Multianticipative Nonlocal Macroscopic Traffic Model

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Abstract: Multianticipative driving behavior, where a vehicle reacts to many vehicles in front, has been extensively studied and modeled using a car-following (i.e., microscopic) approach. A lot of effort has been undertaken to model such multianticipative driving behavior using a macroscopic approach, which is useful for real-time prediction and control applications due to its fast computational demand. However, these macroscopic models have increasingly failed with an increased number of anticipations. To this end, this article puts forward derivation of an improved macroscopic model for multianticipative driving behavior using a modified gas-kinetic approach. First, the basic (microscopic) generalized force model, which has been claimed to fit well with real traffic data, is chosen for the derivation. Second, the derivation method relaxes the condition that deceleration happens instantaneously. Theoretical analysis and numerical simulations of the model are carried out to show the improved performance of the derived model over the existing (multianticipative) macroscopic models.

1 INTRODUCTION

Traffic flow theory is used in a wide range of operational and planning applications, from real-time information/control (Adeli and Samant, 2000; Adeli and Jiang, 2003; Wang and Papageorgiou, 2005; Wang et al., 2006; van Lint and Hoogendoorn, 2009; Ngoduy, 2011; Heilmann et al., 2011) to medium-to-long-term forecasting (Szeto et al., 2011; Balijepalli et al., 2012). In principle, there are three types of traffic flow models in the state-of-the-art of traffic flow theory: microscopic models, mesoscopic models, and macroscopic models. The microscopic approach describes traffic flow at a high level of detail such as the movement of individual vehicles (Treiber et al., 2000; Adeli and Ghosh-Dastidar, 2004; Jiang and Adeli, 2005; Ghosh-Dastidar and Adeli, 2006; Kesting et al., 2007; Yeo and Skabardonis, 2009; Kesting et al., 2010; Tang et al., 2011; Aghabayk et al., 2013), whereas the macroscopic approach represents traffic flow at a low level of detail via aggregate traffic variables such as flow, mean speed, and density (Zhang and Wong, 2006; Boel and Mihaylova, 2006; Tang et al., 2008; Laval and Leclercq, 2010; Xiong et al., 2011; Gning et al., 2011; Zhang et al., 2011; Duret et al., 2011; Tao et al., 2011; Tang et al., 2012). The mesoscopic approach, on the other hand, describes traffic flow at a level of detail between microscopic and macroscopic approach through probabilistic terms. An example of mesoscopic models is the gas-kinetic model, which is used to derive the macroscopic models based on the method of moments (Helbing, 1997; Hoogendoorn, 1999).

There have been recent impressive advances in modeling the dynamics of traffic flow in which the drivers react not only to the behavior of the vehicle directly in front but also to the behavior of those further ahead (i.e., multianticipative driving behavior).
However, most existing work relies on a microscopic modeling approach (Lenz et al., 1999; Hasebe et al., 2003; Ge et al., 2006; Treiber et al., 2006; Hoogendoorn et al., 2006, 2007; Kesting and Treiber, 2008). Basically, the multianticipative driving behavior has been studied both analytically and numerically using either extended intelligent driver models (Treiber et al., 2006; Kesting and Treiber, 2008), extended optimal speed (OV) models (Lenz et al., 1999; Hasebe et al., 2003; Wilson et al., 2004; Ge et al., 2006), or other types of car-following models (Hoogendoorn et al., 2006, 2007). This article is concerned with the development of macroscopic models which are able to capture the multianticipations of the drivers.

In contrast to microscopic models, macroscopic models are preferred for real-time prediction and control applications due to their fast computational demand (Zhang and Wang, 2013). To the best of our knowledge, there are very few advances in incorporating the multianticipative driving behavior in macroscopic models. Although Wilson et al. (2004) have recently discussed ways to consider multianticipative driving behavior in a macroscopic model, their methods are actually based on the extended OV-type models and they have claimed that the derived macroscopic model has increasingly failed with an increased number of anticipations. Subsequently, Ngoduy (2009) has also developed a multianticipative macroscopic model based on the OV-type model and studied the effects of multianticipations on traffic flow instabilities. Nevertheless, as the derivation is somehow similar to those mentioned in Wilson et al. (2004) this model also suffers from the same drawbacks. From our viewpoint, some of these drawbacks are due to the fact that the OV-type model, based on which a macroscopic model is derived, is well known to produce accidents if the initial condition, the optimal speed and the sensitivity parameter are not chosen properly (Helbing, 2009). Moreover, the gradient expansion approach used in the derivation has a weakness: its validity requires implicitly small gradients. Although this weakness could be remedied if we consider higher order terms, these terms consequently lead to a macroscopic model that is not well tractable.

The main objective of this article thus is to improve the current multianticipative macroscopic models. To this end, the contribution of this article is threefold:

1. The generalized force model of Helbing and Tilch (1998) is extended to capture the multianticipative microscopic driving behavior.
2. The proposed multianticipative generalized force model is used to formulate a new macroscopic model using a gas-kinetic approach.
3. The analytical properties of the model are derived and numerical simulations are carried out to show the improvement of the proposed model over the existing multianticipative macroscopic models.

This article is organized as follows. Section 2 introduces an extended generalized force model and the corresponding gas-kinetic equation for multianticipative driving behavior. Section 3 describes the derivation of the multianticipative macroscopic model from the proposed gas-kinetic equation. Section 4 shows the analytical properties of the developed model and some numerical simulation results describing the performance of the developed model compared to the existing multianticipative macroscopic models in several case studies. Finally, we conclude the article in Section 5.

## 2 Generalized Force Model and Gas-Kinetic Equation

### 2.1 Extended generalized force model

We have chosen the Generalized Force Model because it is well suited for the gas-kinetic model as the deceleration (or braking) time is explicitly taken into account and we can derive a macroscopic model which is well tractable and is consistent with the general form of other macroscopic models (Section 3). According to Helbing and Tilch (1998), the amount and direction of a behavioral change such as the acceleration/deceleration is given by a sum of generalized forces reflecting the different motivations which a driver feels at the same time. Because these forces do not fulfill Newton’s laws: \( action = reaction \), they are called generalized forces. Helbing and Tilch (1998) argued that the success of this approach in describing traffic dynamics is based on the fact that driver reactions to typical traffic situations are more or less automatic and determined by the optimal behavioral strategy.

In the model of Helbing and Tilch (1998), the dynamics of a vehicle \( n \) with velocity \( v_n(t) \) at place \( x_n(t) \) and time instant \( t \) is given by the equation of motion:

\[
\frac{dv_n}{dt} = \frac{v_0 - v_n}{\tau_n} + \frac{f_{n,n-1}(v_n, x_n, v_{n-1}, x_{n-1})}{\text{Repulsive interaction}}
\]  

(1)

In Equation (1), \( \tau_n \) and \( v_0 \) denote the acceleration time and the desired speed, respectively, of vehicle \( n \). The \( acceleration \) term presents the motivation of vehicle \( n \) to reach its desired speed \( v_0 \) while the \( repulsive interaction \) term describes the motivation of that vehicle to keep a safe distance from the leader \( n - 1 \). The \( repulsive \)
interaction term is specified as:

$$f_{n, n-1}(v_n, x_n, v_{n-1}, x_{n-1}) = \frac{V_c(s_n) - v_0}{\tau_n} - \frac{(v_n - v_{n-1})H(v_n - v_{n-1})}{\tau_n} e^{-(s_n - s_0)/R}$$  \hspace{1cm} (2)$$

where $s_0$ denotes the speed-dependent safe distance, defined as $s_0 = d + T v_0$. Here, $T$ is the safe time headway measured by the time the follower is needed to stop completely to avoid a collision with a stopped vehicle in front, while $d$ is the safe distance between the follower’s front and the leader’s rear when these two vehicles stop completely. $s_n = x_{n-1} - x_n$ is the distance headway between vehicle $n$ and its direct leader $n-1$. $v_{n-1}$ is the speed of the direct leader. $H(.)$ denotes a Heaviside function, a dimensionless quantity. $R$ is the distance measured by a range of the braking interaction ($m$). $V_c(s_n)$ is the headway-dependent equilibrium speed. $\tau_n$ is the braking (deceleration) time. Typically, $\tau_n < \tau_s$ as deceleration capabilities of vehicles are greater than acceleration capabilities.

Note that, if we neglect the contribution of the finite deceleration time $\tau_n$, the model of Bando et al. (1995) or OV-type model is obtained. There are some important properties of the repulsive interaction force as described in literature (Helbing and Tilch, 1998). First, it will guarantee early enough and sufficient braking in cases of large relative speed ($v_n - v_{n-1}$). Second, this force increases with growing relative speed ($v_n - v_{n-1}$), but will only be effective, if the speed of the follower is larger than that of the leader (i.e., $H(v_n - v_{n-1}) = 1$). Third, it will increase with decreasing distance headway $s_n$, but vanish for a large one (i.e., $s_n \rightarrow \infty$).

Recent empirical studies of Hoogendoorn et al. (2006) have indicated that a vehicle is constrained by the dynamics of not only the direct leader but also the neighboring vehicles, both leading and following ones. This multianticipative driving behavior has also been investigated both theoretically and numerically using microscopic simulation in the literature (Lenz et al., 1999; Ge et al., 2006; Hasebe et al., 2003; Wilson et al., 2004; Treiber et al., 2006; Hoogendoorn et al., 2007; Kesting and Treiber, 2008). To take into account the multianticipative effect where the driver looks further downstream to make his/her decision we extend the general social force Equation (1) as below:

$$\frac{dv_n}{dt} = \frac{v_0 - v_n}{\tau_n} + \sum_{m=1}^{M} \beta_m f_{n, n-m}(v_n, x_n, v_{n-m}, x_{n-m})$$  \hspace{1cm} (3)$$

where $m$ is the index of the $m$th leader, $\beta_m$ is the weight factor which generally satisfies $\beta_1 > \beta_2 > ... > \beta_M$, and $\sum_{m=1}^{M} \beta_m = 1$. Basically, $m$ is related to the traffic conditions as well as the layout of the road. In the scope of this article, we only show the impact of different values of $m$ on traffic dynamics. Recent empirical study by Hoogendoorn et al. (2006) based on data collected at a small section of a freeway in the Netherlands has indicated that $m = 3$ is a reasonable value. However, this is just an indication for a certain freeway section during a certain period of congested traffic. We think there should be more extensive data to be studied to get a good conclusion.

Note that, although we only consider the effect of multiple leading vehicles it is rather straightforward to include the effect of multiple following vehicles on traffic dynamics using the same methodology. To substitute Equation (2) into Equation (3) we obtain the extended generalized force model for multianticipative driving behavior:

$$\frac{dv_n}{dt} = \frac{v_0 - v_n}{\tau_n} + \sum_{m=1}^{M} \beta_m V_c(s_m) - v_0 - \sum_{m=1}^{M} \beta_m (v_n - w_m) H(v_n - w_m) e^{-(s_m - s_0)/R}$$  \hspace{1cm} (4)$$

where $s_m = x_{n-m} - x_n$ is the space headway between the considered vehicle $n$ and its $m$th leader, $w_m$ is the speed of the $m$th leader of vehicle $n$, so by definition $w_m = v_{n-m}$. In Equation (4), the dynamics of a vehicle will be given by two elements: (1) a weighted sum of the distance headway-dependent optimal speed function evaluated at his/her own headway and at the headways of several vehicles in front, and (2) a weighted sum of the relative speed-dependent function evaluated at his/her own speed and the speeds of several vehicles in front. Note that the existing multianticipative models of Wilson et al. (2004) or Ngoduy (2009) neglect the contribution of the second element in their derivations.

2.2 Gas-kinetic equation

In this section, we apply the gas-kinetic equation to vehicular traffic presented in Helbing and Treiber (1998). This equation will serve as the basis for the subsequent derivation of the multianticipative macroscopic model in the next section. We prefer this method because it avoids the requirement of implicitly small gradients as in the gradient expansion approach used in the literature (Wilson et al., 2004). As we will show in the next section, the derivation method uses some exact integration calculations to end up with the macroscopic variables. Let $\rho(x,v,t)$ denote the phase-space density (PSD) distribution of vehicles driving with speed $v$ ($v \in [v, v + dv]$) at location $x$ ($x \in [x, x + dx]$) and time instant $t$, where $dv$ and $dx$ are small deviation of speed and location, respectively. Note that, in a traffic problem, a phase-space quantity represents all possible states of traffic flow in time and location. For a
homogeneous freeway (e.g., without on- and off-ramps), the gas-kinetic model reads:

\[
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} (\rho v) + \frac{\partial}{\partial v} (\rho \frac{dv}{dt}) = 0
\]  

(5)

In Equation (5), the \textit{convection} term describes the changes of the PSD due to the movement of vehicles along the road while the \textit{interaction} term depicts the changes of the PSD due to the acceleration to the desired speed as well as the deceleration.

Let us substitute Equation (4) into the gas-kinetic Equation (5) to obtain:

\[
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} (\rho v) = - \frac{\partial}{\partial v} \left( \frac{\rho v_0 - v}{\tau} \right) - \frac{\partial}{\partial v} \left( \frac{\rho \sum_m \beta_m V_c(s_{nm} - v_0)}{\tau} \right)
\]

\[
\frac{\partial}{\partial v} \left( \rho \sum_m \beta_m \frac{(v_n - w_m)}{\bar{\tau}} H(v_n - w_m) e^{-(s_{nm} - v_0)/R} \right)
\]

(6)

where we have dropped out the vehicle index \( n \) because Equation (6) holds for any vehicle. In Equation (6) the deceleration is determined by the interaction with not only the direct leader but also many other leaders. This is particularly important where vehicles are moving in a platoon so that the leader of the following platoon will adjust his/her speed according to the speeds of all vehicles in the leading platoon. The proposed modeling framework will be extended to take into account the nature of platoon-based dynamics in our future research. This method is different from the other derivations such as the gas-kinetic models of Treiber et al. (1999) because in those models one usually assumes an instantaneous braking to the speed of the slower vehicle in front, if overtaking is not possible. This approximation of the deceleration behavior is justified only if the timescale of the deceleration is significantly smaller than that of the acceleration and therefore is neglected. It leads to the Boltzmann-like braking interaction term as in current gas-kinetic-based models. It is also worth noticing that neither the model of Wilson et al. (2004) nor the model of Ngoduy (2009) consider the contribution of the finite deceleration time (i.e., \( \bar{\tau} \to \infty \)). We will show later that this contribution will improve the model performance over the existing multianticipative macroscopic models.

From Equation (6), we will generate a macroscopic model in which the multianticipative driving behavior has been incorporated. This is the subject of the next section.

## 3 MultiAnticipative macroscopic model

The derivation method in this section is a so-called method of moments. In principle, the method of moments has been applied widely to obtain macroscopic traffic models from gas-kinetic theory in literature (Treiber et al., 1999; Helbing et al., 2001; Hoogendoorn et al., 2002; Ngoduy et al., 2006; Ngoduy, 2008; Ngoduy and Tampere, 2009). By definition, the macroscopic traffic variables are determined as below:

1. Density \( r(x,t) \) describing the number of vehicles per unit road length \([x, x + dx]\) at time \( t \). Let \( g_v(x, v, t) \) denote the probability density function of speed \( v \), the PSD is defined by:

\[
\rho(x, v, t) = g_v(x, v, t) \rho(x, v, t) dv = r(x, t) \int_v g_v(x, v, t) dv
\]

(7)

2. Mean speed \( V(x, t) \). The mean speed \( V \) is defined as \( V = \langle v \rangle = \int_v v g_v(x, v, t) dv \)

\[
V(x, t) = \frac{1}{r(x, t)} \int_v \rho(x, v, t) v dv
\]

(8)

3. Mean speed variance \( \Theta(x, t) \). The mean speed variance \( \Theta \) is defined as \( \Theta = \langle (v - V)^2 \rangle \)

\[
\Theta(x, t) = \frac{1}{r(x, t)} \int_v \rho(x, v, t) (v - V)^2 dv
\]

(9)

where \( \langle \cdot \rangle \) denotes the so-called mean operator, defined as below. For any function \( y(\alpha) \) where \( \alpha \) is a variable:

\[
\langle y(\alpha) \rangle = \int_y y(\alpha) \psi(\alpha) d\alpha
\]

with \( \psi \) being the distribution function of \( \alpha \).

Let us multiply both sides of Equation (6) with \( v^k G_m^m(w_m, s_m|v, x, t) \) \((k = 0, 1, 2, \ldots)\), then integrate them over all possible values of \( v, w_m, \) and \( s_m \). Here \( G_m^m(w_m, s_m|v, x, t) \) is the probability that, given a vehicle with speed \( v \) at location \( x \) and time instant \( t \), the \( m \) th leader drives with a speed \( w_m \) at a headway distance \( s_m \). Let us apply the factorization approximation \( G_m^m(w_m, s_m|v, x, t) = g_m^m(w, x + s_m, t) g_m^m(s_m, x, t) \), which means that distributions of the speed of the \( m \)th
leader \( w_m \) and the corresponding distance headway \( s_m \) are statistically independent, and independent of the speed \( v \) of the considered vehicle. Accordingly, the aggregated left-hand side of Equation (6) becomes:

\[
A_k = \frac{\partial}{\partial t} \left( \rho \int_v^{} dv \int_{w_m}^{w} dw_m \int_{s_m}^{s} ds_m g_v^m g_{w_m}^m g_{s_m}^m \right) = \frac{\partial r}{\partial t} (v^k) \tag{10}
\]

and the aggregated right-hand side of (6) reads:

\[
C_k^m = -r k \int_v^{} dv \int_{w_m}^{w} dw_m \int_{s_m}^{s} ds_m g_v^m g_{w_m}^m g_{s_m}^m dv dw_m ds_m
\]

where \( dv/dt \) is determined from Equation (4).

To set \( k = 0 \) and \( k = 1 \), respectively, we obtain the dynamic equations for the density (i.e., the conservation law):

\[
\frac{\partial r}{\partial t} + \frac{\partial (rv)}{\partial x} = 0 \tag{11}
\]

and for the flow as detailed below. To close the model, we need an assumption for the dynamics of the speed variance, which has been justified to follow an empirical function of density as in the literature (Treiber et al., 1999; Treiber and Kesting, 2013). Consequently, the main difficulty is to determine function \( C_1 = \sum_m C_1^m \), which results in the dynamic equation of the flow. That is:

\[
\frac{\partial q}{\partial t} + \frac{\partial [r(V^2 + \Theta)]}{\partial x} = C_1 \tag{12}
\]

From the multi-anticipative gas-kinetic Equation (6):

\[
C_1 = C_1^a + C_1^b \tag{13}
\]

where:

\[
C_1^a = r \sum_m \beta_m \int_v^{} dv \int_{w_m}^{w} dw_m \int_{s_m}^{s} ds_m g_v^m g_{w_m}^m g_{s_m}^m V_e(s_m) - v dv dw_m ds_m
\]

and

\[
C_1^b = -r \sum_m \beta_m \int_v^{} dv \int_{w_m}^{w} dw_m \int_{s_m}^{s} ds_m g_v^m g_{w_m}^m (v - w_m) e^{-\int_{s_m}^{s} V_e(w, x + s_m, t) dw_m} \frac{dv dw_m ds_m}{\bar{\tau}} \tag{14}
\]

It is straightforward to show that:

\[
C_1^a = \frac{r}{\tau} \left( \sum_m \beta_m \int_{s_m}^{s} ds_m g_v^m g_{w_m}^m (s_m) - V \right) \tag{15}
\]

By definition:

\[
\int_{s_m}^{s} g_v^m (s_m, x, t) = 1 \quad \text{and} \quad \int_{s_m}^{s} h(s_m) g_v^m (s_m, x, t) = h(s_m) \tag{16}
\]

Let \( \bar{s} \) denote the total mean distance gap between two consecutive vehicles. A plausible assumption that the density \( r(x, t) \) is a constant between \( x \) and \( x + \bar{s} \) leads to: \( \bar{s} = \int_{x}^{x+\bar{s}} s g_v (s, x, t) = 1/r \). Let us apply the first Taylor expansion to the mean distance gap between a vehicle and its \( m \)th leader:

\[
\langle s_m \rangle = \bar{s} + (m - 1) \frac{\partial \bar{s}}{\partial x} = \frac{1}{r} - \frac{m - 1}{r^3} \frac{\partial r}{\partial x} \tag{17}
\]

Applying Equations (15) and (16) to the gap-dependent equilibrium speed gives:

\[
\int_{s_m}^{s} ds_m g_v^m (s_m, x, t) V_e(s_m) = V_e(s_m)
\]

\[
= V_e(r) + \frac{m - 1}{r} \frac{dV_e(r)}{dr} \frac{\partial r}{\partial x} \tag{18}
\]

which leads to:

\[
C_1^a = \frac{r (V_e(r) - V)}{\tau} + \sum_m \beta_m (m - 1) \frac{dV_e(r)}{dr} \frac{\partial r}{\partial x} \tag{19}
\]

Let us expand \( s_0(v) \) around \( s_0(V) \) and substitute it to equation the above formula \( C_1^a \):

\[
C_1^b = -\frac{r}{\bar{\tau}} \sum_m \beta_m \int_v^{} dv \int_{w_m}^{w} dw_m \int_{s_m}^{s} ds_m g_v^m g_{w_m}^m (v - w_m) e^{-\left[\langle s_m - s_0(V)\rangle\right] R} ds_0(V) \frac{dv}{R} dv dw_m ds_m
\]

By definition: \( s_0(V) = d + TV \) we have:

\[
C_1^b = -\frac{r}{\bar{\tau} R} \sum_m \beta_m e^{-\left[\langle s_m - s_0(V)\rangle\right] R} \int_v^{} dv \int_{w_m}^{w} dw_m \int_{s_m}^{s} ds_m g_v^m g_{w_m}^m (v - w_m) \frac{dv dw_m ds_m}{\bar{\tau}}
\]

\[
+ \frac{r}{\bar{\tau} R} \sum_m \beta_m e^{-\left[\langle s_m - s_0(V)\rangle\right] R} \int_v^{} dv \int_{w_m}^{w} dw_m \int_{s_m}^{s} ds_m g_v^m g_{w_m}^m (v - w_m) \frac{dv dw_m ds_m}{\bar{\tau}} \tag{20}
\]
Let us apply the first Taylor expansion to the exponential term:

\[ e^{-[v_0 - s_0(V)]/r} = e^{-[1/r - s_0(V)]/r} \left(1 + \frac{m - 1}{R} \frac{1}{r^2} \frac{\partial}{\partial x}\right) \]

Due to a lengthy but rather straightforward calculation, we only show here the main results. Details of these exact calculations of the integrals \( I_m^1 \) and \( I_m^2 \) are described in the Appendix:

\[
\begin{align*}
C_1^b &= -\frac{r}{\tilde{v}} \frac{T}{R} \sum_m \beta_m e^{-[1/r - s_0(V)]/r} \\
&\quad \left(1 + \frac{m - 1}{R} \frac{1}{r^2} \frac{\partial}{\partial x}\right) (I_m^1 - V I_m^2) \\
&= -\frac{r}{\tilde{v}} T \Theta \frac{1}{R} e^{-[1/r - s_0(V)]/r} \sum_m \beta_m \Phi(z_m) \\
&\quad - \frac{1}{\tilde{v} R^2} \frac{T}{r^2} e^{-[1/r - s_0(V)]/r} \frac{\partial}{\partial x} \sum_m \beta_m (m - 1) \Phi(z_m)
\end{align*}
\]

(20)

where \( \Phi(z_m) \) denotes the Normal error function: \( \Phi(z_m) = \int_{-\infty}^{z_m} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy \) and \( z_m = \frac{v_m - V}{\sqrt{\Theta / \tilde{v}}} \).

To substitute Equations (18) and (20) into Equation (11) we obtain the flow dynamic equation:

\[
\frac{\partial q}{\partial t} + \frac{\partial}{\partial x} \left[ r(V^2 + \Theta) \right] = r \left( \frac{V_e(r) - V}{\tau} + \sum_m \beta_m (m - 1) \right)
\]

\[
\left[ \frac{1}{\tilde{v}} \frac{dr}{dr} - \frac{1}{\tilde{v} R} e^{-[1/r - s_0(V)]/r} \Phi(z_m) \right] \frac{\partial}{\partial x}
\]

\[
- \frac{r}{\tilde{v}} T \Theta \frac{1}{R^2} e^{-[1/r - s_0(V)]/r} \sum_m \beta_m \Phi(z_m)
\]

(21)

Further linearization of the error function \( \Phi(z_m) \) leads to:

\[
\Phi\left(\frac{V - V_m}{\sqrt{\Theta / \tilde{v}}} \right) = 0.5 \left(1 - \frac{m}{\sqrt{\tilde{v} \Theta}} \frac{d}{dr} \frac{V}{\partial x}\right)
\]

which is then substituted into Equation (21) to obtain a speed dynamic equation as:

\[
\frac{\partial V}{\partial t} + V \frac{\partial V}{\partial x} = \frac{\dot{V}_e(r, V) - V}{\tau} - \frac{\dot{P}_r}{r} \frac{\partial}{\partial x} - \frac{\ddot{P}_V}{r} \frac{\partial V}{\partial x}
\]

(22)

in which

\[
\dot{P}_r = \frac{\partial P}{\partial r} - \frac{\sum_m \beta_m (m - 1) dV_e(r)}{\tau} + \frac{1}{2\tilde{v} R^2} e^{-[1/r - s_0(V)]/r} \sum_m \beta_m (m - 1)
\]

4 MODEL PROPERTIES

4.1 Stability conditions

This section investigates analytically the property of the introduced model to reflect the effects of multianticipations on traffic flow stability. To do so, we derive stability conditions based on the linear method for the introduced (macroscopic) model. The linear
method refers to linear Taylor approximations, which are used throughout the analysis. The consequence of these approximations is that the conditions that are stable according to this analysis might actually still show nonlinear instability (Ngoduy, 2013). However, in general the linear analysis gives sound insights in the general behavior of the model.

To follow the derivation in Helbing and Johansson (2009), we have replaced all nonlocalities in our model Equation (22) by corresponding gradients and adopted the stability condition in Helbing and Johansson (2009) for our specific functional form of the equilibrium speed function and traffic pressure terms developed in this article. To this end, we can obtain the neutral stability condition:

\[(r \hat{V}_e^r)^2 - \hat{V}_e^r \hat{P}_e (1 + |\hat{V}_e^r|) - \hat{P}_r (1 + |\hat{V}_e^r|)^2 = 0\]

or equivalently the linear stability condition:

\[r|\hat{V}_e^r| - (1 + |\hat{V}_e^r|) \left( \sqrt{\hat{P}_r + \left( \frac{\hat{P}_e}{2r} \right)^2 + |\hat{P}_e|} \right) \leq 0 \]

where

\[\hat{V}_e^r = \frac{dV_e(r)}{dr} + \frac{\tau}{2r} T \Theta \left( \frac{1}{Rr} \right)^2 e^{-|1/r - s_0(V)|/R} \hat{V}_e^V\]

This condition can be further simplified to get the OV-like stability condition if we set:

\[P_r = -\frac{1}{2r} V_e^r, \quad P_e^V = 0, \quad V_e^V = 0, \quad \bar{r} \to \infty,\]

that is, the finite deceleration term is not considered, and convert the density-dependent terms to the distance headway-dependent ones:

\[\hat{V}_e^r = -\bar{s}^2 V_e^s, \quad \text{here} \quad V_e^s = \frac{dV_e(s)}{ds}\]

\[\hat{P}_e^r = \bar{s}^2 \frac{2V_e^s}{2r} \left[ 1 + 2 \sum_m \beta_m (m-1) \right] \]

Then condition (24) is simplified to:

\[V_e^s \leq \frac{1}{\bar{r}} \left[ 0.5 + \sum_m \beta_m (m-1) \right] \]

Let us graphically illustrate the derived linear stability conditions using the equilibrium speed formula in Helbing and Johansson (2009): $V_e(s) = V_0 \left[ \tanh \left( (s - d)/s_c - 1.2 \right) + \tanh(1.2) \right]$, where $V_0 = 115 \text{ km/h}, \ s_c = 50 \text{ m}, \ d = 4 \text{ m}$. Other parameters are $T = 1.2 \text{ seconds}, \ \tau = 3 \text{ seconds}, \ R = 75 \text{ m}, \ \bar{r}$ can vary between 0.5 and 2 seconds to test the impact of the finite deceleration time, $\beta_1 = 0.6, \ \beta_2 = 0.4$ for $m = 2$, $\beta_1 = 0.6, \ \beta_2 = 0.3, \ \beta_3 = 0.1$ for $m = 3$. These model parameters will also be used in the ensuing article. Figure 1 shows the natural stability curves for different sets of model parameters of the proposed model using the linear stability condition (24), where the stability function is measured by

\[y(r) = r|\hat{V}_e^r| - (1 + |\hat{V}_e^r|) \left( \sqrt{\hat{P}_r + \left( \frac{\hat{P}_e}{2r} \right)^2 + |\hat{P}_e|/2r} \right)\]

According to condition (24), for any values of the space headway $s = 1/r$ which produce positive values of function $y(r)$, traffic will become linearly unstable. This figure illustrates that multianticipative driving behavior leads to stabilization of traffic flow, which is consistent with findings in some recent microscopic models (Treiber et al., 2006; Hoogendoorn et al., 2007;
Kesting and Treiber, 2008) and in existing multianticipative macroscopic models (Wilson et al., 2004; Ngoduy, 2009). It also indicates that the finite deceleration term contributes to the stabilization of traffic flow and has a larger impact on the stability with an increased number of anticipations than the OV-type model. This is because the finite deceleration time contributes to smoothing out the sharp front shock which then reduces the amplitude of instabilities.

4.2 Anisotropic property

Model Equations (10) and (21) can be written in the following vector form:

$$\frac{\partial U}{\partial t} + J \frac{\partial U}{\partial x} = S$$

(27)

where $U = [r, rV]$ and $S = [0, rV - V]$. $J$ denotes a Jacobian matrix with the elements below:

$$J = \begin{bmatrix} 0 & -V & \frac{\rho_v}{2} \frac{\rho_v}{2} + V \frac{\rho_v}{2} \\ \frac{\rho_v}{2} - V & \frac{\rho_v}{2} \frac{\rho_v}{2} + V \frac{\rho_v}{2} \\ 2V + \frac{\rho_v}{2} \end{bmatrix}$$

(28)

The Jacobian matrix $J$ has two distinct eigenvalues

$$\lambda_{1,2} = V + \frac{\rho_v}{2} \pm \sqrt{\left( \frac{\rho_v}{2} \right)^2 + \rho_r}$$

(29)

which indicate that the model Equations (10) and (21) belong to a strictly hyperbolic partial differential equation system so that any standard numerical solutions can be used for the approximation, for example, the first-order Harten-Lax-van Leer-Einfeldt (HLLE) scheme in Ngoduy et al. (2004) or the higher order Weighted Essentially Non-Oscillatory (WENO) scheme in Zhang et al. (2003). In this article, we adopt the HLLE scheme to simulate our model. Such scheme has been shown robust and stable with the mesh size which satisfies the CFL (Courant–Friedrich–Lewy) condition (Sod, 1985) for any second-order macroscopic traffic models in Ngoduy et al. (2004): cell length/time step ≥ desired speed. To this end, we choose the cell length of 100 m and time step of 2 seconds for the approximation. Note that the discretized form of the developed model can easily be integrated into the Kalman filter-based real-time traffic prediction and control framework of Wang and Papa-georgiou (2005) and Wang et al. (2006).

4.2.1 Removal of a blockage. We simulate the developed model with the open boundary condition using the following initial conditions: $r(x, 0) = 0$ for $x < 5,000$ m and $x > 7,000$ m; $r(x, 0) = r_{\text{jam}} = 160$ veh/km otherwise, $v(x, 0) = 0 \forall x$. There is no traffic coming from the upstream of the blockage. Figure 2 shows that after the removal of the blockage the speed of vehicles at the head of the queue relaxes to the free-flow velocity corresponding to the empty downstream, while vehicles at the tail of the queue remain standstill at their location (until the vehicles at the head of the queue all move away). Although the traffic conditions upstream are free-flow we observe from this figure that vehicles do not flow backwards into the empty region, a fact that allows us to say that our model satisfies the anisotropy condition. This is consistent with the nonlocal model of Treiber et al. (1999).

4.2.2 Dynamics of queue tails. We simulate the developed model with the open boundary condition using the following initial conditions: $r(x, 0) = 0$ for $x < 5,000$ m; $r(x, 0) = r_{\text{jam}} = 160$ veh/km otherwise, $v(x, 0) = 0 \forall x$. There is neither traffic coming from the upstream nor traffic going out at the downstream (e.g., due to the red-light settings). Figure 3 shows that our model produces the density dynamics that satisfy $r(x, t) = r(x, 0)$. This property indicates that the jump in the vehicular density remains in its original location as time evolves. Therefore, we can conclude that: drivers’ anisotropy is met. This is because the additional braking term strongly increases with the density and speed gradient (positive contribution in the quantity $\dot{\rho}_r$ and negative
contribution in the quantity $\tilde{P}_V$ in the stability condition). That means that this braking term will prevent vehicles from accelerating into denser regions and decelerating when driving into regions of lighter traffic.

We have also carried out the simulations for OV-type multianticipative model (i.e., model of Wilson et al., 2004) in which the finite deceleration time is not considered. The results are described in Figure 4. It illustrates that $r(x, t) \neq r(x, 0)$, indicating that vehicles at the queue tail do not remain standstill but accelerate into denser regions. That means the OV-type model of Wilson et al. (2004) violates the anisotropic property. This is the main drawback of this model due to the neglected finite deceleration time which contributes to the nonlocal terms in the proposed model. Such nonlocal terms will maintain the anisotropy property.

4.3 Stabilization effects

This section investigates numerically the effects of the increased number of anticipations on traffic flow instabilities. To this end, we simulate the proposed model with an open freeway with an on-ramp bottleneck at KM6.0. The main traffic is operating under high-flow condition (i.e., $r(0, t) = 30$ veh/km and $q(0, t) = 1,900$ veh/h). That means traffic is generally meta-stable in this situation so a small perturbation of the on-ramp flow can trigger stop-and-go waves. We have chosen the on-ramp flow $q_{ramp}(t) = 300$ veh/h so that the stop-and-go waves are obtained for $m = 1$ (Figure 5a), corresponding to the single look-ahead model equation. Then we simulate the model again using the same inputs but with an increased number of anticipations ($m = 2$ and $m = 3$). Figures 5b and c illustrate that the multianticipative driving behavior leads to the suppression of stop-and-go waves due to a small perturbation of the on-ramp flow to an already heavy main traffic stream. It shows that when drivers look further downstream, the amplitude of stop-and-go traffic is reduced, leading to stabilized traffic flow as expected. It is clear that the initial density of 30 veh/km or space headway of approximately 33 m lines inside the stability curves (i.e., the second curve, the fourth curve, and the sixth curve from the top in Figure 1a, respectively, for Figures 5a–c) which indicates that traffic is linearly unstable upstream of the bottleneck. Figure 5d describes the stop-and-go waves upstream of the on-ramp for the multianticipative
driving behavior (e.g., \( m = 3 \)) with different model parameters (e.g., increased relaxation times). That indicates that stop-and-go waves do exist in reality for any type of driving behavior. This again confirms the linear stability results obtained above that multianticipations lead to a stabilization of traffic flow but there exists a certain combination of model parameter set which lead to (linear) traffic instabilities.

In summary, our numerical results confirm that the derived model overcomes the drawbacks of existing multianticipative macroscopic models in preserving the anisotropic properties and is consistent with microscopic models in describing the effects of multianticipative driving behavior on traffic flow instabilities. It shows the combined effects of multianticipation factor and the finite deceleration time on the stability diagram because the latter contributes to smoothing out the sharp front shock which consequently reduces the amplitude of instabilities.

4.3.1 Model verification. This article just focuses on the mathematical analysis of the developed model while the full model calibration/validation in a large freeway will require a lot of extra work which should deserve a separate paper. Nevertheless, we will show the performance of the proposed model in replication of traffic congestion on a freeway using empirical data, which was collected on a small section of freeway M25 in England for illustration purposes. To this end, the data were collected on a 5-km section of freeway M25 between Junction 10 (J10) and Junction 11 (J11) as described in Figure 6.

The data were used in our recent publication (Ngoduy, 2011; Ngoduy and Maher, 2012) for the calibration of the model of Treiber et al. (1999). More specifically, the duration of our simulation is 4 hours in a peak period. The chosen section is subject to recurrent traffic congestion, which is induced by the bottleneck at J11. To model the backward propagation of the congestion, we have imposed the speed at location 4909A as a boundary condition at the exit (Figure 7a).
Furthermore, the flow and speed at location 4860A and 4888A are used as an input to the model (e.g., Figures 7b and c). The other inputs are inflow and outflow at the on- and off-ramp (Figure 7d). In this article, all parameters take the (macroscopic) global values that have been calibrated in other research (Treiber et al., 1999; Hoogendoorn et al., 2006; Ngoduy, 2011; Ngoduy and Maher, 2012): $m = 3, \beta_1 = 0.6, \beta_2 = 0.3, \beta_3 = 0.1, V_0 = 108 \text{ km/h}, T = 0.75 \text{ seconds}, \tau = 2.5 \text{ seconds}, \tilde{\tau} = 0.75 \text{ seconds}, s = 50 \text{ m}, d = 4 \text{ m}, R = 100 \text{ m}$. The time and space interval for our simulation are chosen as 6 seconds and 250 m, respectively, so as the CFL condition is satisfied ($\Delta x / \Delta t > V_0 = 30 \text{ m/s}$). It is worth noting that the cell interfaces are placed at the detector locations.

Figures 8–10 show that the proposed model is able to capture well where and when traffic becomes congested. Table 1 supports the results in Figures 8–10, which depicts the total root mean square errors (RMSE) between the simulation results and the data obtained at
the detectors. The RMSE is calculated by the following equation:

$$RMSE_r = \sqrt{\frac{1}{K} \sum_{k=1}^{K} (r_i(k) - \hat{r}_i(k))^2}$$

$$RMSE_q = \sqrt{\frac{1}{K} \sum_{k=1}^{K} (V_i(k) - \hat{V}_i(k))^2}$$

where \([r_i(k), V_i(k)]\) denote, respectively, the predicted density and speed at time instant \(k\), location \(i\). \([\hat{r}_i(k), \hat{V}_i(k)]\) are the density and speed, respectively, obtained from loop detectors at time instant \(k\) and location \(i\). It is worth mentioning that we only used the model parameters which were calibrated before in our test. However, the model still produces reasonably accurate results, particularly the propagation of stop-and-go waves. The prediction errors at just upstream of the on-ramp (i.e., detectors 4883A and 4888A) are higher than those at the other locations. This is due to the complexity of the merging traffic at the on-ramp area.

### Table 1

<table>
<thead>
<tr>
<th>Detector</th>
<th>(RMSE_r) (veh./km)</th>
<th>(RMSE_q) (km/h)</th>
</tr>
</thead>
<tbody>
<tr>
<td>4866A</td>
<td>4.64</td>
<td>7.58</td>
</tr>
<tr>
<td>4871A</td>
<td>5.98</td>
<td>9.02</td>
</tr>
<tr>
<td>4876A</td>
<td>6.58</td>
<td>11.97</td>
</tr>
<tr>
<td>4879A</td>
<td>6.03</td>
<td>13.93</td>
</tr>
<tr>
<td>4883A</td>
<td>8.35</td>
<td>25.91</td>
</tr>
<tr>
<td>4888A</td>
<td>8.37</td>
<td>23.38</td>
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<tr>
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<td>12.74</td>
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<tr>
<td>4898A</td>
<td>6.58</td>
<td>11.16</td>
</tr>
<tr>
<td>4902A</td>
<td>7.77</td>
<td>11.84</td>
</tr>
</tbody>
</table>

### 5 CONCLUDING REMARKS

Although much effort has been undertaken to develop microscopic models taking into account the multianticipative driving behavior, where the drivers react not only to the behavior of the vehicle directly in front but also to the behavior of those further ahead, there has been few advances in developing macroscopic models which
can capture such driving behavior. This article has presented an alternative approach to develop an improved multianticipative macroscopic model. Particularly, our model has been developed from a gas-kinetic model in which the multianticipative driving behavior is explicitly described through an extended (microscopic) generalized force model. The proposed method does not use either the gradient expansion method or the popular Boltzmann-like formula as in existing macroscopic models. Linear stability analysis and numerical simulation results have shown that the proposed model overcomes some drawbacks of the current multianticipative macroscopic models. In addition, the developed model is nonlocal in its derivation process (i.e., thanks to the speed-dependent safe distance and Normal error function terms) and has a structure similar to a generalized macroscopic model (Helbing and Johansson, 2009) with modified pressure terms accounting for the finite deceleration time and multianticipations so it could also capture the transitions between meta-stability and the stop-and-go waves in real traffic flow. The proposed methodology could be generalized to the multiclass model to investigate the effect of the intelligent vehicles which allow for wider anticipations with roadside infrastructure.

Numerical results in comparison to the real data using model parameters which were calibrated in the literature have indicated that our model is able to reflect reasonably accurately the traffic operation on a 5-km section of freeway M25 in England. Nevertheless, it would still need further real-life experiments to establish a unique and calibrated/validated model performance. This calibration/validation work will be left in our future research. Although this article is only limited to the (macroscopic) modeling of the forward looking effects, it is rather straightforward to formulate the backward looking effects using the proposed approach. Our future research will be put along this direction.

ACKNOWLEDGMENT

The first author wishes to thank the UK Engineering and Physical Sciences Research Council (EPSRC) for funding project EP/J002186/1.

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APPENDIX: EXACT INTEGRATION CALCULATIONS

This section describes how to derive the exact integration terms \( I_{1m} \) and \( I_{2m} \) in the formula \( C_{1m} \) in Equation (19) in Section 3. Using the convolution integrals of Normal distribution functions and the definition of the Normal error function leads to the following two basic calculations:

\[
\int_v \int_0^v dv G(v, x, t) g_m^m(v, x + s_m, t) = \int_v dv G(v, V, \Theta) g(V, m, \Theta_m) = \frac{1}{\sqrt{2\pi} (\Theta + \Theta_m)} \frac{1}{\sqrt{\Theta + \Theta_m}} N(z_m) \tag{A.1}
\]

and

\[
\int_v \int_0^v dw_m g_m(v, x, t) g_m^m(w_m, x + s_m, t) = \Phi(z_m) \tag{A.2}
\]

where \( G(v, V, \Theta) \) denotes the Normal distribution of variable \( v \), having mean value \( V \) and variance value \( \Theta \), that is \( G(v, V, \Theta) = \frac{1}{\sqrt{2\pi} \Theta} e^{-\frac{(v-V)^2}{2\Theta}} \).

\[
z_m = \frac{V - V_m}{\sqrt{\Theta + \Theta_m}}, \quad N(z_m) = \frac{e^{-z_m^2/2}}{\sqrt{2\pi}},
\]

\[
\Phi(z_m) = \int_{-\infty}^{z_m} N(y)dy,
\]

defined as the Normal error function.

These two basic integrals are used throughout the algebraic calculations in the rest of this section.

Let us consider

\[
I_{2m}^m = \int_v \int_0^v dv g_m(v, x, t) \int_0^v dw_m g_m^m(w_m, x + s_m, t)
\]

and let us denote

\[
\Phi(z_m) = \int_v \int_0^v dv g_m(v, x, t) \int_0^v dw_m g_m^m(w_m, x + s_m, t)
\]

\[
= \int_v \int_0^v dv (v - V) g_m(v, x, t) \int_0^v dw_m g_m^m(w_m, x + s_m, t)
\]

\[
+ V \int_0^v dv g_m(v, x, t) \int_0^v dw_m g_m^m(w_m, x + s_m, t)
\]

Using the rule of integration by parts gives:

\[
I_{2m}^m = \Theta \int_v \int_0^v dv g_m(v, x, t) g_m^m(v, x + s_m, t) + V \Phi(z_m)
\]

Similarly, let us denote

\[
I_{1m}^m = \int_v \int_0^v dv g_m(v, x, t) \int_0^v dw_m g_m^m(w_m, x + s_m, t)
\]

\[
= \int_v \int_0^v dv g_m(v, x, t) \int_0^v dw_m (w_m - V_m) g_m^m(w_m, x + s_m, t)
\]

\[
+ V_m \int_v \int_0^v dv g_m(v, x, t) \int_0^v dw_m g_m^m(w_m, x + s_m, t)
\]

Using the rule of integration by parts gives:

\[
I_{1m}^m = - \Theta_m \int_v \int_0^v dv g_m(v, x, t) g_m^m(v, x + s_m, t) + V_m \Phi(z_m)
\]

Hence,

\[
I_{2m}^m = I_{2a}^m - I_{2b}^m = \sqrt{\Theta + \Theta_m} N(z_m) + (V - V_m) \Phi(z_m)
\]

Let us consider

\[
I_{1m}^m = \int_v \int_0^v dv g_m(v, x, t) \int_0^v dw_m g_m^m(w_m, x + s_m, t)
\]

\[
- \int_v \int_0^v dv g_m(v, x, t) \int_0^v dw_m w_m g_m^m(w_m, x + s_m, t)
\]

and let us denote

\[
I_{1a}^m = \int_v \int_0^v dv g_m(v, x, t) \int_0^v dw_m w_m g_m^m(w_m, x + s_m, t)
\]

\[
= \int_v \int_0^v dv g_m(v, x, t) \int_0^v dw_m (w_m - V_m) g_m^m(w_m, x + s_m, t)
\]

\[
+ V_m \int_v \int_0^v dv g_m(v, x, t) \int_0^v dw_m g_m^m(w_m, x + s_m, t)
\]

\[
= - \Theta_m \int_v \int_0^v dv g_m(v, x, t) g_m^m(v, x + s_m, t) + V_m I_{1a}^m
\]

(A.4)
Let us denote
\[ I_3^m = \int_v dv v g_v(v, x, t) g_v^m(v, x + s_m, t) \]
\[ = \int_v dv (v - V) g_v(v, x, t) g_v^m(v, x + s_m, t) \]
\[ + V \int_v dv g_v(v, x, t) g_v^m(v, x + s_m, t) \]
Using the rule of integration by parts gives:
\[ I_3^m = -\Theta \int_v dv g_v(v, x, t) g_v^m(v, x + s_m, t) \]
\[ + \frac{V}{\sqrt{\Theta + \Theta_m}} N(z_m) \]
\[ = -\frac{\Theta}{\Theta_m} I_3^m + \frac{V}{\Theta_m} N(z_m) \]
\[ + \frac{V}{\Theta_m} \int_v dv g_v(v, x, t) g_v^m(v, x + s_m, t) \]
\[ + \frac{V}{\sqrt{\Theta + \Theta_m}} N(z_m) \]
\[ = -\frac{\Theta}{\Theta_m} I_3^m + \frac{V}{\Theta_m} \frac{N(z_m)}{\sqrt{\Theta + \Theta_m}} \]
\[ (A.5) \]
Hence,
\[ I_3^m = \frac{V}{\Theta_m} + \frac{V}{\Theta_m} \frac{N(z_m)}{\sqrt{\Theta + \Theta_m}} \]
\[ (A.6) \]
From Equations (A.4) and (A.6) we have:
\[ I_3^m = -\Theta_m I_3^m + V_m I_3^{m*} \]
\[ = \frac{V_m \Theta^2 - V \Theta_m^2}{(\Theta + \Theta_m) \sqrt{\Theta + \Theta_m}} N(z_m) + V V_m \Phi(z_m) \]
\[ (A.7) \]
Let us denote
\[ I_{1b}^m = \int_v dv v^2 g_v(v, x, t) \int_0^v dw_m g_{w}^m(w_m, x + s_m, t) \]
\[ = \int_v dv (v - V) v g_v(v, x, t) \int_0^v dw_m g_{w}^m(w_m, x + s_m, t) \]
\[ + V \int_v dv g_v(v, x, t) \int_0^v dw_m g_{w}^m(w_m, x + s_m, t) \]
Using the rule of integration by parts for \( I_4^m \) gives:
\[ I_4^m = \Theta \int_v dv g_v \left[ \int_0^v dw_m g_{w}^m + v g_{w}^m \right] \]
\[ = \Theta \int_v dv g_v \int_0^v dw_m g_{w}^m + \Theta I_3^m \]
\[ = \frac{V \Theta_m + V_m \Theta}{(\Theta + \Theta_m) \sqrt{\Theta + \Theta_m}} N(z_m) + \Theta \Phi(z_m) \]
\[ (A.8) \]
Hence,
\[ I_{1b}^m = I_4^m + V I_3^{m*} \]
\[ = \Theta \frac{V \Theta_m + V_m \Theta + V (\Theta + \Theta_m)}{(\Theta + \Theta_m) \sqrt{\Theta + \Theta_m}} N(z_m) \]
\[ + (V^2 + \Theta) \Phi(z_m) \]
\[ (A.9) \]
From Equations (A.7) and (A.9) we obtain:
\[ I_4^m = I_{1b}^m - I_3^{m*} \]
\[ = \frac{V}{\sqrt{\Theta + \Theta_m}} N(z_m) \]
\[ + (V^2 + \Theta - V V_m) \Phi(z_m) \]