Gorenstein n-$\mathscr{X}$-injective and n-$\mathscr{X}$-flat modules with respect to a special finitely presented module

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Abstract. Let R be a ring, $\mathscr{X}$ a class of R-modules and $n \geq 1$ an integer. In this paper, via special finitely presented modules, we introduce the concepts of Gorenstein n-$\mathscr{X}$-injective and n-$\mathscr{X}$-flat modules. And aside, we obtain some equivalent properties of these modules on n-$\mathscr{X}$-coherent rings. Then, we investigate the relations among Gorenstein n-$\mathscr{X}$-injective, n-$\mathscr{X}$-flat, injective and flat modules on $\mathscr{X}$-FC-rings (i.e., self n-$\mathscr{X}$-injective and n-$\mathscr{X}$-coherent rings). Several known results are generalized to this new context.

Keywords: n-$\mathscr{X}$-coherent ring; Gorenstein n-$\mathscr{X}$-injective module; Gorenstein n-$\mathscr{X}$-flat module.

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1 Introduction

In 1995, Enochs et al. introduced the concept of Gorenstein injective and Gorenstein flat modules. Then, these modules have become a vigorously active area of research. For background on Gorenstein modules, we refer the reader to [8, 9, 12]. In 2012, Gao and Wang introduced and studied in [11] Gorenstein FP-injective modules. They established various homological properties of Gorenstein FP-injective modules mainly over a coherent ring (for more details, see [13]).

Recall that coherent rings were first appeared in Chase's paper [3] without being mentioned by name. The term coherent was first used by Bourbaki in [2]. Then, n-coherent rings were introduced by Costa in [5]. Let n be a non-negative integer. An R-module M is said to be *n-presented* if there is an exact sequence $F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to$ of R-modules, where each F_i is finitely generated free. And a ring R is called left *n-coherent* if every n-presented R-module is $(n+1)$-presented. Thus, for $n = 1$, left n-coherent rings are nothing but coherent rings (see [5, 7, 17]). Chen and Ding in [4] introduced, by using n-presented modules, n-FP-injective and n-flat modules. Bennis in [1] introduced n-$\mathscr{X}$-injective and n-$\mathscr{X}$-flat modules and n-$\mathscr{X}$-coherent rings for any class $\mathscr{X}$ of R-modules. Then, in 2018, Zhao et al. in [20] introduced n-FP-gr-injective graded modules, n-gr-flat graded right modules and n-gr-coherent graded rings on a class of graded R-modules. Moreover, they defined special finitely presented graded modules via projective resolutions of n-presented graded left modules, where if U is n-presented graded module, then in the exact sequence $F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to U \to 0$, $K_{n-1} = \mathrm{Im}(F_{n-1} \to F_{n-2})$ is called a special finitely presented module. In this paper, we unify and extend various homological notions, including the ones cited above, to a more general context. Namely, we define special finitely presented modules via projective resolutions of n-presented modules in a given $\mathscr{X}$ of R-modules. Then, we introduce and study Gorenstein n-$\mathscr{X}$-injective and n-$\mathscr{X}$-flat modules with respect to special finitely presented modules.

The paper is organized as follows:

In Section 2, some fundamental concepts and some preliminary results are stated.

In Section 3, we give some characterizations of n-$\mathscr{X}$-injective and n-$\mathscr{X}$-flat modules.

In Section 4, we introduce the notions of Gorenstein n-$\mathscr{X}$-injective and n-$\mathscr{X}$-flat modules. We generalize some results of [20] to the context of n-$\mathscr{X}$-injective and n-$\mathscr{X}$-flat modules as well as some results of [11] to the context of Gorenstein n-$\mathscr{X}$-injective modules. Then, we give some characterizations of Gorenstein n-$\mathscr{X}$-injective and $\mathscr{X}$-flat modules on n-$\mathscr{X}$-coherent rings.

In Section 5, we introduce and investigate $\mathscr{X}$-FC rings (i.e., self n-$\mathscr{X}$-injective and n-$\mathscr{X}$-coherent rings) whose every module is Gorenstein n-$\mathscr{X}$-injective and every Gorenstein n-$\mathscr{X}$-injective right module is Gorenstein n-$\mathscr{X}$-flat. Furthermore, examples are given which show that the Gorenstein m-$\mathscr{X}$-injectivity (resp., the m-$\mathscr{X}$-flatness) does not imply, in general, the Gorenstein n-$\mathscr{X}$-injectivity (resp., the n-$\mathscr{X}$-flatness) for any $m > n$.

2 Preliminaries

Throughout this paper R will be an associative (non necessarily commutative) ring with identity, and all modules will be unital left R-modules (unless specified otherwise).

In this section, some fundamental concepts and notations are stated.

Let n be a non-negative integer, M an R-module and $\mathscr{X}$ a class of R-modules. Then, M is said to be *Gorenstein injective* (resp., *Gorenstein flat*) [8, 9] if there is an exact sequence $\cdots \to I_1 \to I_0 \to I^0 \to I^1 \to \cdots$ of injective (resp., flat) modules with $M = \ker(I^0 \to I^1)$ such that $\mathrm{Hom}(U, -)$ (resp., $U \otimes_R -$) leaves the sequence exact whenever U is an injective left (resp., right) module. The Gorenstein projective modules are defined dually. Recall also that M is said to be *n-FP-injective* [4] if $\mathrm{Ext}^n_R(U, M) = 0$ for any n-presented R-module U. In case $n = 1$, n-FP-injective modules are nothing but the well-known FP-injective modules. A right module N is called *n-flat* if $\mathrm{Tor}^R_n(N, U) = 0$ for any n-presented R-module U. Also, M is said to be *Gorenstein FP-injective* [11] if there is an exact sequence $\mathbf{E} = \cdots \to E_1 \to E_0 \to E^0 \to E^1 \to \cdots$ with $M = \ker(E^0 \to E^1)$ such that $\mathrm{Hom}_R(P, \mathbf{E})$ is an exact sequence whenever P is finitely presented with $\mathrm{pd}_R(P) < \infty$. A graded R-module M is called *n-FP-gr-injective* [20] if $\mathrm{EXT}^n_R(N, M) = 0$ for any n-presented graded R-module N. A graded right R-module M is called *n-gr-flat* [20] if $\mathrm{Tor}^n_R(M, N) = 0$ for any n-presented graded R-module N (for more details about graded modules, see [10, 14, 15]).

From now on, $\mathscr{X}_{\mathrm{k}}$ is a non empty class of k-presented R-module in a given class $\mathscr{X}$ for any integer $k \geq 0$.

An R-module M is said to be *n-$\mathscr{X}$-injective* [1] if $\mathrm{EXT}^n_R(U, M) = 0$ for any $U \in \mathscr{X}_{\mathrm{n}}$. A right R-module N is called *n-$\mathscr{X}$-flat* [1] if $\mathrm{Tor}^n_R(N, U) = 0$ for any $U \in \mathscr{X}_{\mathrm{n}}$. We use $\mathscr{X}\mathscr{I}_n$ (resp., $\mathscr{X}\mathscr{F}_n$) to denote the class of all n-$\mathscr{X}$-injective left R-modules (resp., n-$\mathscr{X}$-flat right R-modules).

A ring R is called left *n-$\mathscr{X}$-coherent* if every n-presented R-module in $\mathscr{X}$ is $(n+1)$-presented.

It is clear that when $n = 0$ (resp., $n = 1$) and $\mathscr{X}$ is a class of all cyclic R-modules, then R is Noetherian (resp., coherent).

3 n-$\mathscr{X}$-injective, n-$\mathscr{X}$-flat and special $\mathscr{X}$-presented modules

In this section, we state a relative version of [20, Definition 3.1] and provide several characterizations of n-$\mathscr{X}$-injective and n-$\mathscr{X}$-flat modules.

Let us introduce the following notions. Let $n \geq 0$ be an integer and $U \in \mathscr{X}_{\rm n}$ for a class $\mathscr{X}$ of R-modules. Then, an exact sequence of the form

$$F_n \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow U \longrightarrow 0,$$

where each F_i is a finitely generated free R-module, exists. Let $K_{n-1} = \mathrm{Im}(F_{n-1} \to F_{n-2})$ and $K_n = \mathrm{Im}(F_n \to F_{n-1})$. The short exact sequence $0 \to K_n \to F_n \to K_{n-1} \to 0$ is called a special short exact sequence of U. It is clear that K_n and K_{n-1} are finitely generated and finitely presented, respectively. We call K_n a special $\mathscr{X}$-generated R-module and K_{n-1} a special $\mathscr{X}$-presented R-module.

Also, a short exact sequence $0 \to A \to B \to C \to 0$ of R-modules is called special $\mathscr{X}$-pure, if for every special $\mathscr{X}$-presented K_{n-1}, there exists the following exact sequence:

$$0 \to \mathrm{Hom}_R(K_{n-1}, A) \to \mathrm{Hom}_R(K_{n-1}, B) \to \mathrm{Hom}_R(K_{n-1}, C) \to 0.$$

The R-module A is said to be a special $\mathscr{X}$-pure in B. Also, the exact sequence $0 \to C^* \to B^* \to A^* \to 0$ is called a split special exact sequence. If M is an n-$\mathscr{X}$-injective (resp., a flat) R-module, then $\mathrm{Ext}^1_R(K_{n-1}, M) \cong \mathrm{Ext}^n_R(U, M) = 0$ (resp., $\mathrm{Tor}^R_1(M, K_{n-1}) \cong \mathrm{Tor}^R_n(M, U) = 0$) for any $U \in \mathscr{X}_{\rm n}$.

In particular, if $\mathscr{X}$ is a class of graded R-modules, then every special $\mathscr{X}$-generated and every special $\mathscr{X}$-presented module are special finitely generated and special finitely presented graded R-modules, respectively. Also, every n-$\mathscr{X}$-injective R-module and every n-$\mathscr{X}$-flat right R-module are n-FP-gr-injective and n-gr-flat, respectively; see [20].

Proposition 3.1. *Let $\mathscr{X}$ be a class of R-modules and M an R-module. Then, the following statements are equivalent:*

(1) *M is n-$\mathscr{X}$-injective;*

(2) *Every short exact sequence $0 \to M \to A \to C \to 0$ is special $\mathscr{X}$-pure;*

(3) *M is special $\mathscr{X}$-pure in any injective R-module containing it;*

(4) *M is special $\mathscr{X}$-pure in $E(M)$.*

Proof. (1) $\Longrightarrow$ (2) Let $U \in \mathscr{X}_{\mathrm{n}}$. Then, there exists the following exact sequence

$$F_n \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow U \longrightarrow 0,$$

where each F_i is a finitely generated free R-module. If $K_{n-1} = \mathrm{Im}(F_{n-1} \to F_{n-2})$ is special $\mathscr{X}$-presented, then $\mathrm{Ext}^1_R(K_{n-1}, M) \cong \mathrm{Ext}^n_R(U, M) = 0$ since M is n-$\mathscr{X}$-injective. Hence (2) follows.

(2) $\Longrightarrow$ (3) Consider the canonical short exact sequence $0 \to M \to E \to \frac{E}{M} \to 0$, where E is an injective R-module containing M. So by (2), M is special $\mathscr{X}$-pure in E.

(3) $\Longrightarrow$ (4) is trivial.

(4) $\Longrightarrow$ (1) Assume that $U \in \mathscr{X}_{\mathrm{n}}$ and K_{n-1} is special $\mathscr{X}$-presented. By (4), the short exact sequence $0 \to M \to E(M) \to \frac{E(M)}{M} \to 0$ is special $\mathscr{X}$-pure. Therefore, $\mathrm{Ext}^1_R(K_{n-1}, M) = 0$, and so from $\mathrm{Ext}^1_R(K_{n-1}, M) \cong \mathrm{Ext}^n_R(U, M)$ we get that M is n-$\mathscr{X}$-injective. ■

The following lemma is a generalization of [18, Exercise 40].

Lemma 3.2. *Let $\mathscr{X}$ be a class of R-modules and $0 \to A \to B \to C \to 0$ be a short exact sequence of R-modules . Then, the following statements are equivalent:*

(1) *The exact sequence $0 \to A \to B \to C \to 0$ is special $\mathscr{X}$-pure;*

(2) *The sequence $0 \to \mathrm{Hom}_R(K_{n-1}, A) \to \mathrm{Hom}_R(K_{n-1}, B) \to \mathrm{Hom}_R(K_{n-1}, C) \to 0$ is exact for every special $\mathscr{X}$-presented K_{n-1};*

(3) *The short exact sequence $0 \to C^* \to B^* \to A^* \to 0$ is a split special exact sequence.*

Proposition 3.3. *Let $\mathscr{X}$ be a class of R-modules. Then:*

(1) *Every special $\mathscr{X}$-pure submodule of an n-$\mathscr{X}$-flat right R-module is n-$\mathscr{X}$-flat.*

(2) *Every special $\mathscr{X}$-pure submodule of an R-module is n-$\mathscr{X}$-injective.*

Proof. (1) Let A be a special $\mathscr{X}$-pure submodule of an n-$\mathscr{X}$-flat right R-module B. Then, by Lemma 3.2, the sequence $0 \to (\frac{B}{A})^* \to B^* \to A^* \to 0$ is a split special exact sequence. By [1, Lemma 2.8], B^* is n-$\mathscr{X}$-injective. Then, from [1, lemma 2.7] and Lemma 3.2, we deduce that A is n-$\mathscr{X}$-flat.

(2) Let A be a special $\mathscr{X}$-pure submodule of an R-module B. Then, the exact sequence $0 \to A \to B \to \frac{B}{A} \to 0$ is special $\mathscr{X}$-pure. So, by Proposition 3.1, A is n-$\mathscr{X}$-injective. ∎

Remark 3.4. *(1) Every flat right R-module is n-$\mathscr{X}$-flat.*

(2) Every injective left (resp., right) R-module is n-$\mathscr{X}$-injective.

(3) If $U \in \mathscr{X}_{\mathrm{m}}$, then $U \in \mathscr{X}_{\mathrm{n}}$ for any $m \geq n$.

A ring R is called self left n-$\mathscr{X}$-injective if R is an n-$\mathscr{X}$-injective left R-module.
Let $\mathcal{F}$ be a class of R-modules and M an R-module. Recall that a morphism $f : F \to M$ is called an $\mathcal{F}$-precover of M if $F \in \mathcal{F}$ and $\mathrm{Hom}_R(F', F) \to \mathrm{Hom}_R(F', M) \to 0$ is exact for all $F' \in \mathcal{F}$. Moreover, if whenever a morphism $g : F \to F$ such that $fg = f$ is an automorphism of F, then $f : F \to M$ is called an $\mathcal{F}$-cover of M. Dually, the notions of $\mathcal{F}$-preenvelopes and $\mathcal{F}$-envelopes are defined.

Theorem 3.5. *Let R be a left n-$\mathscr{X}$-coherent ring and $\mathscr{X}$ be a class of R-modules. Then, the following statements are equivalent:*

(1) *R is self left n-$\mathscr{X}$-injective;*

(2) *For any R-module, there is an epimorphism $\mathscr{X}\mathscr{I}$-cover;*

(3) *For any right R-module, there is a monomorphic $\mathscr{X}\mathscr{F}$-preenvelope;*

(4) *Every injective right R-module is n-$\mathscr{X}$-flat;*

(5) *Every 1-$\mathscr{X}$-injective right R-module is n-$\mathscr{X}$-flat;*

(6) *Every n-$\mathscr{X}$-injective right R-module is n-$\mathscr{X}$-flat;*

(7) *Every flat R-module is n-$\mathscr{X}$-injective.*

Proof. (1) $\Longrightarrow$ (3) By [1, Theorem 2.16], every right R-module N has an n-$\mathscr{X}$-flat preenvelope $f : N \to F$. By [1, Theorem 2.13], R^* is n-$\mathscr{X}$-flat, and so $\prod R^*$ is n-$\mathscr{X}$-flat by [1, Theorem 2.6].

On the other hand, R^* is a cogenerator, so a monomorphism of the form $g : N \to \prod R^*$ exists. Hence, there exists a homomorphism $h : F \to \prod R^*$ such that $hf = g$ which implies that f is monic.

(3) $\Longrightarrow$ (4) Let E be an injective right R-module. By (3), there is $f : E \to F$ a monic n-$\mathscr{X}$-flat preenvelope of E. So, the sequence $0 \to E \to F \to \frac{F}{E} \to 0$ splits, hence E is n-$\mathscr{X}$-flat.

(3) $\Longrightarrow$ (5) The proof is similar to the one of (3) $\Longrightarrow$ (4).

(4) $\Longrightarrow$ (6) Let N be an n-$\mathscr{X}$-injective right R-module. Then, by Proposition 3.1, the exact sequence $0 \to N \to E(N) \to \frac{E(N)}{N} \to 0$ is special $\mathscr{X}$-pure. Since by (3) $E(N)$ is n-$\mathscr{X}$-flat, then from Proposition 3.3, we deduce that N is n-$\mathscr{X}$-flat.

(5) $\Longrightarrow$ (4) is clear by Remark 3.4.

(4) $\Longrightarrow$ (1) By (4), R^* is n-$\mathscr{X}$-flat since R^* is injective. So, R is self left n-$\mathscr{X}$-injective by [1, Theorem 2.13].

($6 \Rightarrow 7$) Let F be a flat R-module, then F^* is injective, so F^* is n-$\mathscr{X}$-flat by (6), and hence F is n-$\mathscr{X}$-injective.

($7 \Rightarrow 2$) For any R-module M, there is an $\mathscr{X}\mathscr{I}_n$-cover $f : C \to M$. Notice that R is an n-$\mathscr{X}$-injective R-module, so f is an epimorphism.

($2 \Rightarrow 1$) By hypothesis, R has an epimorphism $\mathscr{X}\mathscr{I}_n$-cover $f : D \to R$, then we have a split exact sequence $0 \to Kerf \to D \to R \to 0$ with D is n-$\mathscr{X}$-injective. Then, R is n-$\mathscr{X}$-injective as a left R-module. ∎

Proposition 3.6. *Let R be a left n-$\mathscr{X}$-coherent ring and $\mathscr{X}$ be a class of R-modules. If $\{A_i\}_{i \in I}$ is a family of R-modules, then $\bigoplus_{i \in I} A_i$ is n-$\mathscr{X}$-injective if and only if every A_i is n-$\mathscr{X}$-injective.*

Proof. Assume that $U \in \mathscr{X}_{\mathrm{n}}$. So, there exists a special exact sequence $0 \to K_n \to F_n \to K_{n-1} \to 0$ of $\mathscr{X}_{\mathrm{n}}$. Since R is n-$\mathscr{X}$-coherent, we conclude that $U \in \mathscr{X}_{\mathrm{n+1}}$ and K_n is special $\mathscr{X}$-presented. So, if $\{A_i\}_{i \in I}$ is a family of n-$\mathscr{X}$-injective R-modules, we have that

$$\mathrm{Hom}(K_n, \bigoplus_{i \in I} A_i) \cong \bigoplus_{i \in I} \mathrm{Hom}(K_n, A_i).$$

One easily gets that

$$\mathrm{Ext}^n_R(U, \bigoplus_{i \in I} A_i) \cong \mathrm{Ext}^1_R(K_n, \bigoplus_{i \in I} A_i) \cong \bigoplus_{i \in I} \mathrm{Ext}^1_R(K_n, A_i) \cong \bigoplus_{i \in I} \mathrm{Ext}^n_R(U, A_i).$$

∎

4 Gorenstein n-$\mathscr{X}$-injective and n-$\mathscr{X}$-flat modules

In this section, we investigate Gorenstein n-$\mathscr{X}$-injective and Gorenstein n-$\mathscr{X}$-flat modules which are defined below. Then, by using of results of Section 3, some characterizations of them are given.

Definition 4.1. *Let R be a ring and $\mathscr{X}$ be a class of R-modules. Then:*

(1) *An R-module G is called Gorenstein n-$\mathscr{X}$-injective, if there exists an exact sequence of n-$\mathscr{X}$-injective R-modules:*

$$\mathbf{A} = \cdots \longrightarrow A_1 \longrightarrow A_0 \longrightarrow A^0 \longrightarrow A^1 \longrightarrow \cdots$$

with $G = \ker(A^0 \to A^1)$ such that $\mathrm{Hom}_R(K_{n-1}, \mathbf{A})$ is an exact sequence whenever K_{n-1} is special $\mathscr{X}$-presented with $\mathrm{pd}_R(K_{n-1}) < \infty$.

(1) *An R-module G is called Gorenstein n-$\mathscr{X}$-flat right R-module if there exists an exact sequence of n-$\mathscr{X}$-flat right R-modules:*

$$\mathbf{F} = \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow F^0 \longrightarrow F^1 \longrightarrow \cdots$$

with $G = \ker(F^0 \to F^1)$ such that $\mathbf{F} \otimes_R K_{n-1}$ is an exact sequence whenever K_{n-1} is special $\mathscr{X}$-presented with $\mathrm{fd}_R(K_{n-1}) < \infty$.

For example, if $\mathscr{X}$ is a class of all cyclic R-modules, then every Gorenstein 1-$\mathscr{X}$-injective R-module is Gorenstein FP-injective, and every Gorenstein 1-$\mathscr{X}$-flat right R-module is Gorenstein flat, see [1, 11].

Remark 4.2. *(1) Every n-$\mathscr{X}$-flat right R-module is Gorenstein n-$\mathscr{X}$-flat.*

(2) Every n-$\mathscr{X}$-injective R-module is Gorenstein n-$\mathscr{X}$-injective.

(3) In Definition 4.1, one easily gets that each $\ker(A_i \to A_{i-1})$, $\ker(A^i \to A^{i+1})$ and $\ker(F_i \to F_{i-1})$, $K^i = \ker(F^i \to F^{i+1})$ are Gorenstein n-$\mathscr{X}$-injective and Gorenstein n-$\mathscr{X}$-flat, respectively.

Lemma 4.3. *Let R be a left n-$\mathscr{X}$-coherent ring and $\mathscr{X}$ be a class of R-modules. If K_{n-1} is a special $\mathscr{X}$-presented R-module with $\mathrm{fd}_R(K_{n-1}) < \infty$, then $\mathrm{pd}_R(K_{n-1}) < \infty$.*

Proof. If $\mathrm{fd}_R(K_{n-1}) = m < \infty$, then there exists $U \in \mathscr{X}_{\mathrm{n}}$ such that $\mathrm{fd}_R(U) \leq n + m$. We show that $\mathrm{pd}_R(U) \leq n + m$. Since R is n-$\mathscr{X}$-coherent, the projective resolution $\cdots \to F_{n+1} \to F_n \to \cdots \to F_0 \to U \to 0$, where any F_i is finitely generated free, exists. On the other hand, the above exact sequence is a flat resolution. So by [16, Proposition 8.17], $(n + m - 1)$-syzygy is flat. Hence, the exact sequence $0 \to K_{n+m-1} \to F_{n+m-1} \to \cdots \to F_0 \to U \to 0$ is a flat resolution. Now, a simple observation shows that if $n \geq m$ or $n < m$, K_{n+m-1} is finitely presented and consequently by [16, Theorem 3.56], K_{n+m-1} is projective and therefore, $\mathrm{pd}_R(U) \leq n + m$ if and only if $\mathrm{pd}_R(K_{n-1}) \leq m$. ■

In the following theorem, we show that in the case of left n-$\mathscr{X}$-coherent rings, Gorenstein n-$\mathscr{X}$-flat and Gorenstein n-$\mathscr{X}$-injective are determined via only the existence of the corresponding exact complexes.

Theorem 4.4. *Let R be a left n-$\mathscr{X}$-coherent ring and $\mathscr{X}$ be a class of R-modules. Then:*

(1) *A right R-module G is Gorenstein n-$\mathscr{X}$-flat if and only if there is an exact sequence*

$$\mathbf{F} = \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow F^0 \longrightarrow F^1 \longrightarrow \cdots$$

of n-$\mathscr{X}$-flat right R-modules such that $G = \ker(F^0 \to F^1)$.

(2) *An R-module G is Gorenstein n-$\mathscr{X}$-injective if and only if there is an exact sequence*

$$\mathbf{A} = \cdots \longrightarrow A_1 \longrightarrow A_0 \longrightarrow A^0 \longrightarrow A^1 \longrightarrow \cdots$$

of n-$\mathscr{X}$-injective R-modules such that $G = \ker(A^0 \to A^1)$.

Proof. (1) ($\Longrightarrow$) follows by definition.

($\Longleftarrow$) By definition, it suffices to show that $\mathbf{F} \otimes_R K_{n-1}$ is exact for every special $\mathscr{X}$-presented K_{n-1} with $\mathrm{fd}_R(K_{n-1}) < \infty$. By Lemma 4.3, $\mathrm{pd}_R(K_{n-1}) < \infty$. Let $\mathrm{pd}_R(K_{n-1}) = m$. We prove by induction on m. The case $m = 0$ is clear. Assume that $m \geq 1$. There exists a special exact sequence $0 \to K_n \to P_n \to K_{n-1} \to 0$ of $U \in \mathscr{X}_{\mathrm{n}}$, where P_n is projective. Now, from the n-$\mathscr{X}$-coherence of R, we deduce that K_n is special $\mathscr{X}$-presented. Also, $\mathrm{pd}_R(K_n) \leq m - 1$. So, the following short exact sequence of complexes exists:

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F_1 \otimes_R K_n & \longrightarrow & F_1 \otimes_R P_n & \longrightarrow & F_1 \otimes_R K_{n-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F_0 \otimes_R K_n & \longrightarrow & F_0 \otimes_R P_n & \longrightarrow & F_0 \otimes_R K_{n-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F^0 \otimes_R K_n & \longrightarrow & F^0 \otimes_R P_n & \longrightarrow & F^0 \otimes_R K_{n-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F^1 \otimes_R K_n & \longrightarrow & F^1 \otimes_R P_n & \longrightarrow & F^1 \otimes_R K_{n-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \| & & \| & & \| & & \\
0 & \longrightarrow & \mathbf{F} \otimes_R K_n & \longrightarrow & \mathbf{F} \otimes_R P_n & \longrightarrow & \mathbf{F} \otimes_R K_{n-1} & \longrightarrow & 0.
\end{array}$$

By induction, $\mathbf{F} \otimes_R P_n$ and $\mathbf{F} \otimes_R K_n$ are exact, hence $\mathbf{F} \otimes_R K_{n-1}$ is exact by [16, Theorem 6.10].

(2) ($\Longrightarrow$) This is a direct consequence of the definition.

($\Longleftarrow$) Let K_{n-1} be a special $\mathscr{X}$-presented R-module with $\mathrm{pd}_R(K_{n-1}) < \infty$. Then, similar proof to that of (1) shows that $\mathrm{Hom}_R(K_{n-1}, \mathbf{A})$ is exact and hence G is Gorenstein n-$\mathscr{X}$-injective. ∎

Corollary 4.5. *Let R be a left n-$\mathscr{X}$-coherent ring and $\mathscr{X}$ be a class of R-modules. Then, for any R-module G, the following assertions are equivalent:*

(1) *G is Gorenstein n-$\mathscr{X}$-injective;*

(2) *There is an exact sequence $\cdots \to A_1 \to A_0 \to G \to 0$ of R-modules, where every A_i is n-$\mathscr{X}$-injective;*

(3) *There is a short exact sequence $0 \to L \to M \to G \to 0$ of R-modules, where M is n-$\mathscr{X}$-injective and L is Gorenstein n-$\mathscr{X}$-injective.*

Proof. $(1) \Longrightarrow (2)$ and $(1) \Longrightarrow (3)$ follow from the definition.

$(2) \Longrightarrow (1)$There is an exact sequence

$$0 \longrightarrow G \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots$$

where every I^i is injective for any $i \geq 0$. By Remark 3.4, each I^i is n-$\mathscr{X}$-injective. So, an exact sequence

$$\cdots \longrightarrow A_1 \longrightarrow A_0 \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots$$

of n-$\mathscr{X}$-injective modules exists, where $G = \ker(I^0 \to I^1)$. Therefore, G is Gorenstein n-$\mathscr{X}$-injective by Theorem 4.4.

(3) $\Longrightarrow$ (2) Assume there is an exact sequence

$$0 \longrightarrow L \longrightarrow M \longrightarrow G \longrightarrow 0, \quad (1)$$

where M is n-$\mathscr{X}$-injective and L is Gorenstein n-$\mathscr{X}$-injective. Since L is Gorenstein n-$\mathscr{X}$-injective, there is an exact sequence

$$\cdots \longrightarrow A_2^{'} \longrightarrow A_1^{'} \longrightarrow A_0^{'} \longrightarrow L \longrightarrow 0, \quad (2)$$

where every $A_i^{'}$ is n-$\mathscr{X}$-injective. Assembling the sequences (1) and (2), we get the exact sequence

$$\cdots \longrightarrow A_2^{'} \longrightarrow A_1^{'} \longrightarrow A_0^{'} \longrightarrow M \longrightarrow G \longrightarrow 0,$$

where M and $A_i^{'}$ are n-$\mathscr{X}$-injective, as desired. ■

Corollary 4.6. *Let R be a left n-$\mathscr{X}$-coherent ring and $\mathscr{X}$ be a class of R-modules. Then, for any right R-module G, the following assertions are equivalent:*

(1) *G is Gorenstein n-$\mathscr{X}$-flat;*

(2) *There is an exact sequence $0 \to G \to B^0 \to B^1 \to \cdots$ of right R-modules, where every B^i is n-$\mathscr{X}$-flat;*

(3) *There is a short exact sequence $0 \to G \to M \to L \to 0$ of right R-modules, where M is n-$\mathscr{X}$-flat and L is Gorenstein n-$\mathscr{X}$-flat.*

Proof. (1) $\Longrightarrow$ (2) and (1) $\Longrightarrow$ (3) follow from the definition.

(2) $\Longrightarrow$ (1) For any right R-module G, there is an exact sequence

$$\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow G \longrightarrow 0,$$

where any P_i is flat for any $i \geq 0$. By Remark 3.4, every P_i is n-$\mathscr{X}$-flat. Thus, there is an exact sequence

$$\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow B^0 \longrightarrow B^1 \longrightarrow \cdots$$

of n-$\mathscr{X}$-flat right modules, where $G = \ker(B^0 \to B^1)$. Therefore, by Theorem 4.4, G is Gorenstein n-$\mathscr{X}$-flat.

(3) $\Longrightarrow$ (2) Assume there is an exact sequence

$$0 \longrightarrow G \longrightarrow M \longrightarrow L \longrightarrow 0, \ \ (1)$$

where M is n-$\mathscr{X}$-flat and L is Gorenstein n-$\mathscr{X}$-flat. Since L is Gorenstein n-$\mathscr{X}$-flat, there is an exact sequence

$$0 \longrightarrow L \longrightarrow F^0 \longrightarrow F^1 \longrightarrow F^2 \longrightarrow \cdots, \ \ (2)$$

where every F^i is n-$\mathscr{X}$-flat. Assembling the sequences (1) and (2), we get the exact sequence

$$0 \longrightarrow G \longrightarrow M \longrightarrow F^0 \longrightarrow F^1 \longrightarrow F^2 \longrightarrow \cdots,$$

where M and any F^i are n-$\mathscr{X}$-flat, as desired. ∎

Proposition 4.7. *Let $\mathscr{X}$ be a class of R-modules. Then:*

(1) *Every direct product of Gorenstein n-$\mathscr{X}$-injective R-modules is a Gorenstein n-$\mathscr{X}$-injective R-module.*

(2) *Every direct sum of Gorenstein n-$\mathscr{X}$-flat right R-modules is a Gorenstein n-$\mathscr{X}$-flat R-module.*

Proof. (1) Let $U \in \mathscr{X}_{\mathrm{n}}$ and $\{A_i\}_{i\in I}$ be a family of n-$\mathscr{X}$-injective R-modules. Then, by [1, Lemma 2.7], $\prod A_i$ is n-$\mathscr{X}$-injective. So, if $\{G_i\}_{i\in I}$ is a family of Gorenstein n-$\mathscr{X}$-injective R-modules, then the following corresponding exact sequences of n-$\mathscr{X}$-injective R-modules

$$\mathbf{A_i} = \cdots \longrightarrow (A_i)_1 \longrightarrow (A_i)_0 \longrightarrow (A_i)^0 \longrightarrow (A_i)^1 \longrightarrow \cdots,$$

where $G_i = \ker((A_i)^0 \to (A_i)^1)$, induce the following exact sequence of n-$\mathscr{X}$-injective R-modules:

$$\prod_{i\in I} \mathbf{A_i} = \cdots \longrightarrow \prod_{i\in I}(A_i)_1 \longrightarrow \prod_{i\in I}(A_i)_0 \longrightarrow \prod_{i\in I}(A_i)^0 \longrightarrow \prod_{i\in I}(A_i)^1 \longrightarrow \cdots,$$

where $\prod_{i\in I} G_i = \ker(\prod_{i\in I}(A_i)^0 \to \prod_{i\in I}(A_i)^1)$. If K_{n-1} is special $\mathscr{X}$-presented, then

$$\mathrm{Hom}_R(K_{n-1}, \prod_{i\in I} \mathbf{A_i}) \cong \prod_{i\in I} \mathrm{Hom}_R(K_{n-1}, \mathbf{A_i}).$$

By hypothesis, $\mathrm{Hom}_R(K_{n-1}, \mathbf{A_i})$ is exact, and consequently $\prod_{i\in I} G_i$ is Gorenstein n-$\mathscr{X}$-injective.

(2) Let $U \in \mathscr{X}_{\mathrm{n}}$ and $\{I_i\}_{i\in J}$ be a family of n-$\mathscr{X}$-flat right R-modules. Then, by [1, Lemma 2.7], $\bigoplus_{i\in J} I_i$ is n-$\mathscr{X}$-flat. So, if $\{G_i\}_{i\in J}$ is a family of Gorenstein n-$\mathscr{X}$-flat right R-modules, then the following corresponding exact sequences of n-$\mathscr{X}$-flat right R-modules

$$\mathbf{I_i} = \cdots \longrightarrow (I_i)_1 \longrightarrow (I_i)_0 \longrightarrow (I_i)^0 \longrightarrow (I_i)^1 \longrightarrow \cdots,$$

where $G_i = \ker((I_i)^0 \to (I_i)^1)$, induces the following exact sequence of n-$\mathscr{X}$-flat right R-modules:

$$\bigoplus_{i\in J} \mathbf{I_i} = \cdots \longrightarrow \bigoplus_{i\in J}(I_i)_1 \longrightarrow \bigoplus_{i\in J}(I_i)_0 \longrightarrow \bigoplus_{i\in J}(I_i)^0 \longrightarrow \bigoplus_{i\in J}(I_i)^1 \longrightarrow \cdots,$$

where $\bigoplus_{i\in J} G_i = \ker((\bigoplus_{i\in J} I_i)^0 \to (\bigoplus_{i\in J} I_i)^1)$. If K_{n-1} is special $\mathscr{X}$-presented, then

$$(\bigoplus_{i\in J} \mathbf{I_i} \otimes_R K_{n-1}) \cong \bigoplus_{i\in J}(\mathbf{I_i} \otimes_R K_{n-1}).$$

By hypothesis, $(\mathbf{I_i} \otimes_R K_{n-1})$ is exact, and consequently $\bigoplus_{i\in J} G_i$ is Gorenstein n-$\mathscr{X}$-flat. ∎

Now, we study the Gorenstein n-$\mathscr{X}$-injectivity and Gorenstein n-$\mathscr{X}$-flatness of modules in short exact sequences.

Proposition 4.8. *Let R be a left n-$\mathscr{X}$-coherent ring and $\mathscr{X}$ be a class of R-modules. Then:*

(1) *Let $0 \to A \to G \to N \to 0$ be an exact sequence of R-modules. If A and N are Gorenstein n-$\mathscr{X}$-injective, then G is Gorenstein n-$\mathscr{X}$-injective.*

(2) *Let $0 \to K \to G \to B \to 0$ be an exact sequence of right R-modules. If K and B are Gorenstein n-$\mathscr{X}$-flat, then G is Gorenstein n-$\mathscr{X}$-flat.*

Proof. (1) Since A and N are Gorenstein n-$\mathscr{X}$-injective, by Corollary 4.5, there exist exact sequences $0 \to K \to A_0 \to A \to 0$ and $0 \to L \to N_0 \to N \to 0$ of R-modules, where A_0 and N_0 are n-$\mathscr{X}$-injective and also, K and L are Gorenstein n-$\mathscr{X}$-injective. Now, we consider the following commutative diagram:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & K \oplus L & \longrightarrow & L & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_0 & \longrightarrow & A_0 \oplus N_0 & \longrightarrow & N_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & G & \longrightarrow & N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

The exactness of the middle horizontal sequence where A_0 and N_0 are n-$\mathscr{X}$-injective, implies that $A_0 \oplus N_0$ is n-$\mathscr{X}$-injective by [1, Lemma 2.7]. Also, $K \oplus L$ is Gorenstein n-$\mathscr{X}$-injective by Proposition 4.7(1). Hence, from the middle vertical sequence and Corollary 4.5, we deduce that G is Gorenstein n-$\mathscr{X}$-injective.

(2) Since K and B are Gorenstein n-$\mathscr{X}$-flat, by Corollary 4.6, there exist exact sequences $0 \to K \to K_0 \to L_1 \to 0$ and $0 \to B \to B_0 \to L_1^{'} \to 0$ of R-modules, where K_0 and B_0 are n-$\mathscr{X}$-flat and also, L_1 and $L_1^{'}$ are Gorenstein n-$\mathscr{X}$-flat. Now, we consider the following commutative diagram:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & G & \longrightarrow & B & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_0 & \longrightarrow & K_0 \oplus B_0 & \longrightarrow & B_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L_1 & \longrightarrow & L_1 \oplus L_1^{'} & \longrightarrow & L_1^{'} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

The exactness of the middle horizontal sequence with K_0 and B_0 are n-$\mathscr{X}$-flat, implies that $K_0 \oplus B_0$ is n-$\mathscr{X}$-flat by [1, Lemma 2.7]. Also, $L_1 \oplus L_1^{'}$ is Gorenstein n-$\mathscr{X}$-flat by Proposition 4.7(2). Hence from the middle vertical sequence and Corollary 4.6, we deduce that G is Gorenstein n-$\mathscr{X}$-flat. ∎

The left n-$\mathscr{X}$-injective dimension of an R-module M, denoted by $\mathrm{id}_{\mathscr{X}_\mathrm{n}}(M)$, is defined to be the least non-negative integer m such that $\mathrm{Ext}_R^{n+m+1}(U, M) = 0$ for any $U \in \mathscr{X}_\mathrm{n}$. The left n-$\mathscr{X}$-flat dimension of a right R-module M, denoted by $\mathrm{fd}_{\mathscr{X}_\mathrm{n}}(M)$, is defined to be the least non-negative integer m such that $\mathrm{Tor}^R_{n+m+1}(M, U) = 0$ for any $U \in \mathscr{X}_\mathrm{n}$. If G is Gorenstein n-$\mathscr{X}$-injective, then $\mathrm{id}_{\mathscr{X}_\mathrm{n}}(G) = m$ if there is an exact sequence

$$0 \longrightarrow A_m \longrightarrow \cdots \longrightarrow A_1 \longrightarrow A_0 \longrightarrow G \longrightarrow 0$$

or an exact sequence

$$0 \longrightarrow G \longrightarrow A^0 \longrightarrow A^1 \longrightarrow \cdots \longrightarrow A^m \longrightarrow 0$$

of n-$\mathscr{X}$-injective R-modules. Similarly, if G is Gorenstein n-$\mathscr{X}$-flat and $\mathrm{fd}_{\mathscr{X}_\mathrm{n}}(G) = m$, then the above exact sequences for n-$\mathscr{X}$-flat right R-modules exists.

The following theorems are generalizations of Corollaries 4.5 and 4.6 and Proposition 4.8.

Theorem 4.9. *Let R be a left n-$\mathscr{X}$-coherent ring and $\mathscr{X}$ be a class of R-modules which is closed under kernels of epimorphisms. Then, for every R-module G, the following statements are equivalent:*

(1) *G is Gorenstein n-$\mathscr{X}$-injective;*

(2) *There exists an n-$\mathscr{X}$-injective resolution of G:*

$$\cdots \xrightarrow{f_3} A_2 \xrightarrow{f_2} A_1 \xrightarrow{f_1} A_0 \xrightarrow{f_0} G \longrightarrow 0$$

such that $\bigoplus_{i=0}^{\infty} \mathrm{Im}(f_i)$ is Gorenstein n-$\mathscr{X}$-injective;

(3) *There exists an exact sequence*

$$\cdots \xrightarrow{f_3} A_2 \xrightarrow{f_2} A_1 \xrightarrow{f_1} A_0 \xrightarrow{f_0} G \longrightarrow 0$$

of R-modules, where A_i has finite n-$\mathscr{X}$-injective dimension for any $i \geq 0$, such that $\bigoplus_{i=0}^{\infty} \mathrm{Im}(f_i)$ is Gorenstein n-$\mathscr{X}$-injective.

Proof. (1) $\Longrightarrow$ (2) By Corollary 4.5, there is an exact sequence

$$\cdots \xrightarrow{f_3} A_2 \xrightarrow{f_2} A_1 \xrightarrow{f_1} A_0 \xrightarrow{f_0} G \longrightarrow 0,$$

where every A_i is n-$\mathscr{X}$-injective. Consider the following exact sequences:

$$\begin{array}{c}\cdots \longrightarrow A_2 \longrightarrow A_1 \longrightarrow A_0 \longrightarrow \mathrm{Im}(f_0) \longrightarrow 0,\\ \cdots \longrightarrow A_3 \longrightarrow A_2 \longrightarrow A_1 \longrightarrow \mathrm{Im}(f_1) \longrightarrow 0,\\ \vdots \qquad \vdots \qquad \vdots \qquad\qquad \vdots\end{array}$$

By Proposition 3.6, $\bigoplus_{i\in I} A_i$ is n-$\mathscr{X}$-injective. Thus, there exists an exact sequence

$$\cdots \longrightarrow \bigoplus_{i\geq 2} A_i \longrightarrow \bigoplus_{i\geq 1} A_i \longrightarrow \bigoplus_{i\geq 0} A_i \longrightarrow \bigoplus_{i=0}^{\infty} \mathrm{Im}(f_i) \longrightarrow 0$$

of n-$\mathscr{X}$-injective R-modules. Consequently, Corollary 4.5 implies that $\bigoplus_{i=0}^{\infty} \mathrm{Im}(f_i)$ is Gorenstein n-$\mathscr{X}$-injective.

(2) $\Longrightarrow$ (3) trivial.

(3) $\Longrightarrow$ (1) Let

$$\cdots \xrightarrow{f_3} A_2 \xrightarrow{f_2} A_1 \xrightarrow{f_1} A_0 \xrightarrow{f_0} G \longrightarrow 0$$

be an exact sequence of R-modules, where A_i has finite n-$\mathscr{X}$-injective dimension. By Corollary 4.5, it is sufficient to prove that A_i is n-$\mathscr{X}$-injective for any $i \geq 0$. Consider, the short exact sequence $0 \to \mathrm{Im}(f_{i+1}) \to A_i \to \mathrm{Im}(f_i) \to 0$ for any $i \geq 0$. Therefore, the short exact sequence $0 \to \bigoplus_{i=0}^{\infty} \mathrm{Im}(f_{i+1}) \to \bigoplus_{i=0}^{\infty} A_i \to \bigoplus_{i=0}^{\infty} \mathrm{Im}(f_i) \to 0$ exists. By (3) and Proposition 4.8(1), $\bigoplus_{i=0}^{\infty} A_i$ is Gorenstein n-$\mathscr{X}$-injective. Also, $\bigoplus_{i=0}^{\infty} A_i$ has finite n-$\mathscr{X}$-injective dimension. If $\mathrm{id}_{\mathscr{X}_{\mathrm{n}}}(\bigoplus_{i=0}^{\infty} A_i) = k$, then there exists an n-$\mathscr{X}$-injective resolution of $\bigoplus_{i=0}^{\infty} A_i$:

$$0 \longrightarrow B_k \longrightarrow B_{k-1} \longrightarrow \cdots \longrightarrow B_0 \longrightarrow \bigoplus_{i\in I} A_i \longrightarrow 0.$$

Let $L_{k-1} = \ker(B_{k-1} \to B_{k-2})$ and $U \in \mathscr{X}_{\mathrm{n}}$. Then, the exact sequence $0 \to B_k \to B_{k-1} \to L_{k-1} \to 0$ induces the following exact sequence:

$$0 = \mathrm{Ext}_R^n(U, B_{k-1}) \longrightarrow \mathrm{Ext}_R^n(U, L_{k-1}) \longrightarrow \mathrm{Ext}_R^{n+1}(U, B_k) \longrightarrow \cdots .$$

By hypothesis, B_k is $(n+1)$-$\mathscr{X}$-injective, and also $U \in \mathscr{X}_{\mathrm{n+1}}$ since R is n-$\mathscr{X}$-coherent. So $\mathrm{Ext}_R^{n+1}(U, B_k) = 0$, and hence $\mathrm{Ext}_R^n(U, L_{k-1}) = 0$. Thus, L_{k-1} is n-$\mathscr{X}$-injective. Then, with the same process, we get that $\bigoplus_{i=0}^{\infty} A_i$ is n-$\mathscr{X}$-injective, and so by Proposition 3.6, A_i is n-$\mathscr{X}$-injective for any $i \geq 0$. ∎

For the following theorem, the proof is similar to that of $(1) \Longrightarrow (2)$, $(2) \Longrightarrow (3)$ and $(3) \Longrightarrow (1)$ in Theorem 4.9.

Theorem 4.10. *Let R be a left n-$\mathscr{X}$-coherent ring and $\mathscr{X}$ be a class of R-modules which is closed under kernels of epimorphisms. Then, for every right R-module G, the following statements are equivalent:*

(1) *G is Gorenstein n-$\mathscr{X}$-flat;*

(2) *There exists the following right n-$\mathscr{X}$-flat resolution of G:*

$$0 \longrightarrow G \xrightarrow{f^0} I^0 \xrightarrow{f^1} I^1 \xrightarrow{f^2} \cdots$$

such that $\bigoplus_{i=0}^{\infty} \mathrm{Im}(f^i)$ is Gorenstein n-$\mathscr{X}$-flat;

(3) *There exists an exact sequence*

$$0 \longrightarrow G \xrightarrow{f^0} I^0 \xrightarrow{f^1} I^1 \xrightarrow{f^2} \cdots$$

of right R-modules, where I_i has finite n-$\mathscr{X}$-flat dimension for any $i \geq 0$, such that $\bigoplus_{i=0}^{\infty} \mathrm{Im}(f^i)$ is Gorenstein n-$\mathscr{X}$-flat.

5 $\mathscr{X}$-FC-rings

A ring R is called left $\mathscr{X}$-FC-ring if R is self left n-$\mathscr{X}$-injective and left n-$\mathscr{X}$-coherent. In this section, we investigate properties of Gorenstein n-$\mathscr{X}$-injective and n-$\mathscr{X}$-flat modules over $\mathscr{X}$-FC-rings, thus generalizing several classical results. Notice that the notion of $\mathscr{X}$-FC-ring generalizes the classical notions of quasi-Frobenius and FC (i.e., IF) rings.

It is well-known that quasi-Frobenius (resp., FC) rings can be seen as rings over which all modules are Gorenstein injective (resp., Gorenstein FP-injective). Here, we extend this fact as well as other known ones to our new context.

Proposition 5.1. *Let $\mathscr{X}$ be a class of R-modules. Then, every R-module is Gorenstein n-$\mathscr{X}$-injective if and only if every projective R-module is n-$\mathscr{X}$-injective and for any R-module N, $\mathrm{Hom}_R(-, N)$ is exact with respect to all special short exact sequences of $\mathscr{X}_{\mathrm{n}}$ with modules of finite projective dimension.*

Proof. ($\Longrightarrow$) Let M be a projective R-module. Then, by hypothesis, M is Gorenstein n-$\mathscr{X}$-injective. So, the following n-$\mathscr{X}$-injective resolution of M exists:

$$\cdots \longrightarrow A_1 \longrightarrow A_0 \longrightarrow M \longrightarrow 0.$$

Since M is projective, M is n-$\mathscr{X}$-injective as a direct summand of A_0. Also, by hypothesis and Definition 4.1, $\mathrm{Hom}_R(-, N)$ is exact with respect to all special short exact sequences with modules of finite projective dimension since every R-module N is Gorenstein n-$\mathscr{X}$-injective.

($\Longleftarrow$) Choose an injective resolution of G: $0 \to G \to E^0 \to E^1 \to \cdots$ and a projective resolution of G: $\cdots \to F_1 \to F_0 \to G \to 0$, where every F_i is n-$\mathscr{X}$-injective by hypothesis. Assembling these resolutions, we get, by Remark 3.4, the following exact sequence of n-$\mathscr{X}$-injective R-modules:

$$\mathbf{A} = \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow E^0 \longrightarrow E^1 \longrightarrow \cdots ,$$

where $G = \ker(E^0 \to E^1)$, $K^i = \ker(E^i \to E^{i+1})$ and $K_i = \ker(F_i \to F_{i-1})$ for any $i \geq 1$. Let K_{n-1} be a special $\mathscr{X}$-presented module with $\mathrm{pd}_R(K_{n-1}) < \infty$. Then, by hypothesis, we have:

$$\mathrm{Ext}^1_R(K_{n-1}, G) = \mathrm{Ext}^1_R(K_{n-1}, K_i) = \mathrm{Ext}^1_R(K_{n-1}, K^i) = 0.$$

So, $\mathrm{Hom}_R(K_{n-1}, \mathbf{A})$ is exact, and hence G is Gorenstein n-$\mathscr{X}$-injective. ■

Proposition 5.2. *Let $\mathscr{X}$ be a class of R-modules. Then, every right R-module is Gorenstein n-$\mathscr{X}$-flat if and only if every injective right R-module is n-$\mathscr{X}$-flat and for any R-module N, $N \otimes_R -$ is exact with respect to all special short exact sequences of $\mathscr{X}_{\mathrm{n}}$ with modules of finite projective dimension.*

Proof. Similar to the proof of Proposition 5.1. ■

Theorem 5.3. *Let R be a left n-$\mathscr{X}$-coherent ring and $\mathscr{X}$ be a class of R-modules. Then, the following statements are equivalent:*

(1) *Every R-module is Gorenstein n-$\mathscr{X}$-injective;*

(2) *Every projective R-module is n-$\mathscr{X}$-injective;*

(3) *R is self left n-$\mathscr{X}$-injective.*

Proof. (1) $\Longrightarrow$ (2) and (2) $\Longrightarrow$ (3) hold by Proposition 5.1.

(3) $\Longrightarrow$ (1) Let G be an R-module and $\cdots \to F_1 \to F_0 \to G \to 0$ be any free resolution of G. Then, by Proposition 3.6, each F_i is n-$\mathscr{X}$-injective. Hence, Corollary 4.5 completes the proof. ■

Examples 5.4. *Let $R = k[x^3, x^2, x^2y, xy^2, xy, y^2, y^3]$ be a ring and $\mathscr{X}$ a class of all 1-presented R-modules. We claim that R is not 1-$\mathscr{X}$-injective. Suppose to the contrary, R is 1-$\mathscr{X}$-injective. We have $\frac{R}{Rx^2}$ is special $\mathscr{X}$-presented since $Rx^2 \cong R$ is special $\mathscr{X}$-generated. Also, $\mathrm{pd}_R(\frac{R}{Rx^2}) < \infty$. So by Proposition 5.1 and Theorem 5.3, $\frac{R}{Rx^2}$ is projective. Therefore, the exact sequence $0 \to Rx^2 \to R \to \frac{R}{Rx^2} \to 0$ splits. Thus, Rx^2 is a direct summand of R and so, x^2 is an idempotent, a contradiction.*

Let $\mathscr{X}$ be a class of graded R-modules. Then, a graded ring R will be called n-gr-regular if and only if it is n-$\mathscr{X}$-regular if and only if every n-presented R-module in $\mathscr{X}$ is projective if and only if every R-module in $\mathscr{X}$ is n-$\mathscr{X}$-injective if and only if every right R-module in $\mathscr{X}$ is n-$\mathscr{X}$-flat. This is a generalization of [19, Proposition 3.11]. Notice that, when $n = 1$, then R is gr-regular if and only if 1-$\mathscr{X}$-regular, see [18].

The following example show that, for some of class $\mathscr{X}$ of R-modules and any $m > n$, every Gorenstein n-$\mathscr{X}$-injective (resp., flat) module is Gorenstein m-$\mathscr{X}$-injective. But, Gorenstein m-$\mathscr{X}$-injectivity (resp., flatness) does not imply, in general, Gorenstein n-$\mathscr{X}$-injectivity (resp., flatness).

Examples 5.5. *(1) Let R be a graded ring and $\mathscr{X}$ a class of graded R-module. Then, for any $m > n$, every Gorenstein n-$\mathscr{X}$-injective (resp., flat) module is Gorenstein m-$\mathscr{X}$-injective (resp., flat), since by [20, Remark 3.5], every n-$\mathscr{X}$-injective (resp., flat) module is m-$\mathscr{X}$-injective (resp., flat).*

(2) Let $R = k[X]$, where k is a field, and $\mathscr{X}$ a class of graded R-modules. Then, by Remark 4.2, every graded left (resp., right) R-module is Gorenstein 2-$\mathscr{X}$-injective (resp., flat) since every 2-presented graded R-module is projective. We claim that there is a graded left (resp., right) R-module N so that N is not Gorenstein 1-$\mathscr{X}$-injective (resp., flat). Suppose to the contrary, every graded left (resp., right) R-module is Gorenstein 1-$\mathscr{X}$-injective (resp. flat). If U is a finitely presented graded module, then the special exact sequence $0 \to L \to F_0 \to U \to 0$ of graded modules exists. So by Proposition 5.1 (resp., Proposition 5.2), U is projective and it follows that R is 1- $\mathscr{X}$-regular or $\mathscr{X}$-regular, contradiction, see [20, Example 3.6].

Proposition 5.6. *Let $\mathscr{X}$ be a class of R-modules. Then, the following statements hold:*

(1) *If G is a Gorenstein injective R-module, then $\mathrm{Hom}_R(-, G)$ is exact with respect to all special short exact sequences with modules of finite projective dimension.*

(2) *If G is a Gorenstein flat right R-module, then $G \otimes_R -$ is exact with respect to all special short exact sequences with modules of finite flat dimension.*

Proof. (1) Let $0 \to K_n \to P_n \to K_{n-1} \to 0$ be a special short exact sequence of $U \in \mathscr{X}_{\mathrm{n}}$. It is clear that $\mathrm{pd}_R(U) = m < \infty$ since $\mathrm{pd}_R(K_{n-1}) < \infty$. Also, let G be Gorenstein injective. Then, the following injective resolution of G exists:

$$0 \longrightarrow N \longrightarrow A_{m-1} \longrightarrow \cdots \longrightarrow A_0 \longrightarrow G \longrightarrow 0.$$

So, $\mathrm{Ext}_R^{n+i}(U, A_j) = 0$ for every $0 \leq j \leq m-1$ and any $i \geq 0$. Thus, we deduce that $\mathrm{Ext}_R^{n+i}(U, G) \cong \mathrm{Ext}_R^{m+n+i}(U, N) = 0$ for any $i \geq 0$. So, $\mathrm{Ext}_R^1(K_{n-1}, G) \cong \mathrm{Ext}_R^n(U, G) = 0$.

(2) The proof is similar to the one above. ∎

Now we can state the main result of this section.

Theorem 5.7. *Let R be a left n-$\mathscr{X}$-coherent ring and $\mathscr{X}$ be a class of R-modules. Then, the following statements are equivalent:*

(1) *R is self left n-$\mathscr{X}$-injective;*

(2) *Every Gorenstein n-$\mathscr{X}$-flat R-module is Gorenstein n-$\mathscr{X}$-injective;*

(3) *Every Gorenstein flat R-module is Gorenstein n-$\mathscr{X}$-injective;*

(4) *Every flat R-module is Gorenstein n-$\mathscr{X}$-injective;*

(5) *Every Gorenstein projective R-module is Gorenstein n-$\mathscr{X}$-injective;*

(6) *Every projective R-module is Gorenstein n-$\mathscr{X}$-injective;*

(7) *Every Gorenstein injective right R-module is Gorenstein n-$\mathscr{X}$-flat;*

(8) *Every injective right R-module is Gorenstein n-$\mathscr{X}$-flat;*

(9) *Every Gorenstein 1-$\mathscr{X}$-injective right R-module is Gorenstein n-$\mathscr{X}$-flat;*

(10) *Every Gorenstein n-$\mathscr{X}$-injective right R-module is Gorenstein n-$\mathscr{X}$-flat.*

Proof. (1) $\Longrightarrow$ (2), (1) $\Longrightarrow$ (3), (1) $\Longrightarrow$ (4), (1) $\Longrightarrow$ (5) and (1) $\Longrightarrow$ (6) follow immediately from Theorem 5.3.

(3) $\Longrightarrow$ (4), (4) $\Longrightarrow$ (6) and (5) $\Longrightarrow$ (6) are trivial.

(3) $\Longrightarrow$ (1) Assume that G is a projective R-module. Then, G is flat and so G is Gorenstein n-$\mathscr{X}$-injective by (3). So, similar to the proof of ($\Longrightarrow$) of Proposition 5.1, G is n-$\mathscr{X}$-injective. Thus, the assertion follows from Theorem 5.3.

(6) $\Longrightarrow$ (1) This is similar to the proof of (3) $\Longrightarrow$ (1).

(1) $\Longrightarrow$ (9) By Theorem 3.5, every 1-$\mathscr{X}$-injective right R-module is n-$\mathscr{X}$-flat. Suppose that G is Gorenstein 1-$\mathscr{X}$-injective. So, an exact sequence

$$\mathbf{M} = \cdots \longrightarrow M_1 \longrightarrow M_0 \longrightarrow M^0 \longrightarrow M^1 \longrightarrow \cdots ,$$

of n-$\mathscr{X}$-flat right R-modules exists, where $G = \ker(M^0 \to M^1)$. Let K_{n-1} be special $\mathscr{X}$-presented with $\mathrm{f.d(K_{n-1})} < \infty$. Then, similar to the proof of Theorem 4.4(1), $\mathbf{M} \otimes_R K_{n-1}$ is exact, and hence G is Gorenstein n-$\mathscr{X}$-flat.

(9) $\Longrightarrow$ (7) By Remark 3.4, every injective right R-module is 1-$\mathscr{X}$-injective. So, if G is Gorenstein injective, then an exact sequence

$$\mathbf{E} = \cdots \longrightarrow E_1 \longrightarrow E_0 \longrightarrow E^0 \longrightarrow E^1 \longrightarrow \cdots$$

of 1-$\mathscr{X}$-injective right R-modules exists, where $G = \ker(E^0 \to E^1)$. So, if $U \in \mathscr{X}_1$ with $\mathrm{pd}(U) < \infty$, then U is special $\mathscr{X}$-presented and by Proposition 5.6, $\mathrm{Hom}_R(U, \mathbf{E})$ is exact. Therefore, G is Gorenstein 1-$\mathscr{X}$-injective.

(7) $\Longrightarrow$ (8) is trivial since every injective R-module is Gorenstein injective.

(8) $\Longrightarrow$ (1) Let M be an injective right R-module. Since M is Gorenstein n-$\mathscr{X}$-flat, we have an exact sequence:

$$\mathbf{M} = \cdots \longrightarrow M_1 \longrightarrow M_0 \longrightarrow M^0 \longrightarrow M^1 \longrightarrow \cdots ,$$

where any M_i is n-$\mathscr{X}$-flat and $M = \ker(M^0 \to M^1)$. Then, the split exact sequence $0 \to M \to M^0 \to L \to 0$ implies that M is n-$\mathscr{X}$-flat, and hence by Theorem 3.5, we deduce that R is self left n-$\mathscr{X}$-injective.

$(1) \Longrightarrow (10)$ Suppose that G is a Gorenstein n-$\mathscr{X}$-injective right R-module. By Theorem 3.5(6), every n-$\mathscr{X}$-injective right R-module is n-$\mathscr{X}$-flat. Thus, an exact sequence

$$\mathbf{N} = \cdots \longrightarrow N_1 \longrightarrow N_0 \longrightarrow N^0 \longrightarrow N^1 \longrightarrow \cdots$$

of n-$\mathscr{X}$-flat right R-modules exists, where $G = \ker(N^0 \to N^1)$. Then, similar to the proof of Theorem 4.4(1), (10) follows.

$(10) \Longrightarrow (7)$ is clear. ■

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