GRAPHS WHICH ARE LOCALLY A CUBE

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We prove that there are exactly two connected graphs which are locally a cube: a graph on 15 vertices which is the complement of the (3 x 5)-grid and a graph on 24 vertices which is the 1-skeleton of a certain 4-dimensional regular polytope called the 24-cell.

1. Introduction

Let \( \{e_1, e_2, e_3, e_4\} \) be the standard basis of \( \mathbb{R}^4 \). The 24-cell is a 4-dimensional regular polytope whose vertices are the 24 vectors \( \pm e_i \pm e_j \) \((i \neq j)\) of \( \mathbb{R}^4 \), two vertices being adjacent iff the angle between the corresponding vectors is 60°. It is well known [2] and easy to check that the 1-skeleton (i.e. the graph consisting of the vertices and edges) of this polytope has the following property: for every vertex \( v \), the neighbourhood of \( v \) (i.e. the subgraph induced by \( G \) on the set of vertices adjacent to \( v \)) is isomorphic to the 1-skeleton of a 3-dimensional cube. In other words, the 24-cell is locally a cube.

More generally [3], given a graph \( G' \), we shall say that a graph \( G \) is locally \( G' \) if, for every vertex \( v \) of \( G \), the neighbourhood \( G(v) \) of \( v \) is isomorphic to \( G' \). If \( G' \) is the 1-skeleton of a 3-dimensional cube and if \( G \) is locally \( G' \), we shall say that \( G \) is locally a cube.

It is natural to ask whether the 24-cell is the only connected graph which is locally a cube.

Theorem. If a connected graph \( G \) is locally a cube, \( G \) is isomorphic either to the 1-skeleton of the 24-cell or to the complement of the (3 x 5)-grid.

The \((p \times q)\)-grid is the graph whose vertices are the \( pq \) ordered pairs \((i, j)\) with \( i = 1, \ldots, p \) and \( j = 1, \ldots, q \), two vertices being adjacent iff they have one coordinate in common.

The adjacency relation in a graph \( G \) will be denoted by \( \sim \) and the number of vertices of \( G \) by \( |G| \).

2. Proof of the Theorem

Lemma 1. For any two adjacent vertices of \( G \), there are exactly 3 vertices of \( G \) adjacent to both of them.
This follows immediately from the fact that the cube is a regular graph of degree 3.

Let \( v \) be a fixed vertex of \( G \). We shall denote by \( v_i (i = 1, \ldots, 8) \) the vertices of \( G(v) \) and by \( G_i \) the subgraph induced by \( G \) on the set of vertices adjacent to \( v_i \) and at distance 2 from \( v \). Since the neighbourhood of \( v_i \) is isomorphic to a cube, \( G_i \) is a 3-claw, that is \( G_i \) has 4 vertices \( w_i, i_1, i_2, i_3 \) such that \( w_i \sim i_r \) for every \( r = 1, 2, 3 \) and \( i_t \neq i_r \) for every \( r \neq s \).

**Lemma 2.** If \( v_i \neq v_j \), then \( G_i \neq G_j \) and the subgraph \( G_i \cap G_j \) is not an edge.

**Proof.** If \( G_i = G_j \), the vertices \( v_i, v_j, i_1, i_2, i_3 \) are all in the neighbourhood of \( w_i \), which is isomorphic to a cube. This is a contradiction because \( v_i \) and \( v_j \) are both adjacent to \( i_1, i_2, i_3 \) and the graph of a cube cannot contain 5 such vertices.

If the claws \( G_i \) and \( G_j \) have exactly one edge in common, we may assume without loss of generality that it is the edge \( \{w_i, i_1\} \), so that \( w_i = w_j \) or \( w_i = i_1 \). In any case, the neighbourhood \( G(w_i) \) contains the vertices \( v_i, v_j, i_1, i_2, i_3 \) with \( v_j \sim i_1 \).

Since \( G(w_i) \) is isomorphic to a cube, \( v_j \) must also be adjacent to one of the vertices \( i_2 \) or \( i_3 \), and so \( G_i \) and \( G_j \) have at least two edges in common, contradicting the initial assumption.

**Lemma 3.** If \( v_i \sim v_j \), then \( w_i \neq w_j \), \( w_i \notin G_j \), \( w_j \notin G_i \) and \( |G_i \cap G_j| = 2 \).

**Proof.** Since \( v_i \sim v_j \), there are exactly 3 vertices adjacent to \( v_i \) and \( v_j \) by Lemma 1. One of them is \( v \). There is no vertex adjacent to \( v_i \) and \( v_j \) in \( G(v) \). Therefore the two missing vertices are at distance 2 from \( v \), and so \( |G_i \cap G_j| = 2 \).

If \( w_i \in G_j \), then \( w_i \sim v_j \). This contradicts Lemma 1 because \( v_i \) and \( w_i \) are both adjacent to \( i_1, i_2, i_3, v_j \). Therefore \( w_i \notin G_i \) and similarly \( w_j \notin G_i \). In particular, \( w_i \neq w_j \).

Let \( d(v_i, v_j) \) denote the distance between \( v_i \) and \( v_j \) in the subgraph \( G(v) \).

**Lemma 4.** If \( d(v_i, v_j) = 2 \), then \( G_i \cap G_j \neq \emptyset \), \( \{w_i\} \) and \( \{w_j\} \).

**Proof.** If \( G_i \cap G_j \) is equal to \( \emptyset \), \( \{w_i\} \) or \( \{w_j\} \), then the vertices \( i_1, i_2, i_3, j_1, j_2, j_3 \) are pairwise distinct. Since \( d(v_i, v_j) = 2 \), there is a vertex \( v_k \in G(v) \) adjacent to \( v_i \) and \( v_j \). By Lemma 3, \( w_k \notin G_i \cup G_j \) and we may assume without loss of generality that \( G_i \cap G_k = \{i_1, i_2\} \) and \( G_j \cap G_k = \{j_1, j_2\} \). It follows that \( G(v_k) \) contains at least 9 vertices, a contradiction.

**Lemma 5.** If \( d(v_i, v_j) = 2 \) and \( w_i = w_j \), then the subgraph \( G_i \cap G_j \) is a 2-claw (i.e. the union of two intersecting edges).

**Proof.** This is a direct consequence of Lemmas 2 and 4.
Lemma 6. If $d(v_i, v_j) = 2$ and $w_i \neq w_j$, then $G_i \cap G_j = \{i_r\}$ for some $r \in \{1, 2, 3\}$. Moreover the 8 vertices of $G(i_r)$ are $v_i, v_j, v_{i}, v_{j}, w_i, w_j, w_k, w_l$ where $v_k$ and $v_l$ are the vertices adjacent to $v_i$ and $v_j$ in the subgraph $G(v)$.

Proof. By Lemmas 2 and 4, we already know that $|G_i \cap G_j| = 1, 2$ or 3. In view of Lemma 3, we may assume that $G_i \cap G_k = \{i_1, i_2\}$ and $G_i \cap G_j = \{j_1, j_2\}$.

By hypothesis, $w_i \neq w_j$. Moreover, $w_i \neq i_1, i_2$ because $w_i \notin G_k$ thanks to Lemma 3. If $w_j = i_3$, then $i_r \neq i_k$ for every $r, s \in \{1, 2, 3\}$ and so $i_1, i_2, j_1, j_2$ are 4 distinct vertices adjacent to $v_k$. It follows that $G(v_k)$ contains at least 9 vertices, a contradiction. Therefore $w_i \notin G_i$ and similarly $w_l \notin G_l$, so that $G_i \cap G_l \subseteq \{i_1, i_2, i_3\}$. Observe also that $\{i_1, i_2\} \neq G_i \cap G_l$, because otherwise $i_1$ and $i_2$ would be adjacent to $v_i, v_j, w_k$ which is a contradiction since $i_1, i_2, v_i, v_j, w_k$ are all in $G(v_k)$ and since the graph of a cube cannot contain 5 such vertices.

(i) If $|G_i \cap G_j| = 3$, then $G_i \cap G_l \ni \{i_1, i_2\}$, which is impossible as we have just seen before.

(ii) If $|G_i \cap G_j| = 2$, it is no loss of generality, thanks to the preceding observations, to assume that $G_i \cap G_j = \{i_1, i_3\}$ and $G_k = \{w_k, i_1, i_2, j_2\}$ with $i_1 = j_1$. The neighbourhood $G(i_1)$ contains $v_i, v_j, v_{i}, v_{j}, w_i, w_j, w_k$ with $w_i \sim v_i \sim v_k \sim v_j \neq w_l$. Since $G(i_1)$ is isomorphic to a cube, there must be a vertex $x \in G(i_1)$ adjacent to $v_i$ and $v_j$ but not to $w_l$. Moreover $x \in G(v)$ because $x \sim v_i, x \neq v_i, x \neq w_i$ and $x \neq w_j$. Thus $x = v_i$ and so $v_i \sim i_1$. Using the fact that $G(v_i)$ is a cube, we get $v_i \sim i_3$. Now, in $G(v_i)$, $v_i$ and $v_j$ are both adjacent to $v_i, i_1, i_2$, a contradiction since $G(v_i)$ is a cube.

(iii) Therefore $|G_i \cap G_j| = 1$ and $G_i \cap G_j = \{i_r\}$ for some $r \in \{1, 2, 3\}$. Together with $G_i \cap G_k = \{i_1, i_2\}$ and $G_j \cap G_k = \{j_1, j_2\}$, this implies $r \neq 3$ and so, without loss of generality, $G_i \cap G_j = \{i_1\}$ and $G_k = \{w_k, i_1, i_2, j_2\}$. Using the same type of arguments as in (ii), we get $v_i \sim i_1$, and, because $G(i_1)$ is a cube, $v_j \neq i_1$. Thus $w_l \neq w_k$ and the vertices $w_l, w_j, w_k, w_l$ are pairwise distinct.

Lemma 7. If $d(v_i, v_j) = 3$, then $w_i \neq w_j$.

Proof. Assume that $w_i = w_j$ and let $v_i \sim v_m \sim v_n \sim v_j$ be a path of length 3 joining $v_i$ to $v_j$ in $G(v)$.

If $|G_i \cap G_j| = 1$, then $G_i \cap G_j = \{w_i\}$ and $i_r \neq j_r$ for every $r, s \in \{1, 2, 3\}$. By Lemma 3, we may assume that $G_i \cap G_m = \{i_1, i_3\}$. Moreover, since $w_m \neq w_i = w_j$, we may assume, by Lemma 6, that $G_j = G_m = \{j_3\}$. Therefore $G_m = \{w_m, i_1, i_2, j_3\}$, which implies $i_3 \neq i_1$ and $j_3 \neq i_2$, a contradiction in the cube $G(w_i)$.

If $|G_i \cap G_j| = 2$, then $G_i \cap G_j$ is an edge, contradicting Lemma 2.

If $|G_i \cap G_j| = 3$, then the subgraph $G_i \cap G_j$ is a 2-claw and we may assume that $G_i = \{w_i, i_1, i_2, j_3\}$ with $i_1 = j_1$ and $i_2 = j_2$. By Lemma 3, $|G_i \cap G_m| = 2$ with $w_i \notin G_m$, and so $G_m$ contains at least one of the two vertices $i_1, i_2$. Since $w_m \neq w_i = w_j$, Lemma 6 implies that $G_j \cap G_m = \{i_3\}$ for some $s \in \{1, 2, 3\}$, and so $G_m$ contains at most one of the two vertices $i_1, i_2$. Therefore, without any loss of generality,
$G_i \cap G_m = \{i_2\}$. Now, by Lemma 6 again, $G(i_2)$ has exactly 4 vertices in common with the cube $G(v)$, namely $v_i, v_m$ and the two vertices of $G(v)$ adjacent to $v_i$ and $v_m$. On the other hand, $v_i \in G(i_2) \cap G(v)$. This is a contradiction since $v_i$ is not adjacent to $v_j$.

If $|G_i \cap G_j| = 4$, then $G_i = G_j$, contradicting Lemma 2.

**Proposition 1.** If a graph $G$ is locally a cube and if, for some vertex $v$ of $G$, there are two vertices $v_i, v_j \in G(v)$ such that $d(v_i, v_j) = 2$ and $v_i \neq v_j$, then $G$ is isomorphic to the 1-skeleton of the 24-cell.

**Proof.** It is easy to check that the 1-skeleton of the 24-cell satisfies the above hypothesis. Therefore, it suffices to prove that a graph $G$ satisfying this hypothesis is uniquely determined up to isomorphism.

We shall denote the adjacencies in the cube $G(v)$ by

$$
v_1 \sim v_2 \sim v_3 \sim v_4 \sim v_1, \quad v_5 \sim v_6 \sim v_7 \sim v_8 \sim v_5
$$

and

$$v_i \sim v_{i+4} \text{ for every } i = 1, 2, 3, 4.$$

Suppose that $w_1 \neq w_3$. Then, by Lemma 6, $G_1 \cap G_3 = \{1_1\}$ without any loss of generality and $v_1, v_2, v_3, v_4, w_1, w_2, w_3, w_4$ are the 8 vertices of the cube $G(i_1)$, with

$$w_1 \sim w_2 \sim w_3 \sim w_4 \sim w_1, \quad w_1 \neq w_3, \quad w_2 \neq w_4$$

Since $v_5 \sim v_1$ and $v_5 \neq 1_1$, it follows from Lemma 3 that $G_1 \cap G_5 = \{1_2, 1_3\}$. Moreover, by Lemma 3 again, $G_1 \cap G_2 = \{1_1, 1_2\}$ without any loss of generality, and so $G_i \cap G_4 = \{1_1, 1_3\}$ thanks to Lemma 6. The neighbourhood $G(i_1)$ contains the vertices $v_1, v_2, v_3, w_1, w_2, w_5$ with

$$w_5 \sim w_5 \sim v_1 \sim v_2 \sim w_2 \sim w_1 \sim v_1$$

and $w_3 \neq w_1$ by Lemma 3. Since $G(i_2)$ is a cube, we have $w_2 \neq w_5$ and so, by Lemma 6, $G_2 \cap G_5 = \{1_2\}$ and $v_1, v_2, v_3, v_4, w_1, w_2, w_3, w_4$ are the 8 vertices of the cube $G(i_2)$, with

$$w_1 \sim w_2 \sim w_6 \sim w_5 \sim w_1, \quad w_1 \neq w_6, \quad w_2 \neq w_5$$

By similar arguments, $w_4 \neq w_5$, $G_4 \cap G_5 = \{1_3\}$ and $v_1, v_4, v_5, v_8, w_1, w_4, w_5, w_8$ are the 8 vertices of the cube $G(1_3)$, with

$$w_4 \sim w_6 \sim w_5 \sim w_1, \quad w_1 \neq w_8, \quad w_4 \neq w_5$$

The neighbourhood $G(w_1)$ contains the vertices $v_1, 1_1, 1_2, 1_3, w_2, w_4, w_5$. Since $G(w_1)$ is a cube, the missing vertex $w \in G(w_1)$ must be adjacent to $w_2, w_4, w_5$ and non adjacent to $1_1, 1_2, 1_3$, so that $w \neq w_1, w_2, w_3, w_4, w_5, w_6, w_8$. Note that $w_2 \neq w_8$ (because $w_2$ is already adjacent to 8 vertices distinct from $w_8$) and also $w_4 \neq w_6, w_5 \neq w_3$. 

Using similar arguments, it is now easy (but a little bit tedious) to show that the subgraph induced by $G$ on the set of vertices $w_i$ ($i = 1, \ldots, 8$) is isomorphic to a cube which is precisely the neighbourhood $G(w)$. Moreover, given any 4 vertices $v_i, v_j, v_k, v_l$ in a face of the cube $G(v)$, there is exactly one vertex $f_i$ of $G$ which is adjacent to $v_i, v_j, v_k, v_l$ and to the vertices $w_i, w_j, w_k, w_l$ of the corresponding face of the cube $G(w)$; for example, we have seen that $1_1, 1_2, 1_3$ are three of the vertices $f_1, f_2, f_3, f_4, f_5, f_6$. This shows that the graph $G$ has $1 + 8 + 6 + 8 + 1 = 24$ vertices and, being uniquely determined up to isomorphism by the preceding construction, it is isomorphic to the 1-skeleton of the 24-cell.

**Proposition 2.** If a graph $G$ is locally a cube and if, for some vertex $v$ of $G$, $w_i = w_l$ whenever $v_i, v_l \in G(v)$ with $d(v_i, v_l) = 2$, then $G$ is isomorphic to the complement of the $(3 \times 5)$-grid.

**Proof.** It is easy to check that the complement of the $(3 \times 5)$-grid satisfies the above hypothesis. Therefore, it suffices to show that a graph $G$ satisfying this hypothesis is uniquely determined up to isomorphism.

We use the same notations as in the proof of Proposition 1 to denote the adjacencies in $G(v)$.

The hypothesis implies that $w_1$ is adjacent to $v_1, v_3, v_6, v_8$ and that $w_2$ is adjacent to $v_2, v_4, v_5, v_7$, with $w_1 \neq w_2$ by Lemma 3. Using Lemma 5, we may assume without loss of generality that $G_1 = \{w_1, 1_1, 1_2, 1_3\}$ and $G_3 = \{w_1, 1_2, 1_3, 3_1\}$. Since $G(w_1)$ is a cube, it follows that $3_1$ is adjacent to $v_6$ and $v_8$ and, without loss of generality, $G_6 = \{w_1, 1_1, 1_3, 3_1\}$ and $G_8 = \{w_1, 1_1, 1_2, 3_1\}$. By Lemmas 3 and 5, the subgraphs $G_2, G_4, G_5, G_7$ are then completely determined.

This construction shows that the graph $G$ has 15 vertices and is uniquely determined up to isomorphism.

The proof of the Theorem follows immediately from Propositions 1 and 2.

### 3. Final comments

A. Brouwer [1] proved independently that there are exactly two connected graphs which are locally a cube. After some exchange of information, he could prove a more general result characterizing the graphs which are locally the complement of a $(p \times q)$-grid with $p \geq q \geq 2$ ($q \geq 2$ or $p > 3$). We shall say that these graphs are locally $p \times q$.

**Theorem** (Brouwer [1]). If $G$ is a connected graph which is locally $p \times q$ with $p \geq q \geq 2$ ($q \geq 2$ or $p > 3$), then $G$ is the complement of a $((p + 1) \times (q + 1))$-grid or

(i) $p = 4$, $q = 2$ and $G$ is the 1-skeleton of the 24-cell

(ii) $p = q = 3$ and $G$ is the Johnson scheme $(\xi)$ on 20 vertices (that is the graph...
whose vertices are the 3-subsets of a 6-set, two vertices being adjacent iff the corresponding 3-subsets intersect in a 2-subset).

The remaining cases \((p, q) = (3, 2), (2, 2)\) or \((p, 1)\) with \(p > 1\) allow infinitely many nonisomorphic solutions. \(K_2\) is obviously the unique locally 1×1 graph.

**References**