Haar wavelet approximate solutions for the generalized Lane–Emden equations arising in astrophysics

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A B S T R A C T
This paper provides a technique to investigate the solutions of generalized nonlinear singular Lane–Emden equations of first and second kinds by using a Haar wavelet quasi-linearization approach. The Lane–Emden equation is widely studied and is treated as a challenging equation in the theory of stellar structure for the gravitational potential of a self-gravitating, spherically symmetric polytropic fluid which models the thermal behavior of a spherical cloud of gas acting under the mutual attraction of its molecules and subject to the classical laws of thermodynamics. The proposed method is based on the quasi-linearization approximation and replacement of an unknown function through a truncated series of Haar wavelet series of the function. The method is shown to be very reliable and easy to capture the solutions of generalized nonlinear singular Lane–Emden equations. The applicability of the method is shown by numerical tests on various cases of the generalized Lane–Emden equation and solutions are also reported in the neighborhood of a singular point.

1. Introduction

The generalized nonlinear singular Lane–Emden equation is a very well known equation in the theory of stellar structure and models many phenomena in mathematical physics and astrophysics [1]. It is a nonlinear differential equation which describes the equilibrium density distribution in self-gravitating sphere of polytropic isothermal gas and has a regular singularity at the origin. This equation was first studied by the astrophysicists Jonathan Homer Lane and Robert Emden [2] who considered the temperature variation of a spherical gas cloud under the mutual attraction of its molecules and subject to the laws of classical thermodynamics [3]. The polytropic theory of stars essentially follows out of thermodynamic considerations that deal with the issue of energy transport, through the transfer of materials between different levels of the star and modeling of clusters of galaxies. Mostly problems with regard to the diffusion of heat perpendicular to the surfaces of parallel planes are represented by the heat equation. In particular for a polytropic equation of state, the Lane–Emden equation arises. Consider the generalized Lane–Emden equation

where \( \omega(\xi) \) represents the temperature. For the case of steady-state, consider \( u(\xi) = 0 \) in Eq. (1.1) and it becomes

subject to the conditions:

The value of \( \alpha \) determines geometrically the shape of Eq. (1.1). For well defined geometries \( \alpha = 0 \) represents an infinite slab, \( \alpha = 1 \) an infinite circular cylinder and \( \alpha = 2 \) a sphere. When \( \alpha = 2, f(\xi) = 1 \) and \( g(\omega) = \omega^\alpha \). Eq. (1.2) becomes the standard Lane–Emden equation with a polytropic index \( p \). In this context the physically interesting range of \( p \) is \( 0 \leq p \leq 5 \). Fowler considered a generalization of the Lane–Emden equation called the Emden–Fowler equation [4], where \( f(\xi) = \xi^\alpha \) and \( g(\omega) = \omega^p \). Emden studied Richardson’s thermionic theory [5] and derived the equation of isothermal gaseous sphere.

The generalized Lane–Emden equation of first kind [6] is

The real constants \( \alpha, \kappa, n \) and \( p \) are determined from the physics of the problem under investigation. By assuming \( \kappa \neq 0, \kappa \) and \( p \) can
be scaled to ±1. The generalized Lane–Emden equation of second kind follows [7,8]
\[
d^2\phi + \frac{\alpha}{x} d\phi + \frac{k}{x} e^{\rho(x)} = 0.
\]
(1.6)
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Wong [9] found another generalized version by taking \( f(\xi) = \xi^n \), \( g(\omega) = (\text{sgn})/a(\omega) \) in Eq. (1.2). Depending on values of \( \alpha, k, n, \) and \( p, \) the Eq. (1.6) reduces either to the Thomas Fermi equation and its generalizations or to the one dimensional equation of motion with a force depending on the power of the distance plus. Eq. (1.6) is also encountered when spherically symmetric solutions of the Einstein field equations for perfect fluid matter with shear or without.

We provide basis for derivation of Eq. (1.1) in two cases. Dehghan and Shakeri [10] have derived the standard Lane–Emden equation and solved it in order to address the difficulty of a singular point at \( \xi = 0. \) In astrophysics, the Lane–Emden equation can be expressed in the form of Poisson equation as
\[
\nabla^2 \phi = -4\pi G \rho
\]
wherein \( \nabla^2 = \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr}, \rho \) is the density at a distance \( r \) from the center of a spherical cloud of gas and \( \phi \) is the gravitational potential of gas. Eq. (1.7) is governed by the combination of following relations
\[
g = \frac{GM(r)}{r^2} = -\frac{d\phi}{dr} = -\frac{1}{\rho} \frac{d\rho}{dr}, \quad \frac{dM(r)}{dr} = 4\pi \rho r^2
\]
where \( M(r) \) is the mass of sphere and \( P \) is the pressure at radius \( r. \) \( G \) is the gravitational constant = 6.668 \times 10^{-8} \text{ units}. Further \( P \) follows the relation
\[
P = K \rho^\gamma
\]
where \( \gamma \) and \( K \) are empirical constants.

Consider the condition that \( \phi = \phi_0(0), \phi_0 \) is the value of \( \phi \) at the center of sphere, Eq. (1.7) now reduces in the form
\[
\nabla^2 \phi = -a^2 \phi
\]
by considering \( a^2 = (n + 1)K^{-1}G \) \( \text{and} \) \( n = \frac{1}{\gamma - 1}. \)
Taking \( r = \xi / (\phi_0)\gamma^{-1}, \) Eq. (1.10) reduces to the Eq. (1.12) with \( f(\xi) = 1, g(\omega) = a^2 \). The supplementary conditions specified in Eq. (1.13).

In second situation when \( \phi_0 \) is zero, the Poisson equation is to be replaced by
\[
\nabla^2 \phi = -a^2 \phi
\]
Assuming spherical symmetry with \( \phi = K \omega \) and \( r = (\sqrt{K}/a) \xi, \) this equation reduces in Eq. (1.14) with \( p = -1. \) We have found various types of famous Lane–Emden equation in literature and further generalized forms. Wazew [6] applied adomin decomposition method to the Lane–Emden equations with the functions \( f(\xi)g(\omega) = k e^\omega \) and \( f(\xi)g(\omega) = k \omega^p \) but Aslanov [11] extended these functions \( f(\xi)g(\omega) \) in various contexts.

Recently, many analytic and numerical methods have been used to solve Lane–Emden equations. The main difficulty in the solution arises at the singularity of the equation at the origin. Benko et al. [12] studied the power series method to solve Lane–Emden equation, Mohan and Al-Bayaty [13] used backward Euler method, Harley and Momoniat [14] determined invariant boundary conditions of Lane–Emden equations. Yildirim [15] has used the variational iteration method to solve the Emden–Fowler type of equations. Mandelzeig et al. [16] have used quasi-linearization approach to solve Eq. (1.2), Parand et al. [17] proposed an approximation algorithm for the solution of using Hermite functions. Singh et al. [18] using the homotopy analysis method while Ramos [19] presented a series approach on same and made comparisons with homotopy perturbation method. In recent years the wavelet approach is becoming increasingly popular in the field of numerical approximations. Different types of wavelets and approximating functions have been used in numerical solution of boundary value problems. Out of these, Haar wavelets [20–22] are the simplest orthonormal wavelets which have gained popularity among researchers due to their useful properties such as simple applicability, orthogonality and compact support. In most of the cases, the beauty of wavelet approximation is overshadowed by computational cost of the algorithm. Compact support of the Haar–wavelet basis permits straight inclusion of the different types of boundary conditions in the numerical algorithms, due to the linear and piecewise nature, Haar wavelet basis lacks differentiability and hence the integration approach is used instead of the differentiation for calculation of the coefficients. The attributes of other differentiable wavelets like the wavelets of high order spline basis are overshadowed by the computational cost of the algorithms obtained from these wavelets. Yousefi [23] has applied an integral operator to convert Lane–Emden equation to integral equation and solved by Galerkin and collocation methods with Legendre wavelets. Galerkin method creates numerically complications when nonlinearities are treated in a wavelet subspace for solving differential equations because there integrals of products of wavelets and their derivatives must be computed. This can be done by introducing the connection coefficients [20] which is applicable only for a narrow class of equations. But there is no need of connection coefficients in case of collocation method. The approximation of a solution of the differential equation by Haar wavelets has an error. In order to minimize this error we choose collocation method at the collocation points, where approximation is exact. On the other hand in Galerkin method, the error is orthogonal to each Haar wavelet selected (for more details see [24]). One of the advantages of the wavelet method is its ability to detect singularities, local high frequencies, irregular structure and transient phenomena exhibited by the analyzed function. In 1910, Alfred Haar introduced a Haar function which presents a rectangular pulse pair. It is not possible to apply the Haar wavelet directly for solving differential equations because Haar wavelet is a discontinuous function, so is not a differentiable everywhere. There are some possibilities to come out from this impasse. First the piecewise constant Haar function can be regularized with interpolation splines, this technique has been applied by Cattani [21]. Cattani observed that computational complexity can be reduced if the interval of integration is divided into some segments and a method called piecewise constant approximation can be applied. The second possibility is to make use of the integral method by which the highest derivative appearing in the differential equation is expanded into the Haar series. This approximation is integrated while the boundary conditions are incorporated by integration constants. This approach has been realized by Chen and Hsiao [22] who first derived a Haar operational matrix for the integrals of the Haar functions and put the application for the Haar analysis into the dynamical systems. Wang [25] and Lepik [26] have proposed a method based on Haar wavelets for solving nonlinear stiff differential equations. In this paper we find the Haar wavelet solution of the more generalized versions of first and second kind type of Lane–Emden equation. Nonlinearity of the system also complicates the solutions. So to solve the nonlinear differential equations Harpreet et al. [27] have proposed the Haar wavelet quasi-linearization technique by using the concept of Chen and Hsiao operational matrix with quasi-linearization process.

To find the solution of a nonlinear differential equation in the neighborhood of a singular point is not easy by available every method. In our proposed method nonlinear part is dealt with
Haar wavelet collocation method to find the solution of generalized Lane–Emden equation in the neighborhood of singular point \( \xi = 0 \). Advantage of this technique is that it can capture the solution of the problem in the neighborhood of singular point. Haar wavelets especially, is the simplest class of basis functions that could be used with main advantage that bear some local properties in singular systems analysis which lead to better condition number of the resulting system. Wavelet can be finer to find the sufficient accuracy in the solution due to its local and global properties and Haar wavelets are easy to handle with collocation method and quasi-linearization process for solving nonlinear singular differential equations than Galerkin method because there is no need of connection coefficients. Moreover, in this approach a linear system is generated that is easy to solve in comparison to the nonlinear system of equations obtained otherwise. The proposed method does not require conversion of a boundary value problem into a system of first order ordinary differential equations by using a procedure like shooting and hence the boundary-value problem is not integrated as an initial value problem with guesses for the unknown initial values. This property of Haar wavelet quasi-linearization method eliminates the possibility of unstable solution due to missing initial condition.

The main aim of this paper is to study applications of the Haar wavelets to capture the solutions of generalized nonlinear singular Lane–Emden equation which is very well known equation in astrophysics. We have used quasi-linearization process and collocation method with Haar wavelets for this aim. The proposed method reduces the problem to a system of algebraic equations and successfully captures the solutions for a target problem. This section is devoted to the introduction of the various forms of generalized nonlinear singular Lane–Emden equation and different approaches to find the solutions. Section 2 depicts the fundamentals of Haar wavelets as construction of wavelets, its properties and operational matrix of derivative as a working tool. The Section 3 reveals that how quasi-linearization works with Haar wavelets for nonlinear singular differential equations and Section 4 discusses the convergence of Haar wavelet method. In Section 5 the applicability of Haar wavelet quasi-linearization method is revealed and numerical results are compared with available solutions in the literature. The conclusion is described in the final section.

2. Fundamentals of Haar wavelets and associated matrix

In this section, we summarize the fundamentals of Haar wavelets. The structure of Haar wavelet family is based on multiresolution analysis [28,29]. A multiresolution analysis (MRA) \( K = \{ V_\ell \subset L_2(\mathbb{R}) : \ell \} \) of \( X \) consisting of a sequences of nested spaces on \( V_\ell \subset V_{\ell+1} \) at different levels \( \ell \) whose union is dense in \( L_2(\mathbb{R}) \). Let \( L_2(\mathbb{R}) \) be the space of functions with finite energy defined over a domain \( \Omega \subset \mathbb{R}^d \) and \( \langle \cdot, \cdot \rangle \) be an inner product on \( X \). Bases of the spaces \( V_\ell \) are formed by the sets of scaling bases functions \( \phi_{\ell,j}(k\xi) \) in complete orthonormal system [30], where \( x(\ell,j) \) is an index set defined over all basis functions on level \( \ell \). The strictly nested structure of the \( V_\ell \) implies the existence of difference spaces \( W_\ell \) such that \( V_\ell \cap W_\ell = V_{\ell+1} \). The spaces \( W_\ell \) are spanned by sets of Haar wavelet basis functions \( h_{\ell,j}(k\xi) \). For all levels \( \ell \), \( V_\ell \) and \( W_\ell \) are subspaces of \( V_{\ell+1} \) implying the existence of refinement relationships.

The basic and simplest form of Haar wavelet is the Haar scaling function that appears in the form of a square wave over the interval \( \xi \in [0, 1] \), denoted by \( h_1(\xi) \) and generally written as

\[
h_1(\xi) = \begin{cases} 1, & 0 \leq \xi < 1 \\ 0, & \text{elsewhere.} \end{cases}
\] (2.12)

The above expression, called Haar father wavelet, is the zeroth level wavelet which has no displacement and dilation of unit magnitude.

According to the concept of MRA [31] as an example the space \( V_j \) can be defined like

\[
V_j = sp(h_j, \ell_{j=0,1,\ldots,2^{j-1}} = W_{j-1} \bigoplus V_{j-1}
= W_{j-1} \bigoplus W_{j-2} \bigoplus V_{j-2} \bigoplus \cdots \\
= \bigoplus_{k=0}^{j-1} W_k \bigoplus V_0.
\] (2.13)

The Haar mother wavelet is the first level Haar wavelet written as the linear combination of the Haar scaling function as

\[
h_2(\xi) = h_1(2\xi) + h_1(2\xi - 1).
\] (2.14)

The following definitions illustrate the translation and dilation of wavelet function for making operational matrix.

Definition 1. Let \( h_1 L^2(\mathbb{R}) \). For \( k \in \mathbb{R} \) let \( T_k : L^2(\mathbb{R}) \to L^2(\mathbb{R}) \) be given by \( (T_k h)(\xi) = h(\xi - k) \) and \( D_k : L^2(\mathbb{R}) \to L^2(\mathbb{R}) \) be given by \( D_k h(\xi) = 2^k \). Operators \( T_k \) and \( D_k \) are called translation and dilation operator.

Definition 2. A function \( h L^2(\mathbb{R}) \) is called an orthonormal wavelet for \( L^2(\mathbb{R}) \) if \( D^m T_k h : j, k, m \in \mathbb{Z} \) is an orthonormal basis for \( L^2(\mathbb{R}) \). Index \( j \) refers to dilation and \( k \) refers to translation.

Each Haar wavelet is composed of a couple of constant steps of opposite sign during its subinterval and is zero elsewhere. The term wavelet is used to refer to a set of orthonormal basis functions generated by dilation and translation of a compactly supported scaling function \( h_1(\xi) \) and associated wavelet \( h_2(\xi) \) associated with a multiresolution analysis of \( L^2(\mathbb{R}) \). Thus we can express the Haar wavelet family as

\[
h_1(\xi) = h_1(2^j \xi - k)
= \begin{cases} 1, & \frac{k}{2^j} \leq \xi < \frac{k + 0.5}{2^j} \\
-1, & \frac{k + 0.5}{2^j} \leq \xi < \frac{k + 1}{2^j} \\
0, & \text{elsewhere.} \end{cases}
\] (2.15)

where \( 0 \leq k < 2^j - 1 \). For any fixed level \( m \), there are \( m \) series of \( j \) to fill the interval corresponding to that level and for a provided \( J \), the index number \( i \) can reach the maximum value \( M = 2^m J \), when including all levels of wavelets. For instance, the Haar wavelet matrix of order \( 8 \times 8 \) is given by

\[
H_8(\xi) = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 \\
1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\
1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1
\end{bmatrix}
\]

The operational matrix \( P_{\gamma,\ell}(\xi) \) of order \( 2m \times 2m \) is derived from integration of Haar wavelet family with the aid of following formula:

\[
P_{\gamma,\ell}(\xi) = \int_{\mathbb{A}} \cdots \int_{\mathbb{A}} h_i(\xi) \, d\xi' = \frac{1}{(\gamma - 1)!} \int_{\mathbb{A}} (t - \xi)^{-1} h_i(\xi) \, d\xi.
\] (2.16)
For $\gamma = 2$, $P_n(\xi)$ will get the following form

$$P_{i,2}(\xi) = \begin{cases} 
\frac{1}{4\eta^2} - \frac{1}{2} \left( \frac{\xi - k}{2} \right)^2, & \xi \in \left[ \frac{k + 0.5}{2n}, \frac{k + 1}{2n} \right) \\
\frac{1}{4\eta^2}, & \xi \in \left[ \frac{k + 0.5}{2n}, \frac{k + 1}{2n} \right] \\
0, & \text{elsewhere.}
\end{cases} \quad (2.17)$$

3. Haar wavelet approximation with quasi-linearization technique to the Lane–Emden equation

The Haar basis has the very important property of multiresolution analysis that is $V_{j+1} = V_j \bigoplus W_j$. So a function $\omega(\xi) \in L_2[0, 1]$ may be expanded as (for detail see [29,32]).

$$\omega(\xi) = \sum_{i=0}^{\infty} a_i h_i(\xi). \quad (3.18)$$

The orthogonality property puts a strong limitation on the construction of wavelets and allows us to transform any square integrable function on the interval time $[0, 1]$ into Haar wavelet expansion as the following form.

$$\omega(\xi) = a_0 h_0(\xi) + \sum_{j=0}^{l-1} \sum_{k=0}^{2^j-1} a_{2^j+k} h_{2^j+k}(\xi). \quad (3.19)$$

Similarly the highest derivative can be written as wavelet series

$$\sum_{i=0}^{2m} a_i h_i(\xi) = a^0 h_{2m}(\xi). \quad (3.20)$$

then we have used the quasi-linearization process for linearization. The quasi-linearization process is an application of the Newton Raphson Kantorovich approximation method in function space given by Bellman and Kalaba [34]. The idea and advantage of the method is based on the fact that linear equations can often be solved analytically or numerically while there are no useful techniques for obtaining the general solution of a nonlinear equation in terms of a finite set of particular solutions. Consider an nth order nonlinear ordinary differential equation

$$L^n \omega(\xi) = f(\omega(\xi), \omega^{(1)}(\xi), \omega^{(2)}(\xi), \ldots, \omega^{(n-1)}(\xi), \xi) \quad (3.21)$$

with the initial conditions

$$\omega(0) = \lambda_0, \quad \omega^{(1)}(0) = \lambda_1, \ldots, \omega^{(n)}(0) = \lambda_n. \quad (3.22)$$

where $\omega^p(\xi) = \omega_0(\xi)$. The functions $f_{\omega^{(j)}} = \frac{df}{d\omega^j}$ are functional derivatives of the functions. The zeroth approximation $\omega_0(\xi)$ is chosen from mathematical or physical considerations.

When $f$ is nonlinear function in Eq. (1.1), we linearize by using quasi-linearization process followed by Eq. (3.23), Eq. (1.2) becomes,

$$\omega^{(2)}_{n+1}(\xi) + \frac{\alpha}{\xi} \omega^{(1)}_{n+1}(\xi) + kf(\omega, \omega^{(1)}(\xi), \ldots, \omega^{(n-1)}(\xi), \xi)$$

$$+ \sum_{i=0}^{n-1} (\omega^{(i)}_{n+1} - \omega^{(i)}_n) f_{\omega^{(i)}}(\omega, \omega^{(1)}(\xi), \omega^{(2)}(\xi), \ldots, \omega^{(n-1)}(\xi), \xi)$$

$$= u(\xi). \quad (3.24)$$

Any $\omega(\xi) \in L_2(0, 1)$ can be expressed in the form of truncated Haar wavelet series as

$$\omega^{(n)}_{n+1}(\xi) = \sum_{i=0}^{2m} a_i h_i(\xi). \quad (3.25)$$

Then using the concept of operational matrix and Haar wavelet technique, we can obtain the derivatives from following equation

$$\omega^{(n-n)}_{n+1}(\xi) = \sum_{i=0}^{2m} P_{1,i}(\xi) + \sum_{i=0}^{2m} \sum_{j=0}^{2^i-1} P_{2,i} h_{2^i+j}(\xi)$$

$$+ \sum_{i=0}^{n-1} (\omega^{(i)}_{n+1} - \omega^{(i)}_n) f_{\omega^{(i)}}(\omega, \omega^{(1)}(\xi), \omega^{(2)}(\xi), \ldots, \omega^{(n-1)}(\xi), \xi)$$

$$= u(\xi). \quad (3.27)$$

Finally, we can obtain the Haar wavelet solution of given problem by substituting the values of $a_i$'s in Eq. (3.27).

4. Convergence of Haar wavelet approximation

A function $\omega \in L^2(R)$, MRA of $L^2(R)$ generates a sequence of subspaces $V_j, V_{j+1}, V_{j+2} \ldots$ such that the projections of $\omega$ onto these spaces give finer and finer approximations of the function $\omega$ as $j \rightarrow \infty$. The corresponding error at $j$th level may be defined as

$$e_j(\xi) = |\omega(\xi) - \omega_j(\xi)|$$

$$= \omega(\xi) - \sum_{i=0}^{2^{j+1}-1} a_i h_i(\xi) = \sum_{i=0}^\infty a_i h_i(\xi). \quad (4.28)$$

We can analyze the error for generalized nonlinear singular Lane–Emden equation (3.27), if we know the exact solution of this equation. Convergence of the method may be discussed on the same lines as given in Saeedi et al. [35].
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Theorem. Suppose that $f$ satisfies a Lipschitz condition on $[0, 1]$. For all $x > 0$, $\forall \xi_1, \xi_2 \in [0, 1]$ such that $|f(\xi_1) - f(\xi_2)| \leq M|\xi_1 - \xi_2|$, where $M$ is the Lipschitz constant.

Proof. See Saeedi et al. [35]. □

5. Numerical results and discussion

This section presents numerical results and discussion of the proposed method for solving different types of generalized Lane–Emden equation. Some special cases of Eq. (1.1) are considered to demonstrate the simple applicability and efficiency of the method. All results are computed by C++ and MATLAB R2007b which are reported in Figs. 1–9 and Tables 1–5.

5.1. Homogeneous Lane–Emden type equations

Case 5.1.1: Consider Eq. (1.2) with $f(\xi) = 1$, $g(\omega) = \omega^\alpha$, $\alpha = 2$ and $\kappa = 1$ and subject to initial conditions: $\omega(0) = 1$, $\omega'(0) = 0$.

$$\frac{d^2 \omega}{d\xi^2} + \frac{2}{\xi} \frac{d\omega}{d\xi} + \omega^\alpha(\xi) = 0, \quad \alpha, \xi \geq 0. \quad (5.30)$$
Lane–Emden equation of polytropic index $p$ which describes the sphere shape [1]. Here $p$ is the polytropic index which is related to the ratio of specific heats of the gas comprising the star. In galactic dynamics $p$ is larger than 1 which means that no polytropic stellar system can be homogeneous. However, the polytrope of index $p = 6$ contains some radically different and unexpected characteristics. A polytropic star of index $p = 1$ terminates at a finite radius and the solution for the polytrope of index $p = 6$ contain some radically different and unexpected characteristics. A polytropic star of index $p = 5$ has an infinite radius and in reality cannot exist. It has been claimed in the literature that exact solution is available only for $p = 0, 1, 5$. We can find Haar wavelets solution for any finite polytropic index $p$ and in the neighborhood of singular point $\xi = 0$. Finally, the solution of Eq. (5.30) has been obtained by substituting the value of $a’s$ in approximate Haar wavelet series. The obtained numerical solutions for $m = 32, 128$ and $p = 0, 1, 5, 2.5, 3.25, 3.5, 4.5, 6$ are represented in Fig. 1 and Table 1.

Case 5.1.2: The white dwarf equation

$$\frac{d^2 \omega}{d \xi^2} + 2 \frac{d \omega}{\xi d \xi} + (\omega^2(\xi) - c)^{3/2} = 0, \quad \xi \geq 0 \tag{5.31}$$

subject to the initial conditions: $\omega(0) = 1, \omega'(0) = 0$.

Inserting $f(\xi) g(\omega) = (\omega^2(\xi) - c)^{3/2}, \alpha = 2, \kappa = 1$ into Eq. (1.2) which gives us the white dwarf equation introduced by Chandrasekhar [1,3] in his study of the gravitational potential of the degenerate white dwarf stars. According to Mukremin Kilic [36], “A white dwarf is like a hot stove; once the stove is off, it cools slowly over time. By measuring how cool the stove is, we can tell how long it has been off. The two stars we identified have been cooling for billions of years”. Kilic explains that white dwarf stars are the burned out cores of stars similar to the Sun. In about 5 billion years, the Sun also will burn out and turn into a white dwarf star. It will lose its outer layers as it dies and turn into an incredibly dense star the size of Earth. It is clear, if $c = 0$, this equation becomes a Lane–Emden equation with polytropic index $p = 3$. Wavelet solutions are obtained for $c = 0, 0.1, 0.2, 0.3$ in the interval $[0, 1)$ and in the neighborhood of singular point $\xi = 0$ which are shown in Fig. 2 and Table 2 for $m = 128$.

Case 5.1.3: Consider in Eq. (1.2) with $f(\xi) = \xi^a \log \xi$ and $g(\omega) = \log^{(3)}(\omega), \alpha = 2, \kappa = 1, \beta = 2$, obtained nonlinear singular Emden–Fowler equation.

$$\frac{d^2 \omega}{d \xi^2} + 2 \frac{d \omega}{\xi d \xi} + \xi^a \log^l(\xi) \log^{(3)}(\omega) = 0, \quad \xi \geq 0 \tag{5.32}$$

with initial conditions: $\omega(0) = e, \omega'(0) = 0$.

Haar wavelet solutions are obtained for values of $a = 2, l = 2, n = 3$ and exact solution of this problem is not available in literature. Computed solutions are shown in Fig. 3 and Table 3 for $m = 32$.

Case 5.1.4: For $f(\xi) = e^{\xi - a(\xi)}, g(\omega) = e^\omega, \alpha = 2, \kappa = 1$ in Eq. (1.2) with conditions: $\omega(0) = 0, \omega'(0) = 0$, has the following form of generalized nonlinear singular Lane–Emden equation of
Table 1
Comparison of a present method solution.

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<th>( \xi )</th>
<th>( \text{Exact} )</th>
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<td>( m = 128 )</td>
<td>1.000000</td>
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<tr>
<td>( p = 2.5 )</td>
<td>0.0000001</td>
<td>0.00001</td>
<td>0.00001</td>
<td>0.00001</td>
<td>0.00001</td>
<td>0.00001</td>
</tr>
<tr>
<td>( m = 32 )</td>
<td>1.000000</td>
<td>1.000000</td>
<td>1.000000</td>
<td>1.000000</td>
<td>1.000000</td>
<td>1.000000</td>
</tr>
<tr>
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<td>0.001</td>
<td>0.001</td>
<td>0.001</td>
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</tr>
<tr>
<td>( m = 32 )</td>
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<td>1.000000</td>
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<td>1.000000</td>
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</table>

Table 2
Performance of the Haar wavelet method.

<table>
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<tr>
<th>( \xi )</th>
<th>( \text{Exact} )</th>
<th>( \text{HWS} )</th>
<th>( \text{HWS} )</th>
<th>( \text{HWS} )</th>
<th>( \text{HWS} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \epsilon = 0 )</td>
<td>0.0000001</td>
<td>1.000000</td>
<td>1.000000</td>
<td>1.000000</td>
<td>1.000000</td>
</tr>
<tr>
<td>( \epsilon = 0.1 )</td>
<td>0.0000001</td>
<td>1.000000</td>
<td>1.000000</td>
<td>1.000000</td>
<td>1.000000</td>
</tr>
<tr>
<td>( \epsilon = 0.2 )</td>
<td>0.0000001</td>
<td>1.000000</td>
<td>1.000000</td>
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<td>1.000000</td>
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<tr>
<td>( \epsilon = 0.3 )</td>
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</table>

Table 3
Comparison of Haar wavelet solution.

<table>
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<tr>
<th>( \xi )</th>
<th>( \text{Case: 5.1.3} )</th>
<th>( \text{Case: 5.1.4} )</th>
<th>( \text{HWS} )</th>
<th>( \text{ADIM 11} )</th>
<th>( \text{HWS} )</th>
<th>( \text{Exact} )</th>
<th>( \text{ADIM 11} )</th>
<th>( \text{HWS} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 0.00000001 )</td>
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<td>2.71828</td>
<td>2.71828</td>
<td>2.71828</td>
<td>2.71828</td>
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</tr>
<tr>
<td>( 0.0000001 )</td>
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<td>2.71828</td>
<td>2.71828</td>
<td>2.71828</td>
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<td>2.71828</td>
<td>2.71828</td>
<td>2.71828</td>
<td>2.71828</td>
</tr>
<tr>
<td>( 0.2 )</td>
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<td>2.71828</td>
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<td>2.71828</td>
<td>2.71828</td>
</tr>
<tr>
<td>( 0.3 )</td>
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<td>2.71828</td>
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<tr>
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<td>2.71828</td>
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<td>2.71828</td>
<td>2.71828</td>
<td>2.71828</td>
</tr>
<tr>
<td>( 0.5 )</td>
<td>2.71828</td>
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<td>2.71828</td>
<td>2.71828</td>
<td>2.71828</td>
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<td>2.71828</td>
</tr>
<tr>
<td>( 0.6 )</td>
<td>2.71828</td>
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<td>2.71828</td>
<td>2.71828</td>
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<td>2.71828</td>
<td>2.71828</td>
<td>2.71828</td>
</tr>
<tr>
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<td>2.71828</td>
<td>2.71828</td>
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</tr>
<tr>
<td>( 0.8 )</td>
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<td>2.71828</td>
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<td>2.71828</td>
<td>2.71828</td>
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</tr>
</tbody>
</table>

Case 5.1.6: Bonnor–Ebert Gas Spheres [37].
Consider the Eq. (1.2) by taking \( f(\xi) = \log(\xi), \omega(\xi) = e^{-\alpha(\xi)}, \alpha = 2, l = 0 \) and \( \kappa = -1 \). Initial conditions: \( \omega(0) = 0, \omega'(0) = 0 \).

This equation describes the Bonnor–Ebert Gas spheres model which a more general form of isothermal gas spheres. This model appears in Richardson’s theory of thermionic current when the density and electric force of an electron gas in the neighborhood of a hot body in thermal equilibrium is to be determined. For large radius where the effect of the central conditions is very weak the solution should asymptotically approach the singular isothermal solution. The Bonnor–Ebert gas spheres consisting of an ideal gas has an infinite radius. In order to understand the behavior of the equation at neighborhood of singular point, Haar wavelet solutions
Table 5
Performance of the present method.

<table>
<thead>
<tr>
<th>ξ</th>
<th>Case: 5.1.7 Exact [39]</th>
<th>Case: 5.1.7 HWS</th>
<th>Case: 5.2.1 Exact</th>
<th>Case: 5.2.1 HWS</th>
<th>Case: 5.2.2 Exact</th>
<th>Case: 5.2.2 HWS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/64</td>
<td>0.989257</td>
<td>0.989257</td>
<td>−3.75599e−006</td>
<td>−4.21338e−006</td>
<td>1.00024</td>
<td>1.00024</td>
</tr>
<tr>
<td>5/64</td>
<td>0.974632</td>
<td>0.974632</td>
<td>−0.00439</td>
<td>−0.00423</td>
<td>1.00612</td>
<td>1.00612</td>
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<tr>
<td>9/64</td>
<td>0.966175</td>
<td>0.966175</td>
<td>−0.002389</td>
<td>−0.002362</td>
<td>1.01997</td>
<td>1.01997</td>
</tr>
<tr>
<td>11/64</td>
<td>0.947679</td>
<td>0.947679</td>
<td>−0.0008957</td>
<td>−0.0008921</td>
<td>1.02998</td>
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<tr>
<td>15/64</td>
<td>0.927775</td>
<td>0.927775</td>
<td>−0.018397</td>
<td>−0.018358</td>
<td>1.05647</td>
<td>1.05647</td>
</tr>
<tr>
<td>19/64</td>
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<td>0.907055</td>
<td>−0.029733</td>
<td>−0.029696</td>
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<tr>
<td>23/64</td>
<td>0.885957</td>
<td>0.885957</td>
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<td>1.13786</td>
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<td>31/64</td>
<td>0.843823</td>
<td>0.843823</td>
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<tr>
<td>35/64</td>
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<td>0.823194</td>
<td>−0.088392</td>
<td>−0.088391</td>
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<td>1.34861</td>
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<tr>
<td>39/64</td>
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<td>0.803032</td>
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<td>−0.099534</td>
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<tr>
<td>43/64</td>
<td>0.783422</td>
<td>0.783422</td>
<td>−0.105202</td>
<td>−0.105237</td>
<td>1.57053</td>
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<tr>
<td>47/64</td>
<td>0.764417</td>
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<tr>
<td>51/64</td>
<td>0.746048</td>
<td>0.746048</td>
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<td>−0.089334</td>
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<tr>
<td>55/64</td>
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</tr>
<tr>
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<td>63/64</td>
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<td>−0.0060</td>
<td>2.63529</td>
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</table>

are obtained for m = 128. Comparison of Haar wavelet solutions with those approximate series solutions which are obtained in [38] are shown by Fig. 6 and Table 4.

Case 5.1.7: Consider Eq. (1.2) with \( f(\xi) = -\frac{1}{16} 2^{1/2} (-4 + 5 \xi^{1/2}) \). The exact solution for this problem is \( \omega(\xi) = \sqrt{\left(\frac{1}{16} \xi^{1/2}\right)} \).

5.2. Non-homogeneous Lane–Emden type equations

Case 5.2.1: Consider the Eq. (1.1) with \( f(\xi) g(\omega) = \xi \omega(\xi), u(\xi) = \xi^{5} - \xi^{7} + 44 \xi^{3} - 30 \xi^{2} + 8 \alpha \). The exact solution for this problem is \( \omega(\xi) = 3 \alpha \xi^{2} - 2 \alpha \xi - \frac{3}{16} + \frac{1}{16} \xi^{1/2} \).

5.2.2: Consider Eq. (1.1) with \( \alpha = 2 \).

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References