Caratheodory-Hamilton-Jacobi Theory in Optimal Control

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1. Introduction

In articles published in 1925 [1] and 1926 [2], and culminating in his book in 1935 [3] (see also [4]), Caratheodory introduced a unifying approach to problems in the Calculus of Variations, namely the concept of equivalent problems. This method leads immediately to the classical Hamilton-Jacobi theory. Recently, this approach has been modified to be useful in the solution of optimal control problems [5, 6, 7, 8, 9]. This paper will present and amplify the modified method.

The problem considered is as follows: suppose we are given the system of $n$ first-order ordinary differential equations:

$$
\dot{x}_i = f_i(t, x_1, ..., x_n, u_1, ..., u_m),
$$

where the control vector $u = (u_1, ..., u_m)$ is to be taken from a given set of vectors, $U$, which satisfies:

(i) each $u \in U$ is a continuously differentiable function of $t$;

(ii) for each $u \in U$ and at each time $t$, there are small variations $\delta u \in U$ such that $u + \delta u$ is in $U$ for $\delta u$'s of either sign.

Requirement (ii) here means that the range set is an open set in $E^n$. An example of a suitable set $U$ is the set of all vectors whose components are continuously differentiable functions with $|u_\alpha(t)| < 1$ for each $\alpha$. Since piecewise continuous functions can be approximated by continuously differentiable ones and closed sets can be approximated by open ones, the method described in this paper can be used as an approximation technique for more general cases. Bridgland [10] has generalized the method to include measurable control functions.

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The functions \( f_i \) are assumed to satisfy conditions sufficient to guarantee existence and uniqueness of solutions, \( x(t) = (x_1(t), \ldots, x_n(t)) \) of the differential equations on \( t_0 \leq t \leq T \) for any \( u \in U \), where \( t_0 \) and \( T \) are given numbers. Assumptions of differentiability of the \( f_i \)'s will be stated later in the hypotheses of the theorems.

For each control \( u(t) \) and corresponding trajectory, \( x(t) \), suppose a functional is defined by

\[
J[u] = \int_{t_0}^{T} L(t, x, u) \, dt,
\]

where \( L(t, x, u) \) is a given function satisfying differentiability assumptions to be stated later, and suppose further, that we are given two states:

\[
x_0 = (x_{10}, x_{20}, \ldots, x_{n0}) \quad \text{and} \quad x_T = (x_{1T}, x_{2T}, \ldots, x_{nT}).
\]

**Definition.** A control vector which is in \( U \) and which drives the system from the initial state \( x(t_0) = x_0 \) to the final state \( x(T) = x_T \) will be called an *admissible* control.

The optimal control problem we consider is then: find an admissible control vector which minimizes the functional (1.2).

### 2. Equivalent Problems

Suppose \( S(t, x) \) is any given function that is continuously differentiable. We then define

\[
L^*(t, x, u) = L(t, x, u) - \frac{\partial S}{\partial t}(t, x) - \sum_i \frac{\partial S}{\partial x_i}(t, x)f_i(t, x, u)
\]

and

\[
J^*[u] = \int_{t_0}^{T} L^*(t, x, u) \, dt.
\]

By using any control \( u \) and integrating along the corresponding trajectory,

\[
J^*(u) = J[u] - \int_{t_0}^{T} \frac{dS}{dt} \, dt
\]

since along the trajectory, \( f_i(t, x, u) = dx_i/dt \). Then

\[
J^*[u] = J[u] - S(t, x)|_{t_0}^{T}.
\]

For specified end states \( x_0 \) and \( x_T \), the last term on the right hand side of (2.3) is a known constant for a given \( S(t, x) \) and does not depend on the trajectory.
between \( x_0 \) and \( x_T \) since \( dS/dt \) is a total derivative. Hence, any admissible control \( u \) which minimizes (2.2) also minimizes (1.2) and vice versa. Thus, minimizing (2.2) subject to (1.1) is equivalent to minimizing (1.2) subject to (1.1).

By the above, any continuously differentiable \( S(t, x) \) leads to an equivalent problem. However, certain choices of the function \( S(t, x) \) lead to equivalent problems for which the minimization of \( J^*[u] \) is easily accomplished. This will be the case if \( S(t, x) \) is such that for each \((t, x)\), \( L^*(t, x, u) \) satisfies the following two conditions:

(i) \( L^*(t, x, u) \geq 0 \) for all \( u \in U \), and
(ii) there is an admissible \( \bar{u} = \bar{u}(t, x) \) such that \( L^*(t, x, \bar{u}) = 0 \).

By using such an \( S(t, x) \), the equivalent problem has

\[
J^*[u] \geq 0 \quad \text{and} \quad J^*[\bar{u}] = 0.
\]

Thus \( \bar{u} \) is an optimal control which minimizes (2.2), and hence also (1.2). We will not be able to find such an \( S(t, x) \) for all problems.

3. Hamilton-Jacobi Partial Differential Equation

We now determine the conditions that \( S(t, x) \) must satisfy so that it makes \( L^* \) satisfy (2.4) and (2.5). Conditions (2.4) and (2.5) require that \( L^* \) have a minimum at \( \bar{u} \) as a function of \( u \) and that the value of \( L^* \) at this minimum be zero. Since \( \partial S/\partial t \) does not depend on \( u \), this may be stated as

\[
\min_{u \in U} \left[ L(t, x, u) - \sum_{i} \frac{\partial S}{\partial x_i}(t, x) f_i(t, x, u) \right] - \frac{\partial S}{\partial t}(t, x) = 0. \tag{3.1}
\]

(Note the close relationship to dynamic programming.)

Now suppose \( L \) and the \( f_i \)’s are continuously differentiable with respect to the control functions \( u_\alpha \). Then a minimizing control \( \bar{u} \) must satisfy

\[
\frac{\partial L}{\partial u_\alpha}(t, x, \bar{u}) = \sum_{i} \frac{\partial S}{\partial x_i}(t, x) \frac{\partial f_i}{\partial u_\alpha}(t, x, \bar{u}), \quad \alpha = 1, 2, \ldots, m \tag{3.2}
\]
since the range set of the control vector functions is open. Assumption (ii) on the control set \( U \) in Section 1 was made so that the minimizing control \( \bar{u} \) could be characterized by this relation. We will write this as

\[
\frac{\partial L}{\partial u_\alpha}(t, x, \bar{u}) = \sum_{i} p_i \frac{\partial f_i}{\partial u_\alpha}(t, x, \bar{u}), \quad \alpha = 1, 2, \ldots, m \tag{3.3}
\]
in which we consider the $p_i$'s as independent variables, and make the assumption that we can solve these $m$ relations among the $u_i$'s to find

$$
\tilde{u} = \tilde{\mathcal{U}}(t, x, p),
$$

(3.4)

where $p = (p_1, ..., p_m)$ and $\tilde{u} \in U$. There may be several such solutions.

Having found a function (3.4), we set $\frac{\partial S}{\partial x_i} = p_i$ and use it in (3.1) to obtain

$$
\frac{\partial S}{\partial t} + \sum_i \frac{\partial S}{\partial x_i} f_i(t, x, \tilde{\mathcal{U}}(t, x, \frac{\partial S}{\partial x})) - L(t, x, \tilde{\mathcal{U}}(t, x, \frac{\partial S}{\partial x})) = 0,
$$

(3.5)

or with the Hamiltonian defined by

$$
H(t, x, p) = \sum_i p_i f_i(t, x, \tilde{\mathcal{U}}(t, x, p)) - L(t, x, \tilde{\mathcal{U}}(t, x, p)),
$$

(3.6)

we have

$$
\frac{\partial S}{\partial t} + H(t, x, \frac{\partial S}{\partial x}) = 0.
$$

(3.7)

This is the Hamilton-Jacobi partial differential equation. Thus, for each function $\tilde{\mathcal{U}}(t, x, p)$ in (3.4) which satisfies the minimizing conditions (3.3), there is a corresponding Hamilton-Jacobi equation. We see that if $S(t, x)$ is any function which makes $L^*$ satisfy conditions (2.4) and (2.5), then it must satisfy one of these partial differential equations. It should be emphasized here that obtaining these partial differential equations was only a matter of inverting systems of finite equations, not integrating differential equations.

If $L$ and the $f_i$'s are linear in the controls, conditions (3.3) are independent of the $u_i$'s and this method can not be used directly. Such linear problems are usually referred to as singular problems and require special techniques [11].

The Hamilton-Jacobi partial differential equation is first order, but is usually non-linear. By our differentiability assumptions on $L$, $\tilde{\mathcal{U}}$, and the $f_i$'s, $H(t, x, p)$ is continuously differentiable in its arguments. Then by characteristic theory for first order partial differential equations, the Hamilton-Jacobi equation has infinitely many solutions. Suppose $S(t, x)$ is a particular solution of a Hamilton-Jacobi equation which corresponds to one of the solutions (3.4) of the minimizing conditions (3.3). Using this $S(t, x)$ in the corresponding function (3.4), we obtain a “feedback control”:

$$
\tilde{u} = \tilde{u}(t, x) = \tilde{\mathcal{U}}(t, x, \frac{\partial S}{\partial x}(t, x)).
$$

(3.8)

For each initial state used, this control will drive the system to a final state at time $T$ that depends on the initial state. Along any of these trajectories,
the control $\tilde{u}$ given by (3.8) makes $L^*$ satisfy condition (2.5); that is, $L^*(t, x, \tilde{u}) = 0$ since this is just the Hamilton-Jacobi equation (3.7). Then since $J^*[\tilde{u}] = 0$, (2.3) gives

$$S(t, x)^T_{t_0} = J[\tilde{u}], \quad (3.9)$$

i.e., the value of the functional (1.2) taken along the path given by $\tilde{u}$ between a pair of these corresponding end states is given by the difference of the values of $S(t, x)$ at the end states. Berkovitz [12], and others have shown previously that the "value function" in (3.9) when taken along optimal trajectories satisfies the Hamilton-Jacobi equation. However, we now see that this is true for any control, optimal or not, which arises in the above manner. The control $\tilde{u}$ may not be optimal since we do not know whether $S(t, x)$ makes $L^*$ satisfy condition (2.4); i.e., whether $L^*(t, x, u) \geq 0$ for all $u \in U$. Showing that (2.4) is satisfied is a condition sufficient to guarantee that $\tilde{u}$ is optimal for these end states. (See Section 5 below.)

Since the initial and final states for our problem are given, (1.3), we will only be interested in those solutions of the respective Hamilton-Jacobi equations which lead to admissible controls when used in the corresponding function (3.4); i.e., such that the trajectory starting at $x_0$ ends at $x_T$. For each of these trajectories (optimal or not) between $x_0$ and $x_T$, there will be infinitely many solutions of the corresponding Hamilton-Jacobi equation. The reason for this is that the trajectories (optimal or not) given by the feedback control (3.8) are (shown in Section 6 to be) characteristic curves of the particular Hamilton-Jacobi equation which corresponds to the function (3.4), and infinitely many solution surfaces of a partial differential equation intersect along a characteristic of it.

4. **HAMILTON-JACOBI EQUATION BY PONTRYAGIN MAXIMUM PRINCIPLE**

We have shown by the Caratheodory approach that the value of the functional, when using a feedback control as above, is given by a function (3.9) of the end states which satisfies the Hamilton-Jacobi equation (3.7). We will now show this is true (cf [12]), under the same differentiability assumptions, for any optimal control by means of the necessary condition called the Pontryagin Maximum Principle [13]. This will introduce the adjoint system and variables and allow us to show in the next section the relationship between the Maximum Principle and the Caratheodory approach.

We first define a new Hamiltonian function:

$$H_0(t, x, p, u) = \sum_i p_if_i(t, x, u) - L(t, x, u). \quad (4.1)$$
In this function the variables \( u \) appear explicitly and are to be considered as independent variables. The \( p \)'s here are the adjoint variables and are required to satisfy the adjoint differential equation system:

\[
\dot{p}_i = -\frac{\partial H}{\partial x_i}(t, x, p, u) = -\sum_j p_j \frac{\partial f_j}{\partial x_i}(t, x, u) + \frac{\partial L}{\partial x_i}(t, x, u).
\] (4.2)

The Pontryagin Maximum Principle states that if \( \bar{u}(t) \) is an optimal control which gives \( \bar{x}(t) \) as the optimal trajectory, then there must be a continuous vector \( p = (p_1, \ldots, p_n) \) which satisfies the adjoint system (4.2) with \( \bar{x}(t) \) and \( \bar{u}(t) \) and for which the Hamiltonian, \( H_0(t, x, p, u) \), as a function of \( u \), is maximized by \( \bar{u} \). Then since we have assumed the extremum occurs in the interior of the range of the control set (the range is an open set), we can set the partial derivatives of \( H_0 \) with respect to the \( u \)'s equal to zero to obtain relations that the optimal controls must satisfy:

\[
\frac{\partial H_0}{\partial u_\alpha}(t, x, p, \bar{u}) = 0, \quad \alpha = 1, 2, \ldots, m
\]

or

\[
\sum_j p_j \frac{\partial f_j}{\partial u_\alpha}(t, x, \bar{u}) = \frac{\partial L}{\partial u_\alpha}(t, x, \bar{u}), \quad \alpha = 1, 2, \ldots, m.
\] (4.3)

We note that these conditions are precisely conditions (3.2), which we obtained before in the Caratheodory approach. Solving them gives

\[
\bar{u} = \overline{U}(t, x, p).
\] (4.4)

We now assume that there is a region containing \( x_0 \) in state space which is covered by a field; i.e., through each state \( X \) in the region, other than \( x_0 \), and for each time \( T, t_0 \leq T \leq T_1 \), where \( T_1 \) is given, there is exactly one optimal trajectory which has initial state \( x_0 \) and final state \( x(T) = X \). Let us consider the functional

\[
J[u; T, X] = \int_{t_0}^{T} L(t, x(t), u(t)) \, dt,
\] (4.5)

where \( u(t) \) is any control in \( U \) which drives the system to \( x(T) = X \) from \( x_0 \). This functional is minimized by the \( \bar{u}(t) \) that gives the optimal trajectory \( x(t) \) between the end states. Define

\[
S(T, X) = J[\bar{u}; T, X].
\] (4.6)

By our assumptions of a field, \( S(T, X) \) is a single-valued function defined everywhere in the region.
Now suppose we consider the first variation of functional (4.5) about the optimal trajectory. To do this, we change the control to $\tilde{u} + \delta u$, which determines a new trajectory $\tilde{x} + \delta x$ (not necessarily optimal) near the optimal trajectory $x(t)$, the new trajectory terminating at $X + \delta X$ at time $T + \delta T$. The new trajectory will be near the old one for small changes in the control because of the theorem of continuous dependence of the solution of a differential equation system on the parameters of the system. The first variation, $\delta J$, is the first order terms of the difference between $J$ evaluated along this new trajectory and along the old one which was an optimal trajectory. The total difference is

$$
\Delta J = J[\tilde{u} + \delta u; T + \delta T, X + \delta X] - J[\tilde{u}; T, X]
$$

$$
= \int_{t_0}^{T+\delta T} L(t, \tilde{x} + \delta x, \tilde{u} + \delta u) \, dt - \int_{t_0}^{T} L(t, \tilde{x}, \tilde{u}) \, dt
$$

$$
= \int_{t_0}^{T+\delta T} L(t, \tilde{x} + \delta x, \tilde{u} + \delta u) \, dt
$$

$$
+ \int_{t_0}^{T} \left[ L(t, \tilde{u} + \delta x, \tilde{u} + \delta u) - L(t, \tilde{x}, \tilde{u}) \right] \, dt.
$$

Expanding $L(t, \tilde{x} + \delta x, \tilde{u} + \delta u)$ in a Taylor Series about $\tilde{x}$ and $\tilde{u}$, the first order effects are

$$
\delta J = L(t, \tilde{x}, \tilde{u})|_{t = T} \delta T + \int_{t_0}^{T} \left[ \sum_i \frac{\partial L}{\partial x_i} (t, \tilde{x}, \tilde{u}) \delta x_i(t) + \sum_\alpha \frac{\partial L}{\partial u_\alpha} (t, \tilde{x}, \tilde{u}) \delta u_\alpha(t) \right] \, dt
$$

$$
= L(t, \tilde{x}, \tilde{u})|_{t = T} \delta T
$$

$$
+ \int_{t_0}^{T} \left[ \sum_i \frac{\partial L}{\partial x_i} (t, \tilde{x}, \tilde{u}) \delta x_i(t) + \sum_\alpha \frac{\partial L}{\partial u_\alpha} (t, \tilde{x}, \tilde{u}) \delta u_\alpha(t) \right] \, dt
$$

by the maximizing condition (4.3). The new trajectory satisfies

$$(\tilde{x}_j + \delta x_j) = f_j(t, \tilde{x} + \delta x, \tilde{u} + \delta u), \quad j = 1, 2, \ldots, n.$$ 

Expanding in a Taylor series and neglecting terms higher than first order, we obtain

$$
\delta x_j(t) = \sum_i \frac{\partial f_i}{\partial x_i} (t, \tilde{x}, \tilde{u}) \delta x_i(t) + \sum_\alpha \frac{\partial f_i}{\partial u_\alpha} (t, \tilde{x}, \tilde{u}) \delta u_\alpha(t)
$$

since $\tilde{x}(t)$ satisfies system (1.1) with $\tilde{u}(t)$. By using this, $\delta J$ becomes

$$
\delta J = L(t, \tilde{x}, \tilde{u})|_{t = T} \delta T
$$

$$
+ \int_{t_0}^{T} \left[ \sum_i \left( \frac{\partial L}{\partial x_i} (t, \tilde{x}, \tilde{u}) - \sum_j p_j(t) \frac{\partial f_i}{\partial x_i} (t, \tilde{x}, \tilde{u}) \right) \delta x_i(t) + \sum_j p_j(t) \delta x_j(t) \right] \, dt.
$$
The term in square brackets is just $\hat{p}_i(t)$ by the adjoint system (4.2) with $\bar{u}(t)$ and $\bar{x}(t)$. Thus

$$
\delta J = \left[ L(t, \bar{x}, \bar{u}) \right]_{t\to T} \delta T + \int_{t_0}^{T} \sum_i \frac{d}{dt} \left[ p_i(t) \delta x_i(t) \right] dt
$$

$$
= \left[ L(t, \bar{x}, \bar{u}) \right]_{t\to T} \delta T + \sum_i p_i(t) \left. \delta x_i(t) \right|_{t_0}^{T}.
$$

Since the initial time $t_0$ and initial state $x_{i0}$ are considered fixed, all the $\delta x_i(t_0)$'s are zero. At the terminal end of the trajectory we have

$$
\delta X_i = \delta x_i(T) + \dot{x}_i(T) \delta T
$$

or

$$
\delta x_i(T) = \delta X_i - f_i(t, \bar{x}, \bar{u}) \left. \right|_{t\to T} \delta T.
$$

Using this we obtain

$$
\delta J = \left[ L(t, \bar{x}, \bar{u}) - \sum_i p_i(t) f_i(t, \bar{x}, \bar{u}) \right]_{t\to T} \delta T + \sum_i p_i(T) \delta X_i. \quad (4.7)
$$

We have thus shown that the variation of $J$ about an optimal trajectory (i.e., the first order changes in $J$ due to the varied trajectory) can be expressed in terms of the changes of the final time and state only, and we do not need to consider the changes of the trajectory along the trajectory.

The above result, (4.7), holds for any trajectory (optimal or not) close to the optimal trajectory considered. Now consider the value of the function $S(T, X)$, defined by (4.6), at the point $(T + \delta T, X)$. This will be the value of $J$ when computed along the optimal trajectory to point $(T + \delta T, X)$. Since (4.7) holds for all neighboring trajectories, we have

$$
S(T + \delta T, X) - S(T, X) = \left[ L(t, \bar{x}, \bar{u}) - \sum_i p_i(t) f_i(t, \bar{x}, \bar{u}) \right]_{t\to T} \delta T + 0(\delta T^2).
$$

Then, dividing by $\delta T$ and letting $\delta T \to 0$, we see that

$$
\frac{\partial S(T, X)}{\partial T} = \left[ L(t, \bar{x}, \bar{u}) - \sum_i p_i(t) f_i(t, \bar{x}, \bar{u}) \right]_{t\to T}. \quad (4.8)
$$

By similar reasoning with $S(T, X + \delta X) - S(T, X)$ it can be shown that

$$
\frac{\partial S(T, X)}{\partial X} = \hat{p}_i(T). \quad (4.9)
$$
Now recall that \( a(T) = X \) and, by (4.4), \( \bar{u}(T) = \bar{H}(T, x(T), p(T)) \). Using these and (4.9) and the definition of \( H_0 \) given by (4.1), in (4.8), we obtain

\[
\frac{\partial S(T, X)}{\partial T} + H_0 \left( T, X, \frac{\partial S(T, X)}{\partial X}, \bar{H} \left( T, X, \frac{\partial S(T, X)}{\partial X} \right) \right) = 0. \tag{4.10}
\]

Comparing definitions (4.1) for \( H_0 \) and (3.6) for \( H \), we see that the partial differential equation (4.10) is the Hamilton-Jacobi equation (3.7) with \( (T, X) \) in place of \((t, x)\). By (4.9), the adjoint variables are really the coefficients that express the sensitivity of \( S(T, X) \) to changes in the end conditions \( \delta X_i \).

5. SUFFICIENT CONDITIONS AND THE WEIERSTRASS EXCESS FUNCTION

The Carathéodory method leads quite easily to a set of sufficient conditions. Basically, these are just hypotheses that guarantee that \( L^*(t, x, u) \) satisfies condition (2.4). We will show that they are the optimal control analogs of the classical sufficient conditions (pp. 146–149 of [14] and pp. 83–87 of [15]) which are based on the Weierstrass Excess Function. To facilitate the statement of the theorems, we define:

**HYPOTHESIS A.** Let \( L \) and the \( f_i \)'s of functional (1.2) and system (1.1) be continuously differentiable in all their arguments. Let \( s(t, x) \) (used in \( L^* \)) be a twice continuously differentiable function which is a solution of the Hamilton-Jacobi equation (3.7) where the function \( \bar{H}(t, x, p) \) is continuously differentiable and satisfies the minimizing condition (3.3). Let \( \bar{x}(t) \) be the trajectory starting at \( \bar{x}(t_0) = x_0 \) and ending at \( \bar{x}(T) = x_T \) which results from using

\[
\bar{u}(t, x) = \bar{H} \left( t, x, \frac{\partial S}{\partial x} (t, x) \right)
\]

as the control, and assume that \( \bar{u} \in U \).

**THEOREM 5.1.** Assume Hypothesis A holds. Then if for each point \((t, x)\) and for all \( u \in U \),

\[
L^*(t, x, u) \geq 0, \tag{5.1}
\]

the trajectory \( \bar{x}(t) \) is optimal and \( \bar{u}(t, \bar{x}(t)) \) is an optimal control in \( U \) for the end states given.

**PROOF.** On the basis of these assumptions, the equivalent problem (see Section 2) has \( f^*[u] \geq 0 \) for all \( u \in U \). Since the Hamilton-Jacobi equation
is satisfied, $J^*[\bar{u}] = 0$. Therefore, $\bar{u}(t, x)$ minimizes the functional and hence is optimal. Q.E.D.

By analogy to the classical definition of the Weierstrass Excess Function we define the following function:

$$E(t, x, v, u) = L^*(t, x, u) - L^*(t, x, v) - \sum \frac{\partial L^*}{\partial u_\alpha} (t, x, v)(u_\alpha - v_\alpha)$$

(5.2)

$$= L(t, x, u) - L(t, x, v) - \sum \frac{\partial S}{\partial x_i} (t, x)[f_i(t, x, u) - f_i(t, x, v)]$$

$$- \sum \left[ \frac{\partial L}{\partial u_\alpha} (t, x, v) - \sum \frac{\partial S}{\partial x_i} (t, x) \frac{\partial f_i}{\partial u_\alpha} (t, x, v) \right] (u_\alpha - v_\alpha).$$

(5.3)

By using the $S(t, x)$ of Hypothesis A and replacing $v$ by $\bar{u}$, Eq. (5.2) reduces to

$$E(t, x, \bar{u}, u) = L^*(t, x, u)$$

(5.4)

since $L^*(t, x, \bar{u}) = 0$ and the minimizing condition (3.2) is satisfied. Also, for $S(t, x)$ and $\bar{u}$, Eq. (5.3) becomes

$$E(t, x, \bar{u}, u) = H_0 \left( t, x, \frac{\partial S}{\partial x}, \bar{u} \right) - H_0 \left( t, x, \frac{\partial S}{\partial x}, u \right),$$

(5.5)

where $H_0$ is the (Pontryagin) Hamiltonian defined in (4.1). Hence, for $S(t, x)$ and $\bar{u}(t, x)$:

$$L^*(t, x, u) = H_0 \left( t, x, \frac{\partial S}{\partial x}, \bar{u} \right) - H_0 \left( t, x, \frac{\partial S}{\partial x}, u \right).$$

(5.6)

We now see the relationship between the Carathéodory approach, the Weierstrass Excess Function, and the Pontryagin Maximum Principle. The analog of the Weierstrass necessary condition would be that $E(t, x, \bar{u}, u) \geq 0$ along an optimal trajectory. By Eq. (5.6), this corresponds to the Hamiltonian being maximized along an optimal trajectory for a certain set of adjoint variables, i.e., to the Pontryagin Maximum Principle.

By (5.2), we see that if $L^*$ is twice continuously differentiable in the $u_\alpha$'s, the Excess Function is the difference between the value of the function $L^*(t, x, u)$ (as a function of the $u_\alpha$'s) at the point $u$ and the first two terms of its Taylor Series expansion about the point $v$. Thus, $E(t, x, v, u)$ can also be expressed as the remainder of the Taylor Series, i.e.,

$$E(t, x, v, u) = \frac{1}{2} \sum_{\alpha, \beta} \frac{\partial^2 L^*}{\partial u_\alpha \partial u_\beta} (t, x, \bar{u})(u_\alpha - v_\alpha)(u_\beta - v_\beta).$$
where \( \tilde{u} = u + \theta(u - v) \), \( 0 \leq \theta \leq 1 \). Or, by (5.4) with an appropriate \( S(t, x) \) and \( \tilde{u}(t, x) \):

\[
L^*(t, x, u) = \frac{1}{2} \sum_{\alpha, \beta} \frac{\partial^2 L^*}{\partial u_\alpha \partial u_\beta} (t, x, \tilde{u})(u_\alpha - \tilde{u}_\alpha)(u_\beta - \tilde{u}_\beta),
\]

(5.7)

where \( \tilde{u} = \bar{u} + \theta(u - \bar{u}) \), \( 0 \leq \theta \leq 1 \). Let us consider the matrix of this quadratic form:

\[
\left[ \frac{\partial^2 L^*}{\partial u_\alpha \partial u_\beta} (t, x, u) \right] = \left[ \frac{\partial^2 L}{\partial u_\alpha \partial u_\beta} (t, x, u) - \sum_i \frac{\partial S}{\partial x_i} (t, x) \frac{\partial^2 f_i}{\partial u_\alpha \partial u_\beta} (t, x, u) \right].
\]

(5.8)

Note that this matrix is also given by:

\[
\left[ -\frac{\partial H_0}{\partial u_\alpha \partial u_\beta} (t, x, \frac{\partial S}{\partial x} (t, x, u)) \right].
\]

**Theorem 5.2.** Assume Hypothesis A holds. Furthermore, assume \( L^*(t, x, u) \) is twice continuously differentiable in the \( u \)'s and that the range set \( S \) of the control set \( U \) is convex. Then, if for each \( (t, x) \) in some open region of \( E^{n+1} \) containing \( (t, \bar{x}) \), the matrix (5.8) is positive semi-definite for all controls \( u \in U \), the trajectory \( \tilde{x}(t) \) is an optimal trajectory in the open region and \( \tilde{u}(t, \tilde{x}(t)) \) is an optimal control in \( U \) for this open region.

**Proof.** Positive semi-definiteness of matrix (5.8) guarantees that \( L^*(t, x, u) \geq 0 \) by (5.7) for all \( (t, x) \) in the open region and all \( u \in U \). The convexity of the control range is necessary to insure that \( \tilde{u} \) in (5.7) is in \( U \) whenever \( u \) is. Applying Theorem 5.1 completes the proof. Q.E.D.

This theorem guarantees that \( \tilde{x}(t) \) is a local optimum only unless the open region in \( E^{n+1} \) is all of \( E^{n+1} \) and the Taylor's expansion in (5.7) is valid everywhere. Also, we cannot conclude that \( \tilde{x}(t) \) is unique even in the open region since positive semidefiniteness leaves open the possibility that there are other admissible controls which also make \( L^*(t, x, u) \) vanish. However, under the following conditions we can conclude that the optimal trajectory and control are unique.

**Theorem 5.3.** Assume the hypothesis of Theorem 5.2 hold but that matrix (5.8) is positive definite on the open region. Then \( \tilde{x}(t) \) is the unique optimal trajectory in the open region and \( \tilde{u}(t, \tilde{x}(t)) \) is the unique optimal control in \( U \) for this region. If the open region is all of \( E^{n+1} \), then the optimal trajectory and control are globally unique.

**Proof.** Since matrix (5.8) is positive definite, its determinant is non-vanishing. Then the Implicit Function Theorem guarantees that the
minimizing conditions (3.3) have a unique solution \( \bar{u}(t, x, p) \) and this solution is continuously differentiable. Also, positive definiteness of (5.8) in conjunction with Eq. (5.7) shows that \( L^*(t, x, u) > 0 \) for all \( u \neq \bar{u} \), at each point \((t, x)\) of the open region in \( E^{n+1} \). As in Theorem 5.2, the convexity of the control range is necessary to insure that \( \bar{u} \) is in \( U \) whenever \( u \) is. Hence,

\[
\bar{u}(t, x) = \bar{u} \left( t, x, \frac{\partial S}{\partial x}(t, x) \right)
\]

is the only control which makes \( L^*(t, x, u(t, x)) = 0 \) in the open region. Then \( \bar{x}(t) \) and \( \bar{u}(t, \bar{x}(t)) \) are the unique optimal trajectory and control, respectively. Q.E.D.

In applying the sufficient conditions given in the theorems of this section, we must find, or at least show the existence of, a suitable solution \( S(t, x) \) of the Hamilton-Jacobi equation. This is equivalent to showing that the optimal trajectory can be embedded in a field.

6. Characteristic Curves of the Hamilton-Jacobi Partial Differential Equation

The characteristic strips of a partial differential equation consist of curves on solution surfaces of the equation together with a tangent plane at each point of the curve which tangent plane coincides with the tangent plane of the solution surface at that point (see pp. 97-103 of [16]). It is equivalent to consider normal vectors at each point of the curve instead of tangent planes. We will show that the ordinary differential equation system which describes the characteristic strips of the Hamilton-Jacobi equation, Eq. (3.7), is the original system (1.1) plus the adjoint system (4.2) with the function (3.4) used as the control function.

The Hamilton-Jacobi equation is

\[
\frac{\partial S}{\partial t} + H \left( t, x, \frac{\partial S}{\partial x} \right) = 0,
\]

where

\[
H(t, x, p) = \sum_j p_j f_j(t, x, \bar{u}(t, x, p)) - L(t, x, \bar{u}(t, x, p)).
\]

Suppose \( z = S(t, x) \) is a solution surface of the partial differential equation considered in \( t, x, z \)-space, and let \( P_0 \) be the point \( z_0 = S(t_0, x_0) \) on the surface. We will find conditions so that a strip (curve and normal vectors) which coincides with the surface and normal at \( P_0 \) is a characteristic strip.
To describe a characteristic strip we will determine the characteristic curve \( x(t) \) and the normal or gradient

\[
p(t) = \frac{\partial S}{\partial x}(t, x(t))
\]

of the surface along the curve. Suppose \( x(t) \) satisfies

\[
\dot{x}_i = \frac{\partial H}{\partial p_i}(t, x, \frac{\partial S}{\partial x}(t, x)).
\] (6.3)

We integrate this system using \( x_0 \) as the initial condition to obtain \( x(t) \) and use this in \( S(t, x) \) to obtain

\[
x(t) = S(t, x(t)).
\]

Since

\[
\dot{p}_i(t) = \frac{\partial S}{\partial x_i}(t, x(t)),
\]

we can differentiate to obtain

\[
\dot{p}_i(t) = \frac{\partial^2 S}{\partial t \partial x} + \sum_k \frac{\partial^2 S}{\partial x_k \partial x_i} \dot{x}_k
\]

\[
= \frac{\partial^2 S}{\partial t \partial x} + \sum_k \frac{\partial^2 S}{\partial x_k \partial x_i} \frac{\partial H}{\partial x_i} \frac{\partial S}{\partial x}(t, x, \frac{\partial S}{\partial x}).
\] (6.4)

Since the Hamilton-Jacobi equation is an identity in the \( x_i \)'s, we can differentiate it with respect to the \( x_i \)'s to obtain:

\[
\frac{\partial^2 S}{\partial x_i \partial t} + \frac{\partial H}{\partial x_i} + \sum_k \frac{\partial H}{\partial p_k} \frac{\partial S}{\partial x_i} = 0
\]

or

\[
\frac{\partial^2 S}{\partial x_i \partial t} + \sum_k \frac{\partial^2 S}{\partial x_i \partial x_k} \frac{\partial H}{\partial p_k} = - \frac{\partial H}{\partial x_i}.
\]

Hence (6.4) becomes

\[
\dot{p}_i = - \frac{\partial H}{\partial x_i}(t, x, p). \tag{6.5}
\]

We can then write (6.3) without reference to \( \partial S/\partial x \) by using \( p \):

\[
\dot{x}_i = \frac{\partial H}{\partial p_i}(t, x, p). \tag{6.6}
\]
Thus if $x(t)$ and $p(t)$ satisfy (6.5) and (6.6), they describe a characteristic strip.

Let us now compute $\frac{\partial H}{\partial p_i}$ and $\frac{\partial H}{\partial x_i}$. To do this, we consider $H$ and $\mathcal{W}$ to be functions of the $2n+1$ independent variables $t, x_1, ..., x_n, p_1, ..., p_n$. Then keeping in mind the minimizing conditions (3.3), we obtain from (6.2)

$$\frac{\partial H}{\partial p_i} = f_i(t, x, \mathcal{W}) + \sum_j p_j \frac{\partial f_j}{\partial u_a}(t, x, \mathcal{W}) \frac{\partial \mathcal{W}_a}{\partial p_i}$$

$$- \sum_a \frac{\partial L}{\partial u_a}(t, x, \mathcal{W}) \frac{\partial \mathcal{W}_a}{\partial p_i}$$

$$- f_i(t, x, \mathcal{W}),$$

$$\frac{\partial H}{\partial x_i} = \sum_j p_j \left[ \frac{\partial f_j}{\partial x_i}(t, x, \mathcal{W}) + \sum_a \frac{\partial f_j}{\partial u_a}(t, x, \mathcal{W}) \frac{\partial \mathcal{W}_a}{\partial x_i} \right]$$

$$- \frac{\partial L}{\partial x_i}(t, x, \mathcal{W}) - \sum_a \frac{\partial L}{\partial u_a}(t, x, \mathcal{W}) \frac{\partial \mathcal{W}_a}{\partial x_i}$$

$$= \sum_j p_j \frac{\partial f_j}{\partial x_i}(t, x, \mathcal{W}) - \frac{\partial L}{\partial x_i}(t, x, \mathcal{W}).$$

Hence, we see that the characteristic strips are described by

$$x_i = f_i(t, x, \mathcal{W}(t, x, p))$$

$$\dot{p}_i = - \sum_j p_j \frac{\partial f_j}{\partial x_i}(t, x, \mathcal{W}(t, x, p)) + \frac{\partial L}{\partial x_i}(t, x, \mathcal{W}(t, x, p)),$$

which are the original system (1.1) and the adjoint system (4.2) with a control used that satisfies the minimizing condition (3.3). Thus, if $\mathcal{W}(t, x, p)$ leads to an optimal control, the set of characteristic curves is precisely the set of optimal trajectories.

By the properties of the characteristic strips of a first order partial differential equation, any solution surface of the equation can be "built up" by considering particular sets of characteristics, the usual device being to consider all the characteristic strips passing through a given (noncharacteristic) strip (see Fig. 6.1).

---

**Fig. 6.1.** Solution Surface of a First Order Partial Differential Equation.
This particular solution surface will be the one which passes through the given space strip as a boundary condition. It will be uniquely determined as long as the given space strip is not tangent to a characteristic strip at any point. Since infinitely many solution surfaces of a partial differential equation pass through any characteristic strip, there are infinitely many solutions of a Hamilton-Jacobi equation that lead to one feedback control by (3.8), and hence to the same trajectory between two given end states. Each such solution corresponds to a different problem (see (2.2)) which is equivalent to the original optimal control problem.

7. Solution of the Hamilton-Jacobi Partial Differential Equation

By the previous section, solving the Hamilton-Jacobi equation by the method of characteristics is the same as solving the original system of ordinary differential equations. Even though we now have an expression for the control function to use in these equations, the result is still a two-point boundary value problem and up to the present time no completely satisfactory general method has been found for this problem. We will consider solving the Hamilton-Jacobi partial differential equation by other methods.

A partial differential equation has infinitely many distinct solutions. Finding a particular solution that satisfies specific boundary conditions may be difficult. However, finding complete solutions is frequently much easier. Even so, this step is usually the hardest part of applying this method to a particular problem.

**Definition.** A complete solution of Eq. (6.1) is a solution $S(t, x, a)$, where $a$ is an arbitrary constant $n$-vector and where the matrix $[S_x(t, x, a)]$ is nonsingular.

Jacobi showed that from a complete solution of the partial differential equation the complete integral (i.e., a solution containing $2n$ arbitrary constants) of the ordinary differential equation system for the characteristic strips could be obtained by simple differentiations and eliminations. Furthermore, he showed that this is true for any complete solution of the partial differential equation.

For completeness we will state and prove Jacobi’s Theorem:

**Theorem 7.1 (Jacobi).** If $S(t, x, a)$ is any complete solution of Eq. (6.1), then the equations
\begin{align*}
S_x(t, x, a) &= b_i, \quad (7.1) \\
S_{x}(t, x, a) &= p_i, \quad (7.2)
\end{align*}

where $a$ and $b$ are arbitrary constant vectors, give the complete integral of the canonical system of ordinary differential Eqs. (6.7) and (6.8).
PROOF. Since \([S_{x,\dot{a}}]\) is nonsingular, the Implicit Function Theorem guarantees that we can solve (7.1) for \(x = x(t, a, b)\). Using this in (7.2) we obtain \(p = p(t, a, b)\). We will show that these functions are solutions of Eqs. (6.5) and (6.6).

Differentiating (7.1) with respect to \(t\) and equation (6.1) with respect to \(a_i\) and subtracting we find that

\[
\sum_{i=1}^{n} \frac{\partial^2 S}{\partial a_i \partial x_j} \left[ \dot{x}_j - \frac{\partial H}{\partial p_j} \right] = 0.
\]

Since the matrix is nonsingular we obtain

\[
\dot{x}_j = \frac{\partial H}{\partial p_j} (t, x, p).
\] (7.3)

Now, by differentiation of (7.2) with respect to \(t\) and (6.1) with respect to \(x_i\), we find that

\[
\dot{p}_i = -\frac{\partial H}{\partial x_i} + \sum_{j=1}^{n} \frac{\partial^2 S}{\partial x_j \partial x_i} \left[ \dot{x}_j - \frac{\partial H}{\partial p_j} \right].
\]

Since (7.3) holds, we have

\[
\dot{p}_i = -\frac{\partial H}{\partial x_i} (t, x, p). \tag{7.4}
\]

Since \(x(t, a, b)\) and \(p(t, a, b)\) contain \(2n\) arbitrary constants and satisfy (7.3) and (7.4) they give a complete integral of the \(2n\) ordinary differential equation system.

Q.E.D.

On the basis of this theorem, any complete solution of the Hamilton-Jacobi theorem will serve our purposes. In many of the simpler cases such a solution may be found by separating the variables or by otherwise guessing an appropriate form for a solution and then finding the specific functions. Other methods of solution are also available (see [3] and [17]).

8. AN EXAMPLE PROBLEM

Consider the scalar differential equation

\[
\dot{x} = ax + bu \tag{8.1}
\]

and cost functional

\[
J[u] = \int_{t_0}^{T} (cx^3 + u^2) \, dt. \tag{8.2}
\]
The problem is to determine the control \( u \) which drives the system from \( x(t_0) = x_0 \) to \( x(T) = x_T \) while minimizing \( J[u] \). This example has been used before (see [5, 6, 9]).

We find an equivalent problem by defining

\[
L^*(t, x, u) = cx^2 + u^2 - \frac{\partial S}{\partial t} - (ax + bu) \frac{\partial S}{\partial x}
\]

and

\[
J^*[u] = \int_{t_0}^{T} L^*(t, x, u) \, dt.
\]

Equation (3.2) gives

\[
\bar{u} = b \left( \frac{\partial S}{\partial x} \right)
\]

and the Hamilton-Jacobi equation (3.6) is

\[
\frac{\partial S}{\partial t} + ax \frac{\partial S}{\partial x} + \frac{b^2}{4} \left( \frac{\partial S}{\partial x} \right)^2 = cx^2.
\]

As shown in Section 7 any complete solution of Eq. (8.6) will serve our purposes. Suppose we look for a solution of the form

\[
S(t, x) = At + \phi(x),
\]

where \( A \) is a constant. Using (8.7) in Eq. (8.6) we find that

\[
\phi'(x) = \frac{2}{b^2} \left[ -ax \pm \sqrt{a^2x^2 - b^2(A - cx^2)} \right],
\]

from which we could get \( \phi(x) \) by integrating. From (8.5) the optimal control is given by

\[
b\bar{u} = \frac{b^2}{2} \phi'(x)
\]

\[
= -ax \pm \sqrt{(a^2 + cb^2)x^2 - Ab^2}.
\]

Using this control in the differential equation (8.1) and integrating we find that

\[
x(t) = C_1 \cosh(\sqrt{a^2 + cb^2}t + C_2),
\]

where \( C_1 \) and \( C_2 \) are arbitrary constants which may be chosen to satisfy the two required boundary conditions. The solution could have been found by integrating \( \phi'(x) \) and then using (7.1) and (7.2). The \( A \) in (8.8) is given by

\[Ab^2 = C_1 \gamma(a^2 + cb^2).\]
To verify that the control (8.8) is optimal we determine the $L^*(t, x, u)$ of the equivalent problem by using (8.7) in (8.3). We find that

$$L^*(t, x, u) = \left[u - \frac{b}{2} \phi'(x)\right]^2 \geq 0.$$ 

Hence the equivalent problem obviously has (8.8) as its optimal control and therefore so does the original problem.

Another solution of the Hamilton-Jacobi equation could have been used. For example, suppose we look for a solution of equation (8.6) of the form

$$S(t, x) = f(t)x^2. \quad (8.10)$$

Using this in (8.6) we find that $f(t)$ must satisfy the Riccati differential equation

$$f'(t) + b^2 f(t)^2 + 2af(t) = c. \quad (8.11)$$

By using $f(t) = y'(t)/b^2y(t)$, which transforms this equation into one which is linear second order with constant coefficients, the general solution of equation (8.11) is

$$f(t) = \frac{1}{b^2} \left[-a + \sqrt{a^2 + cb^2 \tanh(\sqrt{a^2 + cb^2} t + C_2)}\right] \quad (8.12)$$

where $C_2$ is an arbitrary constant.

The optimal control from (8.5) is

$$b\ddot{u} = b^2 f(t)x \quad (8.13)$$

which when used in Eq. (8.1) gives (the same trajectory as before)

$$x(t) = C_1 \cosh(\sqrt{a^2 + cb^2} t + C_2). \quad (8.14)$$

We can again verify that (8.13) is an optimal control by using (8.10) in (8.3) which gives

$$L^*(t, x, u) = [u - bf(t)x]^2 \geq 0.$$ 

It is clear that this equivalent problem has (8.13) as its optimal control and therefore so does the original problem.

For this example we have obtained two different equivalent problems by using different complete solutions of the Hamilton-Jacobi equation. Since the sufficiency and uniqueness theorem (Theorem 5.3) holds for this example, the two seemingly different expressions, (8.8) and (8.13), for the optimal control must reduce to the same thing if expressed in terms of the same variables.
9. SUMMARY

In this paper we have presented the Carathéodory approach to the calculus of Variations as modified to suit optimal control problems. This method is by determining and solving a problem equivalent to the original problem, the Hamilton-Jacobi partial differential equation being a key step in the determination of the new problem. We have shown the relationship of the Pontrjagin Maximum Principle to this method, as well as the analogs of the Weierstrass Excess Function and the classical sufficient conditions. We have also discussed the solution of the Hamilton-Jacobi partial differential equation and have illustrated the whole method with an example problem.

The ideas presented in this paper are useful in the solution of optimal control problems, but their greatest value lies in a clearer understanding of the concepts underlying these problems.

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