A numerical sufficiency test for the asymptotic stability of linear time-varying systems

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Abstract
A numerical sufficiency test for the asymptotic stability of linear time-varying Hurwitz systems is proposed. The algorithmic procedure constructs a bounding tube in which the state is guaranteed to stay. The continuous-time system is evaluated at discrete time instants, for which successive quadratic Lyapunov functions are generated. The tube is constructed based on: (i) a conservative estimate of the state evolution, from a discrete time instant to the next, obtained from the corresponding Lyapunov function, and (ii) re-evaluation of the tube diameter at each discrete time instant to account for variations in the plant matrix. The numerical test is illustrated in simulation via both a stable and an unstable system.

Keywords: Time-varying linear system; Lyapunov method; Optimization

1. Introduction

Linear time-varying systems arise in many different contexts. They often appear in specific control techniques such as nonlinear gain scheduling (Guo & Rugh, 1995; Rugh, 1991), linear adaptive control (Krause & Kumar, 1986; Kreisselmeier, 1985), and nonlinear tracking (Khalil, 1996).

Assessing the stability of linear time-varying systems is delicate since, even when the eigenvalues of the system matrix have strictly negative real parts for all times, the linear time-varying system can be unstable (Rosenbrock, 1963; Rugh, 1993). Moreover, a necessary and sufficient stability condition is yet to be found.

Hence, considerable research effort has been undertaken to find good sufficient conditions (Amato, Celentano, & Garofalo, 1993; Desoer, 1969; Ilichmann, Owens, & Prätzel-Wolters, 1987; Krause & Kumar, 1986; Kreisselmeier, 1985; Rosenbrock, 1963). They more or less all rely on the use of a Lyapunov function associated with a linear time-invariant approximation of the system and ensuring that the influence of the approximation is not excessive.

The present paper is to some extent similar, in the sense that linear time-invariant Lyapunov equations are solved. The difference lies essentially in the manner in which the time variation of the plant matrix is handled. Rather than ensuring that this variation is smaller than a prescribed bound for all times, the procedure works on a discrete time grid and estimates the worst-possible scenario only at these discrete time instants. The proposed sufficient condition can be less conservative than others, which has been confirmed by examples.

In the sequel, linear time-varying systems whose eigenvalues have negative real parts at all times (i.e. Hurwitz systems) will be examined. A numerical stability test with the following idea is presented:

- A sequence of constant plant matrices is obtained upon considering the time-varying system at discrete time instants.
- A classical Lyapunov equation is solved for each constant plant matrix. This gives a sequence of Lyapunov functions.
- A bounding tube that is guaranteed to contain the states is constructed based on these Lyapunov functions.
The resulting tube is shown to decrease between two successive time instants. However, since the width of the tube changes at each discrete time instant due to variations in the system matrix, its estimation is a real challenge. A useful result can be obtained by using: (i) a Lyapunov-based conservative approach for estimating the evolution of the system state between two successive time instants, and (ii) constrained quadratic optimization, with analytical solution, for defining the state conditions at the beginning of the next time interval.

The idea of successive approximations using ellipsoids has been recently applied to the approximation of the reachability set of various dynamical systems by Kurzhanski and Varaiya (2002a, 2002b). An increasing number of ellipsoids is used to improve the accuracy of the estimated reachability set. These approximations can be done from a conservative point of view (ensuring that all points external to the approximation set are reachable); interior approximation, Kurzhanski & Varaiya, 2002b) or from an inclusive point of view (ensuring that all points external to the approximation set are not reachable; exterior approximation, Kurzhanski & Varaiya, 2002a). In the present work, a single ellipsoid is used at a given grid point and successive approximations refer to time.

The paper is organized as follows. Section 2 briefly reviews background material. The sufficiency test is presented in Section 3.1. Guidelines for choosing the parameters are given in Section 4. Two examples are investigated in Section 5. Finally, conclusions and future work are presented in Section 6.

2. Preliminaries

The sequel addresses a sufficient condition to assess stability of the linear time-varying system:

\[ \dot{x} = A(t)x, \quad x(t_0) = x_0 \]

with \( x \in \mathbb{R}^n \) and \( A(t) \in \mathbb{R}^{n \times n} \) for all times \( t \). The following assumption is assumed to hold throughout the paper.

Assumption 1. The application \( t \to A(t) \) is continuous in \( t \). \( A(t) \) is Hurwitz for all \( t \), and bounded from above, i.e. \( \exists \xi > 0 \) such that \( \|A(t)\| < \xi, \forall t \geq 0 \).

2.1. Lyapunov theorem for linear time-invariant systems

Lemma 1. Consider the constant real matrix \( \bar{A} \in \mathbb{R}^{n \times n} \). If the \( \bar{A} \) has negative-real-part eigenvalues, then, for each symmetric \( n \times n \) matrix \( Q \), there exists a unique symmetric solution to the Lyapunov equation

\[ \bar{A}^T P + P \bar{A} = -Q \]

given by

\[ P = \int_0^\infty e^{\bar{A}^T t} Q e^{\bar{A} t} \, dt. \]

Furthermore, if \( Q \) is positive definite, then \( P \) is positive definite.

This result with its proof can be found in many references, for instance Rugh (1993).

2.2. Quadratic optimization problems

The optimization of a quadratic cost function with a single quadratic equality constraint is considered here. A solution can be computed by solving a specific eigenvalue/eigenvector problem without resorting to iterative techniques.

Lemma 2. Let \( P \) and \( Q \) be arbitrary \( \mathbb{R}^{n \times n} \) positive-definite matrices, and consider an arbitrary positive real number \( V \). A solution to

\[
\begin{align*}
\max_{v} & \quad v^T P v \\
\text{s.t.} & \quad v^T Q v - V = 0
\end{align*}
\]

is \( v^* = \sqrt{(V/\lambda_n)N A^{-1/2} m_n} \), where \( N \) is the matrix obtained upon diagonalizing \( P = N A N^T \), and \( \lambda_n \) is the smallest eigenvalue of \( A^{-1/2} N^T Q N A^{-1/2} \) (the largest eigenvalue in case of minimization \( \min_{v} v^T P v \)). \( m_n \) is an orthonormal eigenvector associated with \( \lambda_n \).

Proof. Diagonalizing \( P = N A N^T \) and introducing the change of coordinates \( z = A^{1/2} N^T v \) gives max \( z^T z \) subject to \( z^T A^{-1/2} N^T Q N A^{-1/2} z - V = 0 \). A solution to this optimization problem is \( z^* = \sqrt{(V/\lambda_n)N A^{-1/2} m_n} \), with \( x^* = \sqrt{(V/\lambda_n)N A^{-1/2} m_n} \).

2.3. Stability definitions

The definitions refer to System (1), where \( x(t) \) denotes the system state at time \( t \).

Definition 1. Let \( t_0 \) be any real number. If for all \( R > 0 \) there exists \( r(t_0, R) > 0 \) such that \( \|x(t_0)\| < r \) implies \( \|x(t)\| < R \) for all \( t \geq t_0 \), then the system is stable in the Lyapunov sense.

Remark 1. For ordinary differential equations, and particularly for (1), stability for the initial time \( t_0 \) implies stability for any subsequent initial time \( t_1 > t_0 \) (see Hahn, 1967, p. 173 just above Definition 36.5). Therefore, \( t_0 \) can be chosen arbitrarily for checking stability and this need only be done once.

Definition 2. If, in Definition 1, \( r \) does not depend on \( t_0 \) (i.e. \( r(R) \) instead of \( r(R, t_0) \)), then the Lyapunov stability of Definition 1 is said to be uniform.

Definition 3. If a system is uniformly stable according to Definition 2 and there exists an \( \varepsilon > 0 \), independent of \( t_0 \), such that \( \|x(t_0)\| < \varepsilon \) implies \( \lim_{t \to \infty} x(t) = 0 \), then the system is uniformly asymptotically stable.

3. Numerical stability test

This section is split into two subsections; the first subsection proposes to trap the system state in a tube whose shape is
subsumed by a sequence $V_i$, $i = 0, \ldots, \infty$ (Theorem 1); the second subsection assesses stability based on the properties of the resulting sequence (Corollaries 1 and 2).

### 3.1. Construction of the bounding tube

The tube consists of decreasing elements joined at pre-determined points. A time grid is initially chosen where these junctions take place. Evaluating the current tube element and joining it with the next one is done in three steps. The whole procedure is now detailed.

#### 3.1.1. Algorithm description

The algorithm starts by choosing $V_0 > 0$ (for instance $V_0 = 1$) corresponding to the initial time $t_0$. Then, $A(t_0)^T P_0 + P_0 A(t_0) = -I$ is solved for the positive-definite matrix $P_0$. Pick $\delta_i > 0$, $i = 0, 1, \ldots, \infty$, defining $t_{i+1} = t_i + \delta_i$ (time grid). $\delta_i$ should be chosen so that (i) $\sum_{i=0}^{\infty} \delta_i = \infty$, and (ii) $A(t)^T P_i + P_i A(t)$ is positive definite for all $t \in [t_i; t_{i+1}]$, where $P_i$ stands for the solution to the Lyapunov equation $A(t)^T P_i + P_i A(t) = -I$ (Assumption 1 guarantees a solution). Such a choice of $\delta_i$ is possible due to a continuity argument about the operator $I(\alpha) = \int_{t_1}^{\infty} e^{A(t_1)} e^{A(t_1)\alpha} dt$, where $\alpha = A(t_1)$ (Krasnosel’skii, 1964). Guidelines for choosing $\delta_i$ and other parameters of the algorithm is postponed until Section 4.

1. **Tube width estimation**: Compute $\hat{\lambda}_{1,i}$ (the largest eigenvalue of $P_i$), and set

   $$\hat{\lambda}_{i+1} = V_i e^{(1/\hat{\lambda}_{1,i})} \int_{t_i}^{t_{i+1}} \beta_i(t) dt,$$

   where $\beta_i(t) = \max(\text{eig}(A(t)^T P_i + P_i A(t)))$.

2. **Tube shape construction**: Solve the following Lyapunov equation for $P_{i+1}$:

   $$A(t_{i+1})^T P_{i+1} + P_{i+1} A(t_{i+1}) = -I.$$  

3. **Tube junction**: Solve the constrained optimization problem:

   $$V_{i+1} = \max_v v^T P_{i+1} v \quad \text{s.t.} \quad v^T P_i v - \hat{\lambda}_{i+1} v = 0.$$  

A typical step is illustrated in Fig. 1.

#### 3.1.2. Terminology, symbols and nomenclature

The algorithm description just given uses various symbols that are now explained in more detail.

- $V(t) = x(t)^T P_i x(t)$ is a Lyapunov function valid on the time interval $[t_i; t_{i+1}]$.
- $V_i$ denotes the quadratic cost function associated with the grid point at time $t_i$. Its value represents an upper bound on the value of $V(t_i)$.
- $\hat{\lambda}_{1,i}$ and $\beta_i$. The decrease rate of the Lyapunov function depends on two quantities. On the one hand, the current value of the state can change the decrease rate. This can be estimated using the largest eigenvalue $\hat{\lambda}_{1,i}$ of $P_i$. On the other hand, $\beta_i$ gives the current value of the decrease rate which does not depend on the value of the current state. It is $-1$ at $t_i$ and increases over the interval $[t_i; t_{i+1}]$ due to the time variability of the matrix $A(t)$.

- $V_{i+1}$ is a conservative estimate of $x(t_{i+1})^T P_{i+1} x(t_{i+1})$.
- $V_{i+1}$ is the value associated to $P_{i+1}$. It is chosen so that the state $x(t_{i+1})$ is guaranteed to be in the set $\{ v | v^T P_{i+1} v \leq V_{i+1} \}$. To achieve this, the ellipsoid corresponding to $V_{i+1}$ is included in the one defined by $V_{i+1}$ while touching one another.

#### 3.1.3. Main result

The algorithm gives a sequence of numbers that fully characterizes a tube in which the state is guaranteed to stay. The statement is the following:

**Theorem 1.** Let Assumption 1 be satisfied ($A(t)$ Hurwitz for all $t$), and $V_0$ be a given positive number, and $V_i, i = 0, \ldots, \infty$ be as in Algorithm 1. If $x_0^T P_0 x_0 \leq V_0$ then, for all $t$, the solution to $\dot{x} = A(t)x$ will be such that $x(t)^T P_i x(t) \leq V_i$ for $t \in [t_i; t_{i+1}]$, $i = 0, \ldots, \infty$.

**Proof.** The choice of $\delta_i$, guarantees that the Lyapunov function $V(t) = x(t)^T P_i x(t)$ decreases over the time interval $[t_i; t_{i+1}]$, i.e.

$$\dot{V}(t) = x(t)^T [A(t)^T P_i + P_i A(t)] x(t) \leq \tilde{\beta}_i(t) x(t)^T x(t) \leq 0.$$

Define

$$\tilde{\beta}_i(t) = \arg\min_v v^T v \quad \text{s.t.} \quad v^T P_i v - V_i = 0.$$
A solution to this minimization problem is given by
\[ \hat{z}_t = \sqrt{\langle V_i / \lambda_i \rangle} m_1, \]
where \( \lambda_i, i \) is the largest eigenvalue of \( P_i \) and \( m_1 \) is a corresponding orthonormal eigenvector.

Now, let \( x(t) \) be defined such that \( x(t) = x(T) x(t) = x(T) x(t) \). Expressing \( x(t) = \sum_{j=i-1}^{\infty} m_j \), where \( m_j \) are the orthonormal eigenvectors of \( P_i \), it follows that \( x(t) = \sum_{j=1}^{\infty} m_j \). Therefore, after recollecting that \( \lambda_i, i \) holds on the next interval, i.e. \( x(t) = x(T) x(t) \), yields \( x(t) = x(T) x(t) \). Hence, the inequality \( V(t) = x(t) x(t) \leq z(t) z(t) x(t) \) holds. Rewriting
\[ x(t) = z(t) z(t) \]
the previous inequality becomes \( x(t) x(t) \geq 1 \), so that (5) reads \( V(t) \leq \beta(1/\lambda_i,v(t)) \). Thus, \( V(t) \leq V(t) \). This means that \( V(t) \leq \beta(1/\lambda_i,v(t)) \). Now, setting \( \beta(1/\lambda_i,v(t)) \) and solving the optimization problem
\[ \hat{z}_i = \text{argmax} \quad v \quad \text{s.t.} \quad x(t) = x(T) x(t) \]
also guarantees that \( x(t) = x(T) x(t) \), since the maximum possible value of \( v = x(T) x(t) \), is reached somewhere on the boundary defined as \( v = \hat{z}_i \). Therefore, after recollecting that \( x(t) = x(T) x(t) \), and setting \( \beta(1/\lambda_i,v(t)) \), the same property holds on the next interval, i.e. \( x(t) = x(T) x(t) \). Proceeding inductively concludes that, \( V(t) \leq V(t) \). Therefore, Definition 1 holds with \( r \) independent of the initial time \( t_0 \). □

3.2. Stability results

Depending on the properties of the sequence \( V_i(t) \) provided by the algorithm, Theorem 1 can be used to infer sufficient conditions for uniform stability and uniform asymptotic stability. The following notation is used throughout the section: \( t(i) : \mathbb{R} \to \mathbb{N} \), such that
\[ t(i) = i \quad \text{whenever} \quad i \in [t_1,t_1]. \]

Corollary 1. Let \( V_i, i = 1, \ldots, \infty \), designate the values provided by the algorithm starting with a given \( V_0 \). If there exists a finite integer \( M \) such that both \( V_i < \infty \) and \( V_i \geq V_i \) for \( i = 0, \ldots, \infty \), then the system is uniformly stable.

Proof. Assumption 1 guarantees both \( A(t) \) Hurwitz and \( \|A(t)\| < \zeta \), \( \forall \in \mathbb{R} \). Using Eq. (8) in p. 595 of Shapiro (1974) (see also Montemayor & Womack, 1975) applied to \( A(t) \) \( P_i P_i A(t) = -I \) ensures that there exists a \( c > 0 \) such that \( \min \lambda(P_i) > c/\zeta \) if the spectral norm is chosen for \( \|A(t)\| \). Because this bound is the same for all \( i \), setting \( \bar{\lambda} = c/\zeta \) gives the lower estimate
\[ x(t) x(t) < x(t) x(t). \]

Theorem 1 then states that, as long as \( x(0) \) is chosen such that \( x(t_0) x(t_0) < V_0 \),
\[ x(t) x(t) < x(t) x(t) x(t) \leq V \bar{\lambda} \eta \geq t_0. \]

Additionally, let \( \bar{\lambda}_{0,0} \) be the smallest eigenvalue of \( P_0 \), so that
\[ x(t_0) x(t_0) \geq \bar{\lambda}_{0,0} x(t_0). \]

The proof then follows by associating to an arbitrarily chosen \( R > 0 \), the specific \( \bar{\lambda} = (V_0 \bar{\lambda} / \bar{\lambda}_{0,0}) R \). Indeed, by choosing \( x_0 \) such that
\[ x(t) x(t) \leq V \bar{\lambda} \eta \geq t_0. \]

Proof. Theorem 1 guarantees that \( x(t_j) x(t_j) < V_j \), whenever \( x(t_0) x(t_0) < V_0 \). Because a monotonically strictly decreasing sequence of infinitely many positive quantities converges to zero (Knopp, 1956), \( \lim_{j \to \infty} x(t_j) x(t_j) = 0 \). Since \( \sum_{i=1}^{\infty} \delta_i = \infty \), there exists \( c > 0 \) for which \( \delta_i > c, \forall i \). This lower boundedness of \( \delta_i \) together with the fact that the ordering among the chosen \( V_i \)'s is maintained, implies that \( \lim_{j \to \infty} t_j = \infty \). Thus, \( x(t_j) x(t_j) \to 0 \) as \( t \to \infty \). To conclude, it must be shown that \( x \to 0 \) as well. Let \( \hat{\lambda} > 0 \) be any lower bound on all eigenvalues of the positive-definite matrices \( P_i \) \( i = 0, \ldots, \infty \), (which exists thanks to Shapiro’s, 1974, Eq. (8)), inequality). This means that \( x(t_0) x(t_0) \). Therefore, when \( t \to \infty \), both \( x(t_0) x(t) \to 0 \) and \( x(t) \to 0 \). Now, uniform Lyapunov stability is guaranteed by Corollary 1. Finally, choosing \( c > 0 \) independently from \( t_0 \), it is always possible to choose \( V_0 \) at time \( t_0 \) sufficiently large such that \( x(t) x(t) < c/\zeta \), \( x(t) \to 0 \). Thus, \( x(t) \to 0 \) as \( t \to \infty \) concluding on uniform asymptotic stability according to Definition 3. □
Remark 2. As for the second method of Lyapunov, the method does not require integration of a differential equation.

4. Practical aspects

This section is concerned with rules of thumb for choosing the parameters of the algorithm and computing certain quantities using a finite number of operations.

4.1. Integrating \( \beta_i \)

Knowing exactly \( \beta_i(t) \) at every time instant is computationally intractable. Hence, a time grid is chosen (finer than the one for the ellipsoids) where \( \beta_i \) is evaluated. A simple Riemann sum can be used to approximate the integral appearing in the definition of \( \tilde{V}_{i+1} \):

\[
\int_{t_i}^{t_{i+1}} \beta_i(t) \, dt \approx \sum_{j=1}^{N} \beta_i \left( t_i + j \frac{t_{i+1} - t_i}{N} \right) \frac{t_{i+1} - t_i}{N} .
\]

4.2. Choosing the parameters

Three parameters particularize the algorithm.

1. The total time for the algorithm: If the system is periodic with period \( T \), then the total time can be set to any value superior to \( T \). When this is not the case, the method relies on simulating over the whole time interval for which the system is designed to work. This is never infinity for real-life systems, and the paper assumes that a finite upper bound is known. The simulation time is then set to this value.

2. The number \( N \) of points where \( \beta_i \) is evaluated: The resulting tube has been witnessed to converge quickly with respect to this parameter. Therefore, one can simply start with a few points and increase their number until the result becomes insensitive to this parameter.

3. The time spacing \( \delta_i \) between ellipsoids: All eigenvalues of \( A(t)^T P_i + P_i A(t) \) should remain negative for all \( t \in [t_i; t_{i+1}] \) when choosing the time grid. Normally, one starts with a loose grid. Then, checking that \( \beta_i(t) \) remains negative between two successive ellipsoids represented by \( P_i \) and \( P_{i+1} \) ensures that the time grid is suitable. Under this condition, and if a reduction of the tube is observed after running the algorithm, the system can be assessed stable. In case \( \beta_i(t) \) becomes positive, then the time separation is reduced and the algorithm run again. As the grid becomes finer, the resulting tube is less conservative and eventually converges to a given tube, where no improvement can be observed. This will be illustrated in the next section.

5. Examples

5.1. Stable system

The first example is a mass–spring system for which both the damping coefficient and the elastic constant are time varying. \( \alpha \) is a constant parameter that accounts for this variability:

\[
A(t) = \begin{pmatrix} 0 & 1 \\ -(2 - \alpha \sin(t)) & -(2 - \alpha \cos(t)) \end{pmatrix}.
\]

As long as \( \alpha < 2 \), the system is Hurwitz for all \( t \). Hence, both the proposed numerical stability condition and those found in the literature can be applied. Stability assessment depends on the sufficient condition used, and the corresponding maximum value of \( \alpha \) determines the degree of conservativeness involved.

5.1.1. Comparison with literature

The conditions appearing in the literature request that the time variation of the system matrix be small. This can be guaranteed by ensuring either a direct bound, \( \|A(t)\| \leq \delta \) for all \( t \geq 0 \) with \( \delta \) sufficiently small, or a bound on the time average

\[
\frac{1}{T} \int_{0}^{T} \|A\| \, dt \leq \delta,
\]

with \( \delta \) sufficiently small and a suitable \( T > 0 \). Using the time average leads to less conservativeness (Amato et al., 1993; Ichmann et al., 1987; Krause & Kumar, 1986). In the present example, \( \|A(t)\| = \max \lambda(A(t)^T A(t)) = \pi \), for all times \( t \). Therefore, the average (14) cannot decrease, no matter how large \( T \) is chosen. Hence, the sufficient test based on \( \|A(t)\| \) (Rosenbrock, 1963) cannot be improved upon using the aforementioned references. Applying the condition given in Rosenbrock (1963) amounts to performing the following computations: the eigenvalues of \( A(t) \) are computed as \( \lambda_{1,2} = \frac{1}{2} (\pi \cos t - 2) \pm \sqrt{\pi^2 \cos^2 t + 4\pi (\sin t - \cos t) - 4} \). The matrices

\[
H(t) = \begin{pmatrix} 1 & 1 \\ \lambda_1(t) & \lambda_2(t) \end{pmatrix}
\]

and \( S(t) = H(t)H^*(t) \) are formed, where * denotes the conjugate transpose. Then, the sufficient condition corresponds to checking the negative definiteness of

\[
L(t) = A(t)^T S(t) + S(t)A(t) - \dot{S}(t) + \varepsilon I,
\]

where \( \varepsilon \) is a small positive number. Computing the largest eigenvalue of \( L(t) \) for \( \alpha = 0.61 \) fails to assess stability, since it becomes positive at time \( t = 2 \) s.

As a matter of fact, Rosenbrock sufficient condition corresponds to solving the Lyapunov equation

\[
A(t)^T P + P A(t) = -C(t),
\]

where \( C(t) \) is the positive-definite matrix

\[
C(t) = (H^*)^{-1}(D^* + D)H^{-1}
\]

with \( D = \begin{pmatrix} \lambda_1(t) & 0 \\ 0 & \lambda_2(t) \end{pmatrix} \) and then using \( V(t) = x^T P(t)x \) as Lyapunov function. This means that \(-C(t) + \dot{P}(t)\) should remain negative definite to assess stability. This is illustrated in Fig. 2.

However, changing the matrix \( C(t) \) to the identity \( I \) for all \( t \) gives an improvement. Nevertheless, the test fails for \( \alpha = 1.3 \).

5.1.2. Application of the numerical test

The bounding tube is constructed for \( \alpha = 1.4 \), which is greater than the largest estimate obtained in the previous section. Lyapunov equation (3) is solved to determine \( P_t \), which is illustrated in Fig. 3. Fig. 4 shows a three-dimensional view of the
Then, checking that $-C(t) + \dot{P}(t)$ remains negative definite ensures stability. This is the case when $x=0.6$ (solid line), but not when $x=0.61$ (dotted line).

![Fig. 2.](image)

**Fig. 2.** $P(t)$ is obtained as the solution of $A(t)^T P(t) + P(t) A(t) = -C(t)$. Then, checking that $-C(t) + \dot{P}(t)$ remains negative definite ensures stability. This is the case when $x=0.6$ (solid line), but not when $x=0.61$ (dotted line).

The proposed numerical test guarantees stability up to $x = 1.6$ and fails beyond. **Fig. 5** shows the result for $x = 1.57$. It also illustrates the influence of the number of points for the tube evaluation and junction. The time horizon is kept constant (10 s). The number of points where the tube junctions take place is gradually increased. The time grid is equally spaced, meaning that, for a given choice of total tube junctions, $\delta > 0$ is chosen and all the $\delta_i$’s for the particular run of the algorithm are set to $\delta$ (i.e. $\delta$ changes from curve to curve so as to meet the constraint of 10 s). The estimate is less conservative as the number of points augments (as predicted by theory). The convergence is

![Fig. 3.](image)

**Fig. 3.** Solution of the Lyapunov equation $A(t_i)^T P_{i} + P_{i} A(t_i) = -I$ for $x=1.4$ at several time instants. Contours corresponding to the same level are plotted, i.e. $V_i = x^T P_{i} x = 1$. One curve is displayed for every time instant $t_i, i = 0, 1, \ldots, 30, \delta_i = 0.2$. The time-variability of the system is illustrated by the variability of the curves in both amplitude and orientation.

![Fig. 4.](image)

**Fig. 4.** Three-dimensional view of the lower half of the bounding; $x = 1.4$. Trajectories for the real system are also displayed for two different initial conditions $x_{1,0}$ and $x_{2,0}$. The solutions remain inside the bounding tube for all times.

![Fig. 5.](image)

**Fig. 5.** Influence of the number of tube junction evaluations. The value of $x$ is close to the limit where the algorithm fails to imply stability ($x=1.57 < 1.6$). The number of points has been gradually increased by 20 points starting with 30 points (top curve) up to 150 points (bottom curve). Full dots show exactly when the evaluation takes place. They have been omitted in the lower curves so as not to overclutter the figure.
The technique relies on successive optimization problems, the solutions of which are known without having to resort to iterative techniques. A sequence of quadratic Lyapunov functions is obtained, the validity of which is confined to the time interval between them. The optimization problems guarantee that, after joining one tube element to the next, all states are included for the next time interval. If the tube decreases in size, then the system is asymptotically stable.

Nevertheless, only a sufficient condition has been obtained using successive ellipsoidal approximations. Indeed, the estimation is conservative in the sense that, while no trajectory can leave the estimated bounding tube, this tube might be very large and even divergent, although all solutions to the time-varying differential equations could remain within a narrow converging region. However, establishing only a sufficient condition for stability has the advantage that the exact shape of the bounding tube need not be characterized. Therefore, a single ellipsoid is sufficient at each time-grid point.

The interesting aspect of the result presented is that it is based on information obtained on this discrete time grid only. Hence, this stability result is well-suited for the gain-scheduling framework and could potentially enrich this field of research.

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References


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