Theory and Methodology

Minimizing mean squared deviation of completion times with maximum tardiness constraint

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Abstract

We consider a nonpreemptive single-machine scheduling problem to minimize mean squared deviation of job completion times about a common due date with maximum tardiness constraint (MSD/T_{\text{max}} problem), where the common due date is large enough so that it does not constrain the minimization of MSD.

The MSD/T_{\text{max}} problem is classified into three cases according to the value of maximum allowable tardiness $\Delta$: $\Delta$-unconstrained, $\Delta$-constrained and tightly $\Delta$-constrained cases. It is shown that the $\Delta$-unconstrained MSD/T_{\text{max}} problem is equivalent to the unconstrained MSD problem and that the tightly $\Delta$-constrained MSD/T_{\text{max}} problem with common due date $d$ is equivalent to the tightly constrained MSD problem with common due date $\Delta$. We also provide bounds to decide when the MSD/T_{\text{max}} problem is $\Delta$-unconstrained or $\Delta$-constrained. Then a solution procedure to the MSD/T_{\text{max}} problem is presented with several examples. © 2001 Elsevier Science B.V. All rights reserved.

Keywords: Scheduling theory; Earliness and tardiness penalties; Maximum allowable tardiness

1. Introduction

The problems involving regular measures of performance, that are nondecreasing in job completion times (mean flow time, maximum tardiness, number of tardy jobs, etc.) have received much attention for many years. However, ever since just-in-time (JIT) concepts were introduced, models to minimize both earliness and tardiness (E/T) have been navigated by many researchers because both earliness and tardiness are undesirable to manufacturers and customers. Since JIT concepts can be compatible with most practical situations, the performance measures in E/T models are being widely studied. File organization problem (Merten and Muller, 1972), PERT/CPM...
project and production of perishable goods (Siney, 1987), final assembly production (Kanet, 1981a), etc. are typical practical applications of E/T models.

Baker and Scudder (1990) surveyed various types of objective functions related to E/T model. Two distinguished objective functions in E/T model are mean squared deviation (MSD) and mean absolute deviation (MAD) from the due date. However, large deviations of completion times from the common due date usually incur more cost than small deviations in most production systems, therefore MSDs of completion times from the due date might be a more useful performance measure.

Bagchi et al. (1987) considered a scheduling problem to minimize MSDs of completion times from a common due date. They demonstrated the equivalence of the unrestricted version of this problem to the completion time variance problem studied by Eilon and Chowdhury (1977), Kanet (1981b) and Vani and Raghavachari (1987). De et al. (1989) modified a conceptual flaw in Bagchi et al. (1987) and provided tighter bounds to determine the due date classifying the given problem into tightly constrained, constrained and unconstrained cases.

The motivation for this study is that customers may accept the delivery of jobs between the due date and the maximum allowable time point with some tardiness penalties. However, customers would not accept the delivery of jobs after this allowable time period. Manufacturers and customers may determine the maximum allowable time period after negotiation. If the maximum allowable time point within which customers are willing to accept the delivery is set equal to $T_{\text{max}}$ in a JIT production environment, the MSD problem with maximum allowable tardiness (MSD/$T_{\text{max}}$ problem) can be developed.

In the MSD/$T_{\text{max}}$ problem, it is assumed that all customers have the same amount of patience when it comes to tardy jobs. This assumption is compatible with the following cases:

1. All parts processed are to be assembled together in order to make one end item. In this case (i.e., an assemble-to-order system), the due dates of all orders are identical and all jobs have the same amount of acceptable overdue periods.
2. All items produced are exported simultaneously to other countries, or they are shipped to domestic customers by single means of transportation. For example, in the case that all finished goods are exported to another country and distributed to dealers in the several states or provinces, the established model will be valid.
3. Only one customer issues all orders (as in original equipment manufacturing).
4. The management policy set by the manufacturer is that greater amounts of money damages are assessed for missing the due dates by a certain period. Then the proposed model can be employed.
5. All customers have different periods of time that they are willing to accept for being past the due date. A minimum acceptable period for all customers may be set for the maximum tardiness allowed in the model.

Since maximum tardiness can be considered a measure for manufacturer's goodwill as well as customer satisfaction, a number of researchers have considered scheduling problems involving maximum tardiness. Generally, the maximum tardiness measure is considered a secondary criterion in the bicriteria scheduling problem (Van Wassenhove and Gelders, 1980; Sen and Gupta, 1983). With maximum tardiness set to 0 (i.e., no tardy job) Chand and Schneeberger (1988) studied the minimization of weighted earliness. When there is no idle time, they showed the constrained weighted earliness problem is equivalent to the constrained weighted completion time problem.

In this paper, the objective of minimizing MSD of completion times from a common due date is chosen with maximum allowable tardiness constraint. In the MSD/$T_{\text{max}}$ problem, there exist cases where maximum allowable tardiness constraint restricts a solution of the MSD problem. According to the value of maximum allowable tardiness $\Delta$, the MSD/$T_{\text{max}}$ problem is classified into $\Delta$-unconstrained, $\Delta$-constrained and tightly $\Delta$-constrained cases.

In Section 2, we formally define the MSD/$T_{\text{max}}$ problem, describe dominance properties for the problem, and explain the $\Delta$-unconstrained,
\( A \)-constrained and tightly \( A \)-constrained cases. In Section 3, we show that the \( A \)-unconstrained MSD/\( T_{max} \) problem is equivalent to the unconstrained MSD problem and an upper bound of maximum allowable tardiness \( A \) to distinguish the \( A \)-unconstrained cases from the \( A \)-constrained cases is developed. We also prove that the tightly \( A \)-constrained MSD/\( T_{max} \) problem with common due date \( d \) is equivalent to the \( A \)-constrained MSD problem with common due date \( A \). Then, we develop a lower bound of \( A \) to distinguish the tightly \( A \)-constrained cases from the \( A \)-constrained cases. Section 4 provides a branching procedure to obtain a solution to the MSD/\( T_{max} \) problem with several examples. Finally, a summary and further research topics are discussed in Section 5.

2. The MSD/\( T_{max} \) problem

Consider a nonpreemptive single-machine scheduling problem with \( n \) jobs. Job \( j \) has its processing time \( p_j \) and common due date \( d \) for all \( 1 \leq j \leq n \). Without loss of generality, all jobs are numbered so that \( p_1 \leq p_2 \leq \ldots \leq p_n \).

The objective of this paper is to find a schedule minimizing MSD of completion times from a common due date, having maximum tardiness less than or equal to the given maximum allowable tardiness. We define notation used in this paper as follows:

\( \Delta \equiv \) maximum allowable tardiness,
\( C_j \equiv \) completion time of job \( j \),
\( R_j \equiv \) starting time of job \( j \),
\( Z(S) \equiv \) objective function value of schedule \( S \),
\( \bar{C} \equiv \) mean completion time, \( \frac{1}{n} \sum_{j=1}^{n} C_j \),
and
\( MS \equiv \) makespan, \( \sum_{j=1}^{n} p_j \).

By using the notation, the MSD/\( T_{max} \) problem can be mathematically formulated as

\[
\begin{align*}
\text{minimize } & Z(S) = \frac{1}{n} \sum_{j=1}^{n} (C_j - d)^2, \\
\text{subject to } & T_{max} \leq \Delta.
\end{align*}
\]

We assume that the common due date is large enough so that it does not constrain the minimization of MSD. Roughly speaking, the value of \( d \) is large enough if it is greater than or equal to MS. This assumption enables us to analyze pure effects of maximum allowable tardiness on the solution. In the following, we provide dominance properties that an optimal schedule for the MSD/\( T_{max} \) problem has.

**Definition.** A schedule is v-about-d if it satisfies Theorem 7 and Propositions 2, 4 and 5 in Bagchi et al. (1987).

**Property 1.** An optimal schedule for the MSD/\( T_{max} \) problem is v-about-d.

**Proof.** Interchanging argument can prove this property. \( \square \)

**Property 2.** In an optimal schedule for the MSD/\( T_{max} \) problem, the largest job is processed first.

**Proof.** Suppose to the contrary that there is an optimal schedule in which the largest job, i.e., job \( n \), is not processed first. Job \( n \) must be processed last by Property 1. If job \( n \) is interchanged with the first job such that the completion time of job \( n \) is the completion time of the first job in the original schedule, then the resulting schedule has the objective function value smaller than that of the optimal schedule. This is a contradiction. \( \square \)

For a given common due date \( d \), consider MSDs of three different sequences \( S_1, S_2 \) and \( S_3 \) ending at \( d + \Delta \), according to various values of maximum allowable tardiness \( \Delta \) as shown in Fig. 1. The bold line in the figure is formed from dominant schedules.

For \( \Delta > \Delta^* \), \( \Delta \) does not constrain the minimization of MSD, hence the problem is equivalent to the unconstrained MSD problem and its optimal schedule can be obtained by the procedure in
Bagchi et al. (1987). Note that it was assumed that common due date $d$ is large enough, hence does not constrain the minimization of MSD in any case. So, an optimal solution can be obtained by shifting sequence $S_1$ so that it completes at $d + A'$. The problem in this area is called $A$-unconstrained. For $A \leq A''$, $A$ is very small and the optimal schedule should finish at $d + A$. The problem in this area is called tightly $A$-constrained.

For $A'' < A < A^*$, $A$ still constrains the minimization of MSD. Therefore it is possible that local minima occurs to the left of $A$. This is the case when $A = A'$ in Fig. 1, where the optimal schedule is $S_1$ that finishes at $d + A''$. The problem in this area is called $A$-constrained. In the following example, it is shown that there is an instance whose optimal schedule finishes before time $d + A$. Consider the five-job problem shown in Table 1.

Fig. 2 shows the solution obtained from the procedure in Bagchi et al. (1987). The solution violates the maximum allowable tardiness $A = 944$ since $A < T_{\text{max}} = 1003$. Table 2 shows MSD values of all schedules which finish at $d + A$ and satisfy Properties 1 and 2. Consider schedule (5, 4, 2, 1, 3) shifted to the left so that $C_3 = 4943$ which is less than $d + A$ as shown in Fig. 3. Since

**Table 1**
The MSD/T\text{\textsubscript{max}} problem with $d = 4000$ and $A = 944$

<table>
<thead>
<tr>
<th>$j$</th>
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<tbody>
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<td>300</td>
<td>800</td>
<td>900</td>
<td>1000</td>
</tr>
</tbody>
</table>

**Table 2**
All nondominating schedules finishing at $d + A$ and their MSD values

<table>
<thead>
<tr>
<th>Schedule</th>
<th>MSD value</th>
</tr>
</thead>
<tbody>
<tr>
<td>(5 3 2 1 4)</td>
<td>417687</td>
</tr>
<tr>
<td>(5 4 2 1 3)</td>
<td>416567</td>
</tr>
</tbody>
</table>

MSDs of all schedules described in Table 2 are larger than the MSD (416566) of the schedule in Fig. 3, an optimal schedule of the MSD/T\text{\textsubscript{max}} problem should finish before $d + A$.

3. Analysis

By definition of the $A$-unconstrained MSD/T\text{\textsubscript{max}} problem, if the problem is $A$-unconstrained, the maximum tardiness constraint does not affect the minimization of MSD. Thus, the schedule, which is optimal for the unconstrained MSD problem, is also optimal for the MSD/T\text{\textsubscript{max}} problem and vice versa.

**Theorem 1.** The $A$-unconstrained MSD/T\text{\textsubscript{max}} problem is equivalent to the unconstrained MSD problem.

Since the unconstrained MSD problem is equivalent to the completion time variance (CTV) problem as shown in Bagchi et al. (1987), so is the $A$-unconstrained MSD/T\text{\textsubscript{max}} problem. Therefore, any solution procedure for the unconstrained
MSD problem or the CTV problem can be applied to the $\Delta$-unconstrained MSD/T$_{\text{max}}$ problem.

Let $k = \max \{i p_n + \sum_{j=1}^{i} p_j < \theta \}$, where $\theta = (1/n) \{n p_n + (n-1) p_1 + (n-2) p_2 + \cdots + p_{n-1} \}$. De et al. (1989) have shown that any potentially optimal schedule for the unconstrained MSD problem has mean completion time greater than mean completion time $d^*_n$ of schedule $(n, k-1, k-2, \ldots, 2, 1, k, \ldots, n-1)$. By Theorem 1, we observe that any potentially optimal schedule for the $\Delta$-unconstrained MSD problem has maximum tardiness less than or equal to $MS - d^*_n$. The following theorem provides an upper bound on $\Delta^*$ that decides whether or not the MSD/T$_{\text{max}}$ problem is $\Delta$-unconstrained.

**Theorem 2.** Let $\Delta^*_u = \frac{1}{2} (MS - p_n)$. There exists an optimal schedule for the $\Delta$-unconstrained MSD problem so that the maximum tardiness is less than or equal to $\Delta^*_u$. So, $\Delta^*_u$ is an upper bound for $\Delta^*$.

**Proof.** Let $S$ be a schedule of the form $(n, i_2, \ldots, i_n)$ with the smallest CTV, where $i_j$ is the $j$th job in $S$. Let $S'$ be schedule $(n, i_n, \ldots, i_2)$, in which the order of the last $(n-1)$ jobs in $S$ is reversed. Then, the mean completion times of these schedules are as follows:

$$\overline{C}_s = \frac{1}{n} \{ n p_n + (n-1) p_2 + \cdots + p_n \},$$

$$\overline{C}'_s = \frac{1}{n} \{ n p_n + p_2 + \cdots + (n-1) p_n \}.$$

Without loss of generality, it is assumed that $\overline{C}_s \geq \overline{C}'_s$. Note that $\overline{C}_s \geq \frac{1}{2} (\overline{C}_s + \overline{C}'_s) = \frac{1}{2} (MS + p_n)$. By the assumption on $d$, $d$ is greater than or equal to $\overline{C}_s$. An optimal schedule to the $\Delta$-unconstrained MSD problem is obtained by translating schedule $S$ so that the mean completion time coincides with common due date $d$. Let $T_{\text{max}}^*$ be the maximum tardiness of the optimal schedule. Then, we have

$$T_{\text{max}}^* = MS - \overline{C}_s \leq MS - \frac{1}{2} (MS + p_n) = \frac{1}{2} (MS - p_n) = \Delta^*_U.$$

This proves the theorem. □

De et al. (1989) have shown that value \( \frac{1}{2} (MS + p_n) \) is a bound $d^*_n$ of the due date that decides whether or not the MSD problem is unconstrained. As shown in the proof of the theorem above, the tighter the value of $d^*_n$ is, the tighter the value of $\Delta^*_u$ is.

The following theorem shows the relation between the MSD/T$_{\text{max}}$ problem and the MSD problem and it is used to prove that a schedule is optimal for the tightly $\Delta$-constrained MSD/T$_{\text{max}}$ problem with common due date $d$ if and only if its reverse schedule is optimal for the tightly constrained MSD problem with common due date $\Delta$.

**Theorem 3.** A schedule that completes at time $d + \Delta$ is optimal for the MSD/T$_{\text{max}}$ problem with common due date $d$ and maximum allowable tardiness $\Delta$ if and only if its reverse schedule that starts at time zero is optimal for the MSD problem with common due date $\Delta$.

**Proof.** Let $S = (i_1, i_2, \ldots, i_n)$ be an optimal schedule that completes at time $d + \Delta$ for the MSD/T$_{\text{max}}$ problem with a common due date $d$ and maximum allowable tardiness $\Delta$. To prove the result, we show that its reverse schedule $\overline{S} = (i_n, i_{n-1} \cdots i_1)$ that starts at time zero is an optimal schedule for the MSD problem with common due date $\Delta$. The other direction can be shown similarly. Schedules $S$ and $\overline{S}$ are presented in Fig. 4(a) and (b), respectively.

Suppose to the contrary that $\overline{S}$ is not optimal for the MSD problem. Let $\sigma = (j_1, j_2 \cdots, j_n)$ be a schedule for the MSD problem such that $Z(\overline{S}) < Z(\overline{S})$ and its start time is $a$, for $a \geq 0$. Define $\sigma$ as the schedule obtained by reversing the schedule $\sigma$ and making $C_{j_a} = d + \Delta - a$. Schedules $\sigma$ and $\overline{\sigma}$ are shown in Fig. 4(c) and (d), respectively. Now, we have schedule $\sigma$ of jobs 1, 2, \ldots, $n$ for the MSD/T$_{\text{max}}$ problem. It is easily observed that

$$Z(\sigma) = \frac{1}{n} \{ n Z(\sigma) - (MS + a - \Delta)^2 + (\Delta - a)^2 \}$$

$$< \frac{1}{n} \{ n Z(\overline{S}) - (MS - \Delta)^2 + \Delta^2 \}$$

$$= Z(\overline{S}).$$
and hence the mean completion time. The other operation decreases each job’s completion time further. The change in the total completion time caused by moving job \( i \) is \( \sum_{j=1}^{i-1} (p_{j} - p_j) \). After placing jobs \( i_1, i_2, \ldots, i_{m-1} \), just before job 1, the change in the total completion time caused by moving job \( i_m \) before \( i_{m-1} \) is \( \sum_{j=1}^{m-1} (p_{i_j} - p_j) \). Total effect of these operations is

\[
\Delta C = - \sum_{j=1}^{e-1} (p_e - p_j) + \sum_{m=1}^{t} \sum_{j=1}^{i_m-1} (p_{i_j} - p_j)
\]

\[
= - \sum_{j=1}^{e-1} p_e + \sum_{j=1}^{e-1} p_j + \sum_{m=1}^{t} (i_m - 1) p_{i_m}
\]

**Theorem 4.** Let \( S \) be a partial schedule of the form \( (B, \ldots, A) \), where \( B \) is an LPT ordered subsequence, \( A \) is an SPT ordered subsequence and \( B \cup A = \{1+1, l+2, \ldots, n\} \). Define \( k = \min \{ i \mid \sum_{j \in B} p_j + \sum_{j=i}^n p_j < d \} \). Consider schedule \( S' = (B, 1, l-1, \ldots, k, k-1, 1, \ldots, k-2, A) \). Let \( \overline{C}_{v'} \) be the mean completion time of \( S' \). Then any \( v \)-about-\( d \) schedule that may be formed by completing \( S \) has mean completion time smaller than \( \overline{C}_{v'} \).

**Proof.** Note that schedule \( S' \) cannot be \( v \)-about-\( d \) because \( R_{k-1} > d \). To prove the result, it suffices to show that any attempt to make the schedule \( S' \) \( v \)-about-\( d \) decreases the mean completion time.

There are two ways to make \( S' \) \( v \)-about-\( d \). One way is to shift \( S' \) to the left until \( R_{k-1} < d \). But, this operation decreases each job’s completion time and hence the mean completion time. The other way is to shuffle jobs scheduled before and after job 1 so that job 1 moves to the left, which makes the schedule \( S' \) \( v \)-about-\( d \). It will be shown that exchanging one job \( e \) scheduled before job 1 and jobs \( i_1, i_2, \ldots, i_t \) \((0 < i_t \leq v - 2)\) scheduled after job 1 decreases the mean completion time (Fig. 5). We do not need to consider shuffling more than one job scheduled before job 1 since moving more jobs scheduled before job 1 to the end decreases the mean completion time further.

To maintain \( v \)-shape, job \( e \) should be placed at the end. The change in the total completion time caused by moving job \( e \) is \(- \sum_{j=1}^{e-1} (p_e - p_j) \). After placing jobs \( i_1, i_2, \ldots, i_{m-1} \), just before job 1, the change in the total completion time caused by moving job \( i_m \) before \( i_{m-1} \) is \( \sum_{j=1}^{m-1} (p_{i_j} - p_j) \). Total effect of these operations is

\[
\Delta C = - \sum_{j=1}^{e-1} (p_e - p_j) + \sum_{m=1}^{t} \sum_{j=1}^{i_m-1} (p_{i_j} - p_j)
\]

\[
= - \sum_{j=1}^{e-1} p_e + \sum_{j=1}^{e-1} p_j + \sum_{m=1}^{t} (i_m - 1) p_{i_m}
\]
\[ -\sum_{m=1}^{t} \sum_{j=1}^{i_{m} - 1} p_{j} \]

\[ = -(i_{1} - 1) \left( \sum_{m=1}^{t} p_{i_{m}} - p_{e} \right) + \sum_{m=2}^{t} (i_{m} - i_{1}) p_{i_{m}} \]

\[ - (e - i_{1}) p_{e} + \sum_{j=1}^{e-1} p_{j} - \sum_{m=1}^{t} \sum_{j=1}^{i_{m} - 1} p_{j} \]

\[ \leq (i_{1} - 1) \left( \sum_{m=1}^{t} p_{i_{m}} - p_{e} \right) \]

\[ + (i_{1} - i_{1}) \left( \sum_{m=2}^{t} p_{i_{m}} - p_{e} \right) \]

\[ - (e - i_{1}) p_{e} + \sum_{j=1}^{e-1} p_{j} - \sum_{m=1}^{t} \sum_{j=1}^{i_{m} - 1} p_{j} \]

\[ = (i_{1} - 1) \left( \sum_{m=1}^{t} p_{i_{m}} - p_{e} \right) + (i_{1} - i_{1}) \]

\[ \times \left( \sum_{m=2}^{t} p_{i_{m}} - p_{e} \right) - \sum_{j=i_{1}}^{e-1} (p_{e} - p_{j}) - \sum_{m=1}^{t} \sum_{j=1}^{i_{m} - 1} p_{j}. \]

To shift job 1 to the left, \((\sum_{m=1}^{t} p_{i_{m}} - p_{e})\) must be negative. This makes \(AC\) negative and decreases the mean completion time. Hence, the theorem follows.

**Corollary 2.** Let \(\delta\) be the mean completion time of the LPT ordered schedule, i.e., \(\delta = (1/n) \{np_{n} + (n-1)p_{n-1} + \cdots + p_{1}\}\). Let \(k = \min\{i \mid \sum_{j=i}^{n} p_{j} < \delta\}\). Consider schedule \(S = (n,n-1,\ldots,k,k-1,1,\ldots,k-2)\). Let \(\overline{C}_{s}\) be the mean completion time of \(S\). Then, any v-about-d schedule has mean completion time smaller than \(\overline{C}_{s}\).

**Corollary 3.** Let \(A_{r}^{*} = MS - \overline{C}_{s}\), where \(\overline{C}_{s}\) is the value defined in Corollary 2. Any potentially optimal schedule to the \(A\)-constrained \(MSD/T_{\text{max}}\) problem has maximum tardiness greater than \(A_{r}^{**}\). So, \(A_{r}^{**}\) is a lower bound to decide whether or not the \(MSD/T_{\text{max}}\) problem is tightly \(A\)-constrained.

### 4. The solution procedure

In this section, a solution procedure for the \(MSD/T_{\text{max}}\) problem is presented. The following procedures will be used in our solution procedure:

1. **Procedure 1 (unconstrained MSD procedure).** The unconstrained MSD algorithm in Bagchi et al. (1987).


3. **Procedure 3 (A-constrained MSD procedure).** Run Procedure 2 with common due date \(A\) and find an optimal sequence. Then, reverse the optimal sequence. The final schedule completes at time \((d + A)\).

4. **Procedure 4 (A-constrained CTV (A-CCTV) procedure).** Find all feasible sequences which completes at \(d + A\) and whose mean completion time is in between \(d\) and \((d + A)\). Then, choose among them the sequence with minimum CTV value. The start time of the obtained sequence is determined by making the mean completion time of the schedule coincide with the given due date.

In the previous sections, it was shown that the \(MSD/T_{\text{max}}\) problem can be classified into the \(A\)-unconstrained, \(A\)-constrained and tightly \(A\)-constrained cases according to the value of maximum allowable tardiness \(A\). Then, we showed that \(A_{r}^{*}\) is a lower bound to distinguish the tightly \(A\)-constrained MSD/T_{\text{max}} problem from the \(A\)-constrained MSD/T_{\text{max}} problem and \(A_{r}^{**}\) is an upper bound to distinguish the \(A\)-unconstrained MSD/T_{\text{max}} problem from the \(A\)-constrained MSD/T_{\text{max}} problem. Our solution procedure first calculates \(A_{r}^{**}\) and \(A_{r}^{*}\), and check if Corollary 1 or Corollary 3 holds. If Corollary 1 (Corollary 3) holds, Procedure 1 (Procedure 2) is employed to obtain an optimal solution for the MSD/T_{\text{max}} problem. Otherwise, both Procedures 3 and 4 are run and the better solution is chosen. The solution procedure for the MSD/T_{\text{max}} problem is stated as follows:

5. **Procedure 5 (MSD/T_{\text{max}} procedure).**

   **Step 1.** Compute the values of \(A_{r}^{**}\) and \(A_{r}^{*}\).

   **Step 2.** If \(A \geq A_{r}^{*}\), then run Procedure 1 to obtain an optimal solution and stop.

   **Step 3.** If \(A \leq A_{r}^{**}\), then run Procedure 3. Stop.

   **Step 4.** Run both Procedures 3 and 4. Choose the better solution and stop.
Consider the 7-job problem taken from Table 2 in Eilon and Chowdhury (1977) (Table 3).

Now, by varying the value of maximum allowable tardiness, we present four instances of the MSD/T\textsuperscript{max} problem in which an optimal schedule is obtained from different procedures. The values of D\textsubscript{l} and D\textsubscript{u} are 27.571 and 53, respectively.

First, consider an instance of the MSD/T\textsuperscript{max} problem with D\hat{=} 80. In this instance, the maximum allowable tardiness is larger than D\textsubscript{u}. Hence this problem is classified into the D\textsuperscript{-}unconstrained case. According to the solution procedure, the optimal schedule (7 6 1 2 3 4 5) is obtained by Procedure 1 as illustrated in Bagchi et al. (1987). The starting time of the schedule is determined by making the mean completion time of this schedule coincide with the due date.

Next, consider an instance with D\hat{=} 15. The instance is tightly D\textsuperscript{-}unconstrained since D < D\textsuperscript{u}. Hence this problem is classified into the D\textsuperscript{}-unconstrained case. According to the solution procedure, the optimal schedule (7 6 1 2 3 4 5) is obtained by Procedure 1 as illustrated in Bagchi et al. (1987). The starting time of the schedule is determined by making the mean completion time of this schedule coincide with the due date.

The branching procedure creates 10 full schedules, and evaluates three full schedules. The optimal schedule is (4 3 1 2 5 6 7) with Z = 5564.429. Note that this optimal schedule is not for the original problem. To obtain the optimal schedule for the MSD/T\textsuperscript{max} problem, the schedule (4 3 1 2 5 6 7) is reversed and the completion time is set to d + \Delta. Therefore, the schedule (7 6 5 2 1 3 4) with starting time at 27 is an optimal schedule for this problem. The MSD value of this schedule is 1321.

Third, consider an instance with \Delta = 30. Both Procedures 3 and 4 are executed since D\textsuperscript{l} < 30 < D\textsuperscript{u}. From Procedure 3, we obtain schedule (7 6 4 1 2 3 5) with Z = 984.429. Next, consider Procedure 4, a branching procedure using the left bound of potentially optimal schedule. It sets the left bound of potentially optimal schedule on the given due date. As illustrated in Fig. 7, the algorithm starts with the partial schedule (7 x x x x x x) because the job with the longest processing time is located first in an optimal schedule.

This partial schedule completes at d + \Delta and the upper bound of the mean completion time of this partial schedule is 203.571 (the LPT order of rest jobs makes this). As shown in Fig. 7, if the partial schedule is (7 6 4 x x x x) or (7 5 x x x x x), the maximum mean completion time of schedules that can be obtained by completing the partial schedule is 194.429 or 163.000. Since we are looking for a schedule whose mean completion

<table>
<thead>
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<td>6</td>
<td>9</td>
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<td>65</td>
<td>82</td>
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</tbody>
</table>

Table 3
The MSD/T\textsuperscript{max} problem with d = 200

![Fig. 6. Solving the MSD problem with \Delta = 15.](image)
time is in between \( d \) and \( d + \Delta \), all schedules produced from these partial schedules cannot be a candidate for the optimal schedule of this problem. Procedure 4 creates three full schedules, and evaluates only one schedule whose MSD is 1100.980. Since the objective function value of the schedule obtained from Procedure 3 is lower, schedule (7 6 4 1 2 3 5) is an optimal schedule.

Last, consider an instance with \( \Delta = 40 \). Like the previous instance, both Procedures 3 and 4 are executed. The values of the objective functions or schedules obtained from Procedures 3 and 4 are 919.285 and 918.285, respectively. So, in this case, Procedure 4 produces better solution. Finally, the optimal schedule of this MSD/\( T_{\max} \) problem is obtained by determining the starting time of the schedule so that the mean completion of the schedule obtained from Procedure 4 coincides with the given due date.

5. Conclusion

We have investigated the MSD problem with maximum tardiness constraint (MSD/\( T_{\max} \) problem) in this paper. The MSD/\( T_{\max} \) problem was classified into \( \Delta \)-unconstrained, \( \Delta \)-constrained and tightly \( \Delta \)-constrained cases. Also, the upper and lower bounds on maximum allowable tardiness are provided. Similar work on the MSD problem was done in De et al. (1989) and Bagchi et al. (1987), where the problem was classified according to the value of a common due date. But, the MSD/\( T_{\max} \) problem is classified according to the value of newly imposed constraint, i.e., maximum allowable tardiness. It has been shown that the \( \Delta \)-unconstrained MSD/\( T_{\max} \) problem is equivalent to the unconstrained MSD problem and the tightly \( \Delta \)-constrained MSD/\( T_{\max} \) problem with common due date \( d \) is equivalent to the tightly constrained MSD problem with common due date \( \Delta \). An optimal solution in each case is obtained by different procedures. For better illustration, the procedure to solve the MSD/\( T_{\max} \) problem has been presented along with corresponding examples.

Further study will consider the MSD/\( T_{\max} \) problem with different weights on earliness and tardiness and other problems having different objective function with maximum allowable tardiness constraint.

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