dx \, dy, \tag{A.9}$$

which is (A.6) in the case $b = 1$. We thus completed the proof. \qed

Lemmas A.2 and A.3 are also valid in the three dimensional case. We state them as follows. We let $\mathcal{M} = (0, L_1) \times (0, L_2) \times (0, L_3)$ be a subset of $\mathbb{R}^3$, and we consider the space:

$$\mathcal{X}(\mathcal{M}) = \{u \in L^p(\mathcal{M}), Tu = au_x + bu_y + cu_z \in L^p(\mathcal{M})\},$$

endowed with its natural norm $\|u\|_{L^p(\mathcal{M})}^p + \|Tu\|_{L^p(\mathcal{M})}^p = 1/p$, where $L_1, L_2, L_3$ are positive constants, and $a, b, c$ are non-zero constants. Then we have the following trace result.
Lemma A.4. If \( u \in \mathcal{X}(\mathcal{M}) \), then the traces of \( u \) are defined on all of \( \partial \mathcal{M} \), i.e. the traces of \( u \) are defined at \( x = 0, L_1, y = 0, L_2 \) and \( z = 0, L_3 \), and they belong to the respective spaces \( W^{-1,p}_{y,z}((0, L_2) \times (0, L_3)) \), \( W^{-1,p}_{x,z}((0, L_1) \times (0, L_3)) \) and \( W^{-1,p}_{x,y}((0, L_1) \times (0, L_3)) \). Furthermore the trace operators are linear continuous in the corresponding spaces.

Lemma A.5. Assume that \( u \in L^p(\mathcal{M}) \) and \( Tu \in L^q(\mathcal{M}) \) with \( a, b, c > 0 \). Then by Lemma A.4 with \( p = \min\{p, q\} \), we know that the traces of \( u \) on \( \partial \mathcal{M} \) are well-defined. Furthermore, we assume that the traces \( u|_{x=0}, u|_{y=0}, u|_{z=0} \) are \( L^2 \)-functions with respect to the corresponding domains. Then the traces \( u|_{x=L_1}, u|_{y=L_2}, u|_{z=L_3} \) are also \( L^2 \)-functions, and the following identity holds:

\[
\int_{\mathcal{M}} u(Tu) \, dx dy dz = \frac{a}{2} \left( \|u|_{x=L_1}\|_{L^2}^2 - \|u|_{x=0}\|_{L^2}^2 \right) + \frac{b}{2} \left( \|u|_{y=L_2}\|_{L^2}^2 - \|u|_{y=0}\|_{L^2}^2 \right) + \frac{c}{2} \left( \|u|_{z=L_3}\|_{L^2}^2 - \|u|_{z=0}\|_{L^2}^2 \right).
\]

The proof of Lemmas A.4 and A.5 are the same as the proof of of Lemmas A.2 and A.3, using the following coordinate transformation adapted for Lemma A.5:

\[
\begin{pmatrix}
    x' \\
    y' \\
    z'
\end{pmatrix} = \begin{pmatrix}
    a & b & c \\
    0 & c & -b \\
    \frac{b^2+c^2}{a} & b & c
\end{pmatrix} \begin{pmatrix}
    x \\
    y \\
    z
\end{pmatrix}.
\]

Appendix B. Convex analysis and Variational Problems

The following results are taken from [ET76].

Let \( X \) be a reflexive separable Banach space and \( X' \) be the dual space of \( X \), and let \( \mathcal{K} \) be a non-empty convex closed subset of \( X \).

Theorem B.1. Let \( J : \mathcal{K} \to \mathbb{R} \) be a convex and lower semi-continuous functional, and assume that one of the following holds.

1. \( \mathcal{K} \) is bounded; or
2. \( J(u) \to +\infty \) as \( \|u\| \to +\infty \), and \( u \in \mathcal{K} \).

Then \( \inf_{v \in \mathcal{K}} J(v) > -\infty \), and there exists at least one \( u \in \mathcal{K} \) such that

\[
J(u) = \inf_{v \in \mathcal{K}} J(v).
\]

The set of \( u \) satisfying (B.1) is convex and closed. If \( J \) is strictly convex, then there exists one and only one \( u \) satisfying (B.1).

Theorem B.2. Let \( A \) be an operator from \( X \) to \( X' \) satisfying the following properties:

1. \( A \) is weakly continuous over the finite dimensional subspaces of \( X \).
2. \( A \) is bounded: \( \|Au\|_{X'} \) is bounded if \( \|u\|_X \) is bounded.
3. \( A \) is monotone: \( \langle Au - Av, u - v \rangle_{(X',X)} \geq 0 \), \( \forall u, v \in \mathcal{K} \).
4. Either one of the following holds:
   a. \( \mathcal{K} \) is bounded; or
(b) $A$ is coercive, that is:

$$\frac{\langle Au, u \rangle_{X'}}{\|u\|_X} \to +\infty, \quad \text{as } \|u\|_X \to +\infty, \quad u \in K.$$ 

Then for all $\ell \in X'$, there exists at least one $u \in K$ such that

$$\langle Au - \ell, v - u \rangle \geq 0, \quad \forall v \in K.$$  \hspace{1cm} (B.2)

If furthermore, $A$ is strictly monotone, i.e.

$$\langle Au - Av, u - v \rangle_{X'} > 0, \quad \forall u, v \in K, u \neq v,$$

then there exists one and only one $u$ satisfying (B.2).  

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