The \textit{b}-Chromatic Number of Corona Graphs\textsuperscript{*}

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\textbf{Abstract}

A \textit{b}-coloring of a graph \(G\) is a proper coloring of the vertices of \(G\) such that there exists a vertex in each color class joined to at least one vertex in each other color class. The \textit{b}-chromatic number of a graph \(G\), denoted by \(\varphi(G)\), is the maximal integer \(k\) such that \(G\) may have a \textit{b}-coloring with \(k\) colors. This parameter has been defined by Irving and Manlove [5]. They proved that determining \(\varphi(G)\) is NP-hard in general and polynomial for trees. In this paper, we find that the \textit{b}-chromatic number on corona graph of any graph \(G\) with path \(P_n\), cycle \(C_n\) and complete graph \(K_n\). Finally, we generalized the \textit{b}-chromatic number on corona graph of any two graphs, each one on \(n\) vertices.

\textbf{Keywords:} \textit{b}-chromatic number, \textit{b}-monotonic, \textit{b}-continuous, corona

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The \( b \)-chromatic number \( \phi(G) \) [5, 7, 8] of a graph \( G \) is the largest positive integer \( k \) such that \( G \) admits a proper \( k \)-coloring in which every color class has a representative adjacent to at least one vertex in each of the other color classes. Such a coloring is called a \( b \)-coloring. This concept of \( b \)-chromatic number was introduced in 1999 by Irving and Manlove [5], who proved that determining \( \phi(G) \) is NP-hard in general and polynomial for trees.

Effantin and Kheddouci studied [1, 2, 3] the \( b \)-chromatic number for the powers of paths, cycles, complete binary trees, and complete caterpillars.

It has been proved in [6] by showing that if \( G \) is a \( d \)-regular graph with girth 5 and without cycles of length 6, then \( \phi(G) = d + 1 \).

Recently, motivated by the works of Sandi Klavžar and Marko Jakovac [7], who proved that the \( b \)-chromatic number of cubic graphs is four with the exception of Petersen graph, \( K_3, 3 \), prism over \( K_3 \), and sporadic with 10 vertices.

The corona of two graphs \( G_1 \) and \( G_2 \) is the graph \( G = G_1 \circ G_2 \) formed from one copy of \( G_1 \) and \( |V(G_1)| \) copies of \( G_2 \) where the \( i \)th vertex of \( G_1 \) is adjacent to every vertex in the \( i \)th copy of \( G_2 \).

2 Motivation

F. Bonomo et al., [4] defined that a graph \( G \) to be \( b \)-monotonic if \( \phi(H_1) \geq \phi(H_2) \) for every induced subgraph \( H_1 \) of \( G \), and every induced subgraph \( H_2 \) of \( H_1 \).

A graph \( G \) is defined to be \( b \)-continuous [4] if it admits a \( b \)-coloring with \( t \) colors, for every \( t = \chi(G), \ldots, \phi(G) \).

In this present paper, we investigate the \( b \)-chromatic number on corona graph of any two graphs, each one on \( n \) vertices. As a motivation, this work will be extended by the authors to investigate the \( b \)-continuous and \( b \)-monotonic for the generalization of corona graphs.
3  b-chromatic number on corona graph of any graph with path

Theorem 3.1. Let $G$ be a simple graph on $n$ vertices. Then

$$\varphi(G \circ P_n) = \begin{cases} 
    n + 1; & \text{for } n \leq 3, \\
    n; & \text{for } n > 3.
\end{cases}$$

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $V(P_n) = \{u_1, u_2, \ldots, u_n\}$. Let $V(G \circ P_n) = \{v_i : 1 \leq i \leq n\} \cup \{u_{ij} : 1 \leq i \leq n; 1 \leq j \leq n\}$. By the definition of corona graph, each vertex of $G$ is adjacent to every vertex of a copy of $P_n$, i.e., every vertex $v_i \in V(G)$ is adjacent to every vertex from the set $\{u_{ij} : 1 \leq j \leq n\}$.

Assign the following $n$-coloring for $G \circ P_n$ as $b$-chromatic:

- For $1 \leq i \leq n$, assign the color $c_i$ to $v_i$.
- For $1 \leq i \leq n$, assign the color $c_i$ to $u_{1i}$, $\forall i \neq 1$.
- For $1 \leq i \leq n$, assign the color $c_i$ to $u_{2i}$, $\forall i \neq 2$.
- For $1 \leq i \leq n$, assign the color $c_i$ to $u_{3i}$, $\forall i \neq 3$.
- For $1 \leq i \leq n$, assign the color $c_i$ to $u_{4i}$, $\forall i \neq 4$.

- For $1 \leq i \leq n$, assign the color $c_i$ to $u_{ni}$, $\forall i \neq n$.
- For $1 \leq i \leq n$, assign to vertex $u_{ni}$, one of allowed colors - such color exists, because $2 \leq \deg(u_{ni}) \leq 3$ and $n > 3$.

Let us assume that $\varphi(G \circ P_n)$ is greater than $n$, i.e. $\varphi(G \circ P_n) = n + 1$, $\forall n > 3$, there must be at least $n + 1$ vertices of degree $n$ in $G \circ P_n$, all with distinct colors, and each adjacent to vertices of all of the other colors. But then these must be the vertices $v_1, v_2, \ldots, v_n$, since there are only ones with degree at least $n$. This is the contradiction, $b$-coloring with $n + 1$ colors is impossible. Thus, we have $\varphi(G \circ P_n) \leq n$. Hence, $\varphi(G \circ P_n) = n, \forall n > 3$.

Note that $\varphi(G \circ P_1) = 2$ for graph $G$ on one vertex and $\varphi(G \circ P_2) = 3$ for graph $G$ on two vertices. Indeed, let us notice that such graph $G \circ P_2$ inculdes $K_3$. Now, let us define $b$-coloring of $G \circ P_3(|V(G)|) = 3$ with four colors in the following way: for $1 \leq i \leq 3$, assign the color $c_i$ to $v_i$, for
1 \leq l \leq 3$, assign the color $c_l$ to $u_{1l}$, $\forall l \neq 1$, for $1 \leq l \leq 3$, assign the color $c_l$ to $u_{2l}$, $\forall l \neq 2$, for $1 \leq l \leq 3$, assign the color $c_l$ to $u_{3l}$, $\forall l \neq 3$ and for $1 \leq l \leq 3$, assign the color $c_4$ to $u_{4l}$. Therefore, $\varphi(G \circ P_3) \geq 4$. Let us assume that $\varphi(G \circ P_3)$ is greater than 4, i.e. $\varphi(G \circ P_3) = 5$, there must be at least 5 vertices of degree 4 in $G \circ P_3$, all with distinct colors, and each adjacent to vertices of all of the other colors. But then these must be the vertices $v_1, v_2$ and $v_3$, since these are only ones with degree at least 4. This is the contradiction, $b$-coloring with 5 colors is impossible. Thus, we have $\varphi(G \circ P_3) \leq 4$. Therefore, $\varphi(G \circ P_3) = 4$. Hence, $\varphi(G \circ P_n) = n + 1$, $\forall n \leq 3$.

Figure 3.1: $b$-coloring of $G \circ P_3$ with four colors for $G = P_3$.

4 \hspace{1em} \textbf{$b$-chromatic number on corona graph of any graph with cycle}

\textbf{Theorem 4.1.} Let $G$ be a simple graph on $n$ vertices, $n > 3$. Then

$$\varphi(G \circ C_n) = n.$$ 

\textit{Proof.} Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $V(C_n) = \{u_1, u_2, \ldots, u_n\}$. Let $V(G \circ C_n) = \{v_i : 1 \leq i \leq n\} \cup \{u_{ij} : 1 \leq i \leq n; 1 \leq j \leq n\}$. By the definition of corona graph, each vertex of $G$ is adjacent to every vertex of a copy of $C_n$, i.e., every vertex $v_i \in V(G)$ is adjacent to every vertex from the set $\{u_{ij} : 1 \leq j \leq n\}$.

Assign the following $n$-coloring for $G \circ C_n$ as $b$-chromatic:

- For $1 \leq i \leq n$, assign the color $c_i$ to $v_i$.
- For $1 \leq i \leq n$, assign the color $c_i$ to $u_{1i}$, $\forall i \neq 1$.
- For $1 \leq i \leq n$, assign the color $c_i$ to $u_{2i}$, $\forall i \neq 2$.
- For $1 \leq i \leq n$, assign the color $c_i$ to $u_{3i}$, $\forall i \neq 3$.
For $1 \leq i \leq n$, assign the color $c_i$ to $u_{4i}$, $\forall i \neq 4$.

For $1 \leq i \leq n$, assign the color $c_i$ to $u_{ni}$, $\forall i \neq n$.

For $1 \leq i \leq n$, assign to vertex $u_{ii}$ one of allowed colors - such color exists, because $\text{deg}(u_{ii}) = 3$ and $n > 3$.

Therefore, $\varphi(G \circ C_n) \geq n$. Let us assume that $\varphi(G \circ C_n)$ is greater than $n$, i.e., $\varphi(G \circ C_n) = n + 1$, $\forall n > 3$, there must be at least $n + 1$ vertices of degree $n$ in $G \circ C_n$, all with distinct colors, and each adjacent to vertices of all of the other colors. But then these must be the vertices $v_1, v_2, \ldots, v_n$, since these are only ones with degree at least $n$. This is the contradiction, $b$-coloring with $n + 1$ colors is impossible. Thus, we have $\varphi(G \circ C_n) \leq n$. Hence, $\varphi(G \circ C_n) = n$, $\forall n > 3$. Note that $\varphi(G \circ C_3) = 4$, since graph $G \circ C_3$ includes graph $K_4$.

Figure 4.2: $b$-coloring of $G \circ C_3$ with four colors for $G = P_3$

5 $b$-chromatic number on corona graph of any graph with complete graph

Theorem 5.1. Let $G$ be a simple graph on $n$ vertices. Then

$$\varphi(G \circ K_n) = n + 1.$$  

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $V(K_n) = \{u_1, u_2, \ldots, u_n\}$. Let $V(G \circ K_n) = \{v_i : 1 \leq i \leq n\} \cup \{u_{ij} : 1 \leq i \leq n; 1 \leq j \leq n\}$. By the definition of corona graph, each vertex of $G$ is adjacent to every vertex of a copy of $K_n$. i.e., every vertex $v_i \in V(G)$ is adjacent to every vertex from the set
\{u_{ij} : 1 \leq j \leq n\}.

Assign the following \(n + 1\)-coloring for \(G \circ K_n\) as \(b\)-chromatic:

- For \(1 \leq i \leq n\), assign the color \(c_i\) to \(v_i\).
- For \(1 \leq l \leq n\), assign color \(c_l\) to \(u_{1l}\), \(\forall l \neq 1\).
- For \(1 \leq l \leq n\), assign color \(c_l\) to \(u_{2l}\), \(\forall l \neq 2\).
- For \(1 \leq l \leq n\), assign color \(c_l\) to \(u_{3l}\), \(\forall l \neq 3\).
- For \(1 \leq l \leq n\), assign color \(c_l\) to \(u_{4l}\), \(\forall l \neq 4\).
- For \(1 \leq l \leq n\), assign color \(c_l\) to \(u_{nl}\), \(\forall l \neq n\).
- For \(1 \leq l \leq n\), assign the color \(c_l\) to \(u_{nl}\).

Therefore, \(\varphi(G \circ K_n) \geq n + 1\).

Let us assume that \(\varphi(G \circ K_n)\) is greater than \(n + 1\), i.e., \(\varphi(G \circ K_n) = n + 2\), there must be at least \(n + 2\) vertices of degree \(n + 1\) in \(G \circ K_n\), all with distinct colors, and each adjacent to vertices of all of the other colors. But then these must be the vertices \(v_1, v_2, \ldots v_n\), since these are only ones with degree at least \(n + 1\). This is the contradiction, \(b\)-coloring with \(n + 2\) colors is impossible. Thus, we have \(\varphi(G \circ K_n) \leq n + 1\). Hence, \(\varphi(G \circ K_n) = n + 1\). \(\square\)

6 \(b\)-chromatic number on corona graph of star graph with path

**Theorem 6.1.** Let \(n\) be a positive integer. Then

\[
\varphi(K_{1,n} \circ P_n) = n + 1.
\]

**Proof.** Let \(V(K_{1,n}) = \{v_1, v_2, \ldots, v_{n+1}\}\) and \(V(P_n) = \{u_1, u_2, \ldots, u_n\}\). By the definition of star graph, \(v_1\) is adjacent to each \(\{v_i : 2 \leq i \leq n\}\). Let \(V(K_{1,n} \circ P_n) = \{v_i : 1 \leq i \leq n + 1\} \cup \{u_{ij} : 1 \leq i \leq n + 1; 1 \leq j \leq n\}\). By the definition of corona graph, each vertex of \(K_{1,n}\) is adjacent to every vertex of a copy of \(P_n\), i.e., every vertex \(v_i \in V(G)\) is adjacent to every vertex from the set \(\{u_{ij} : 1 \leq j \leq n\}\).

Assign the following \(n + 1\)-coloring for \(K_{1,n} \circ P_n\) as \(b\)-chromatic:
• For $1 \leq i \leq n + 1$, assign the color $c_i$ to $v_i$.
• For $1 \leq l \leq n$, assign color $c_l$ to $u_{1l}$, $\forall \ l \neq 1$.
• For $1 \leq l \leq n$, assign color $c_l$ to $u_{2l}$, $\forall \ l \neq 2$.
• For $1 \leq l \leq n$, assign color $c_l$ to $u_{3l}$, $\forall \ l \neq 3$.
• For $1 \leq l \leq n$, assign color $c_l$ to $u_{4l}$, $\forall \ l \neq 4$.

Therefore, $\varphi(K_{1,n} \circ P_n) \geq n + 1$.

Let us assume that $\varphi(K_{1,n} \circ P_n)$ is greater than $n + 1$, i.e., $\varphi(K_{1,n} \circ P_n) = n + 2$, there must be at least $n + 2$ vertices of degree $n + 1$ in $K_{1,n} \circ P_n$, all with distinct colors, and each adjacent to vertices of all of the other colors. But then these must be the vertices $v, v_1, v_2, \ldots, v_n$, since these are only ones with degree at least $n + 1$. This is the contradiction, $b$-coloring with $n + 2$ colors is impossible. Thus, we have $\varphi(K_{1,n} \circ P_n) \leq n + 1$. Hence, $\varphi(K_{1,n} \circ P_n) = n + 1$.

7 Generalization of $b$-chromatic number on corona graph of any two graphs

Theorem 7.1. Let $G$ and $H$ be simple graphs, each one on $n$ vertices. Then

$$\varphi(G \circ H) = \begin{cases} n; \text{ if } \Delta(H) < n - 1, \\ n + 1; \text{ if } \Delta(H) = n - 1. \end{cases}$$

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $V(H) = \{u_1, u_2, \ldots, u_n\}$. Let $V(G \circ H) = \{v_i : 1 \leq i \leq n\} \cup \{u_{ij} : 1 \leq i \leq n; 1 \leq j \leq n\}$. By the definition of corona graph, each vertex of $G$ is adjacent to every vertex of a copy of $H$, i.e., every vertex $v_i \in V(G)$ is adjacent to every vertex from the set $\{u_{ij} : 1 \leq j \leq n\}$. 

7
Let us rename vertices in $i$th copy of $H$ in $G \circ H$, $i = 1, 2, \ldots, n$, in such a way that a vertex of maximum degree has a label $u_{ii}$.

Assign the following coloring for $G \circ H$ as $b$-chromatic:

- For $1 \leq i \leq n$, assign the color $c_i$ to $v_i$.
- For $1 \leq i \leq n$, assign the color $c_i$ to $u_{1i}$, $\forall$ $i \neq 1$.
- For $1 \leq i \leq n$, assign the color $c_i$ to $u_{2i}$, $\forall$ $i \neq 2$.
- For $1 \leq i \leq n$, assign the color $c_i$ to $u_{3i}$, $\forall$ $i \neq 3$.
- For $1 \leq i \leq n$, assign the color $c_i$ to $u_{4i}$, $\forall$ $i \neq 4$.
- For $1 \leq i \leq n$, assign the color $c_i$ to $u_{ni}$, $\forall$ $i \neq n$.

The following cases completes the proof:

**Case (i): $\Delta(H) < n - 1$**

For $1 \leq i \leq n$, assign to vertex $u_{ii}$ one of allowed colors - such color exists, because $1 \leq \deg(u_{ii}) < n - 1$. Therefore, $\varphi(G \circ H) \geq n$.

Let us assume that $\varphi(G \circ H)$ is greater than $n$, i.e., $\varphi(G \circ H) = n + 1$, there must be at least $n + 1$ vertices of degree $n$ in $G \circ H$, all with distinct colors, and each adjacent to vertices of all of the other colors. But then these must be the vertices $v_1, v_2, \ldots, v_n$, since these are only ones with degree at least $n$. This is the contradiction, $b$-coloring with $n + 1$ colors is impossible. Thus, we have $\varphi(G \circ H) \leq n$. Hence, $\varphi(G \circ H) = n$, if $\Delta(H) < n - 1$.

**Case (ii): $\Delta(H) = n - 1$**

For $1 \leq i \leq n$, assign the color $c_{n+1}$ to $u_{ii}$. Therefore, $\varphi(G \circ H) \geq n + 1$.

Let us assume that $\varphi(G \circ H)$ is greater than $n + 1$, i.e., $\varphi(G \circ H) = n + 2$, there must be at least $n + 2$ vertices of degree $n + 1$ in $G \circ H$, all with distinct colors, and each adjacent to vertices of all of the other colors. But then these must be the vertices $v_1, v_2, \ldots, v_n$, since these are only ones with degree at least $n$. This is the contradiction, $b$-coloring with $n + 2$ colors is impossible. Thus, we have $\varphi(G \circ H) \leq n + 1$. Hence, $\varphi(G \circ H) = n + 1$, if $\Delta(H) = n - 1$. 

\[\square\]
Figure 7.1: \(b\)-coloring of \(G \circ H = P_6 \circ W_6\) with seven colors. The case where \(\Delta(H) = n - 1\).

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