Kalman Filtering in Mobile Consensus Networks

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Abstract—In this paper, we design a Kalman filter for a mobile sensor network that tries to reach consensus based solely on relative measurements. We show, that this problem cannot be solved with state-of-the-art Kalman consensus filters because such a mobile sensor network is not observable. Hence, we propose a system transformation and the concept of a reference agent to obtain an asymptotically stable and observable system description. Based on the transformed system, we design a central Kalman filter and visualize the factorization details with Dynamic Bayesian Networks (DBN). Then, the Kalman filter is modified to obtain one decentral Kalman filter for each agent that uses delayed measurements propagated through the communication topology. This allows each agent to estimate the whole system state using only information sent by its neighbor agents defined by a communication topology. Simulation results demonstrate our method.

I. INTRODUCTION

Recently, efforts were made to solve the target tracking problem in Wireless Sensor Networks (WSNs) [1], [2], [3]. To track a target, each sensor node performs measurements. Due to uncertainties these measurements are distorted, leading to incorrect target estimates. Accurate estimates on each sensor node are commonly obtained by a two-stage approach: First each sensor node locally applies a tracking filter and then the consensus algorithm is used to synchronize the individual target state estimates of all sensor nodes.

In this paper, we try to solve a different problem. Instead of tracking one single target in a static WSN and using consensus to get a distributed estimate, we want to reach consensus of a mobile WSN using only relative measurements between the moving sensor nodes.

Finding consensus is the process of agreeing on a common state as time proceeds [4], [5], [6]. More formally, consider a WSN consisting of \( i = 1, \ldots, N \) sensor nodes with local estimates \( \mathbf{x}_i \in \mathbb{R}^n \) of the target state \( \mathbf{x} \in \mathbb{R}^n \). Consensus is reached if \( \| \mathbf{x}_i(t) - \mathbf{x}_j(t) \| \to 0 \) as \( t \to \infty \) for all \( i, j \).

In [7] a constant parameter vector \( \theta \) is to be estimated by the WSN. Each sensor uses noisy measurements of \( \theta \) to maintain a local estimate \( \hat{\theta}_i \). By applying average consensus through the communication topology of the WSN, the local estimates \( \hat{\theta}_i \) converge to the global maximum-likelihood solution \( \theta_{ML} \) as \( t \to \infty \). The authors in [8] apply the same technique to track a moving target: First, each sensor tracks the target through a local Kalman filter. Next, each sensor adapts the mean and covariance matrix in the average consensus step. This two-stage approach is known as the Kalman Consensus Filter (KCF) [9], [10] and is applied by a huge amount of subsequent research, e.g. [11], [12].

A) Single target tracking in a static WSN. The concept of tracking a single target in a static WSN with the KCF is shown in Fig. 1(a). Each sensor node filters the target measurements individually based on absolute measurements (red \( \sim \rightarrow \)) and adapts its local estimate through consensus with the neighbors in the communication topology (green \( \leftrightarrow \)).

B) Multiple independent target tracking in a static WSN. Multiple independent targets can be tracked with the KCF as well by running a KCF instance for each target in parallel, cf. Fig. 1(b).

C) Multiple dependent target tracking in a static WSN. Next, consider a multi-agent system moving in a formation. Here, the dynamics of the targets may depend on each other (Fig. 1(c)). To apply the KCF, it is necessary to combine all individual states into one state vector. This way, the multi-target tracking problem is reduced to the single target tracking case in Fig. 1(a).

D) Multiple dependent mobile sensors. Contrary to the previous cases, we consider multi-agent systems doing consensus solely based on relative measurements (Fig. 1(d)). We call such networks mobile consensus networks. Each agent is equivalent to a target with an attached sensor. In particular, we assume that the sensing topology equals the consensus topology (black \( \rightarrow \)), and the bidirectional communication topology follows the consensus topology. Due to process and measurement noise, the consensus dynamics needs to be filtered. Since the dynamics is now defined by the consensus topology itself, the system states are coupled. Thus, the KCF is not applicable anymore.

Our solution works as follows: Due to the consensus
dynamics and relative measurements, we first transform the states to obtain an observable system. Based on the transformed system, we design a Kalman filter. The design steps are visualized with DBNs. Additionally, a decentral version of the filter is introduced by propagating measurements with time delays [13] through the communication network.

This paper is organized as follows: The next section provides a proper problem formulation. In Section III we introduce the system of reduced order, which is used as base for the Kalman filters in Section IV. Section V provides simulation results and in Section VI a conclusion is given.

II. PROBLEM STATEMENT

A. Preliminaries

Let \( G = (V, E) \) be a digraph with nodes \( V = \{1, \ldots, N\} \) and directed edges \((i, j) \in E \subseteq V \times V \). With \( a_{ij} = 1 \iff (j, i) \in E \) and \( a_{ij} = 0 \) otherwise, define \( A = [a_{ij}] \) as adjacency matrix of \( G \). Further, a path of length \( l \in \mathbb{N} \) is a sequence of distinct nodes \( c_1, \ldots, c_{l+1} \) such that \((c_i, c_{i+1}) \in E \) for all \( i = 1, \ldots, l \). A path from \( c_1 \) to \( c_{l+1} \) is called a minimal path, if there is no other path from \( c_1 \) to \( c_{l+1} \) with shorter length. The length \( l \) of a minimal path from node \( i \) to node \( j \) is referred to as hop count \( h(i, j) \) [14]. Next, define the diameter of the graph as \( d_{\text{max}} = \max_{i,j \in V}(a_{ij}) \). A digraph contains a directed spanning tree if \( \exists e \in E \) such that there exists a path from \( c \) to all other nodes in \( V \). Further, define the in-degree of node \( i \) as \( d_i = \sum_{j \in V \setminus \{i\}} a_{ij} \) and the maximum degree as \( d_{\text{max}} = \max_{i \in V}(d_i) \). Then, with \( D_m = \text{diag}(d_1, \ldots, d_N) \) define the Laplacian matrix \( L = D_m - A \).

It is known [4], that for graphs containing a directed spanning tree with Laplacian matrix \( L \), the eigenvalues are \( 0 = \lambda_1 < 2 \Re(\lambda_2) \leq \ldots \leq 2 \Re(\lambda_N) \). The right eigenvector of the eigenvalue \( \lambda_1 \) is \( 1_N \), i.e. \( L1_N = 0 \). \( 1_N \) and \( 0_N \) are the column vectors with all ones and zeros of dimension \( N \), respectively. Let \( I_N \) be the \( N \times N \) identity matrix and \( 0_{p \times q} \) the \( p \times q \) matrix of zeros.

B. System Description

For the remainder of the paper, we consider a multi-agent system consisting of \( N \) agents with adjacency matrix \( A \). We assume that \( A \) describes a graph that contains a directed spanning tree. Further, the consensus problem is formulated in terms of a first order dynamical system in discrete time with sample time \( \tau \) and time step \( t_k = t_0 + k \tau \). For simplicity, we use the time index \( k \) instead of \( t_k \) and set \( t_0 = 0 \).

Following [5], the noisy consensus problem is given as

\[
\begin{align*}
\mathbf{x}^{k+1} &= \mathbf{x}^k + \tau \mathbf{u}^k + \mathbf{w}^k, \\
\mathbf{y}^k &= L \mathbf{x}^k, \\
\mathbf{y}^k &= N(\mathbf{w}^k|0, \mathbf{Q}), \quad (1a) \\
\mathbf{y}^k &= N(\mathbf{v}^k|0, \mathbf{R}). \quad (1b)
\end{align*}
\]

The vectors \( \mathbf{x}^k = [x_1^k \cdots x_N^k]^T, \mathbf{y}^k = [y_1^k \cdots y_N^k]^T, \mathbf{u}^k = [u_1^k \cdots u_N^k]^T \in \mathbb{R}^N \) represent state, output and control input for all agents, respectively. \( \mathbf{v} = [v_1 \cdots v_N]^T, \mathbf{w} = [w_1 \cdots w_N]^T \in \mathbb{R}^N \) are additive zero-mean white Gaussian noise processes with diagonal covariance matrices \( \mathbf{Q} = \text{diag}(\sigma_{Q,1}^2, \ldots, \sigma_{Q,N}^2) \) and \( \mathbf{R} = \text{diag}(d_1 \sigma_{R,1}^2, \ldots, d_N \sigma_{R,N}^2) \).

Instead of the noisy linear equations in (1) the system can be equivalently expressed by the transitional probability density function (pdf) and the emission distribution [15]

\[
\begin{align*}
p(x^{k+1}|x^k) &= N(x^{k+1}|x^k + \tau \mathbf{u}^k, \mathbf{Q}), \quad (2a) \\
p(y^k|x^k) &= N(y^k|L \mathbf{x}^k, \mathbf{R}). \quad (2b)
\end{align*}
\]

The Laplacian matrix \( L \) in (1b) defines the output of all agents; for a single agent the output reads

\[
y_i^k = \left[ \sum_{j=1}^{N} a_{ij} (x_j^k - x_i^k) \right] + v_i, \quad (3)
\]

where \( a_{ij} \) is an element of the adjacency matrix \( A \). By using the control input \( u^k = -y^k \), the transition reads

\[
x^{k+1} = (I - \tau L) x^k - \tau v + w = P x^k + \xi. \quad (4)
\]

Obviously, the system dynamics of the consensus problem is governed by the matrix \( P \) apart from the noise \( \xi \). Since \( A \) contains a directed spanning tree, the eigenvalues \( \mu_i \in \mathbb{C} \) of \( P \) satisfy \( 1 = \mu_1 > |\mu_2| \) for sufficiently small sampling time \( \tau \) [5]. Thus, all \( \mu_2 \) lie within the unit circle and the system without noise is stable in the sense of Lyapunov and convergence to a common value is guaranteed.

However, due to the non-deterministic system behavior only approximate convergence with a specific variance is guaranteed, loosely speaking \( \|x_i^k - x_j^k\| \approx 0 \) as \( k \to \infty \) holds. This motivates our estimation problem as follows.

**Problem Statement:** For system (1), find a filter to minimize the disturbances introduced by the noise processes.

**Remark 1:** For simplicity, scalar states \( x_i \in \mathbb{R} \) are considered until Section IV. However, it is straightforward to extend all theorems and the proposed filters to the vectorial case \( x_i \in \mathbb{R}^n \) with the Kronecker product. This way, one obtains an augmented system description, where each component of \( x_i \) follows the scalar properties presented in the following.

III. SYSTEM TRANSFORMATION

A. Introducing a Reference Agent

In order to apply a Kalman filter, system (1) needs to be fully observable. By applying the Kalman observability condition, it is easy to see that the rank of the observability matrix \( M_I = [L^T \ (LI_N)^T \ \cdots \ (LI_N^{N-1})^T]^T \) is \( \text{rank}(M_I) = N - 1 \), since \( \lambda_1 = 0 \) is always an eigenvalue of the Laplacian matrix \( L \) [4]. The null eigenvalue can be explained by realizing that (1b) solely uses differences, and the absolute reference point is lost. For the system dynamics (1a) this is of no interest, since the final consensus value does not matter as long as synchrony is reached.

Hence, we transform (1) to a system of order \( N - 1 \) by eliminating the eigenvalue \( \lambda_1 = 0 \) of \( L \). To this end, we introduce transformation matrices \( D \in \mathbb{R}^{(N-1) \times N} \) and \( U \in \mathbb{R}^{N \times (N-1)} \) [16] as

\[
D = \begin{bmatrix}
I_{r-1} & -1_{r-1} & 0_{(r-1) \times (N-r)} \\
0_{(N-r) \times (r-1)} & -1_{N-r} & I_{N-r}
\end{bmatrix}, \quad (5)
\]

\[
U = \begin{bmatrix}
I_{r-1} & 0_{(r-1) \times (N-r)} \\
0_{(N-r) \times (r-1)} & I_{N-r}
\end{bmatrix}. \quad (6)
\]
We refer to \( r \in \{1, \ldots, N\} \) as reference agent. By using the substitutions \( \Delta^k = \Delta x^k \) and \( x^k = 1_N x^k + U \Delta^k \) one obtains from (1) the \( \Delta \)-system
\[
\Delta^{k+1} = \Delta^k + \tau D u^k + D w, \quad N(w|0, Q),
\]
\[
y^k = L U \Delta^k + v, \quad N(v|0, R),
\]
of order \( N - 1 \). The transformed output reduces to (7b) by exploiting the relation \( L \mathbf{1}_{N} = \mathbf{0}_N \) (cf. Section II-A).

Inserting the control law \( u^k = -y^k \) results in the dynamics
\[
\Delta^{k+1} = (I_{N-1} - \tau DLU) \Delta^k - \tau Dv + Dw, \quad w_t \sim \mathcal{N}(0, \Sigma).
\]

The system matrix of the closed loop reads \( \tilde{P} \). The Gaussian noise \( \tilde{w} \) is given by \( \mathcal{N}(\tilde{w}|0, \Sigma) \) with covariance \( \Sigma = \text{diag}(\sigma^2_{Q,r}) \in \mathbb{R}^{(N-1) \times (N-1)} \).

The \( \Delta \)-system describes the dynamics of the differences of all states with respect to the reference agent. If consensus is reached, all differences are zero, i.e., the \( \Delta \)-system is asymptotically stable. To prove stability of the \( \Delta \)-system we restate the following lemma.

**Lemma 1 ([5]):** For a consensus problem with Laplacian matrix \( L \) with eigenvalues \( 0 = \lambda_1 < \Re(\lambda_2) \leq \cdots \leq \Re(\lambda_N) \), the eigenvalues of the matrix \( \mathbf{P} = I - \tau L \) of the discrete time system are \( \mu_i = 1 - \tau \lambda_i \) for all \( i \in \{1, \ldots, N\} \).

The discrete time system is stable in the sense of Lyapunov as long the sample time \( \tau \in (0, \frac{1}{\max(\lambda_i)}) \).

By using Lemma 1, we are ready to prove the following theorem.

**Theorem 1 (Stability of the \( \Delta \)-system):** Given a consensus problem of the form (1). Then, ignoring the noise, the closed loop \( \Delta \)-system with system matrix \( \mathbf{P} \) in (7) with sample time \( \tau \in (0, \frac{1}{\max(\lambda_i)}) \) is asymptotically stable.

**Proof:** By reordering the system states, we can assume w.l.o.g. that the reference agent is \( r = 1 \). W.l.o.g. \( L_{ij} = [l_{ij}] = T^{-1} LT \) is the Laplacian matrix in Jordan canonical form with \( l_{11} = \lambda_1 = 0 \). Then
\[
DL_{\mathbf{U}} = \begin{bmatrix}
\lambda_2 & *_2 & 0 \\
& \ddots & \ddots \\
& & \lambda_{N-1} & *_{N-1} \\
0 & & & \lambda_N
\end{bmatrix} \in \mathbb{R}^{(N-1) \times (N-1)},
\]
where \( *_{i\geq2} \in \{0, 1\} \) according to the algebraic and geometric multiplicity of the eigenvalues. Thus, the matrices \( D \) and \( \mathbf{U} \) eliminate \( \lambda_1 = 0 \); all remaining eigenvalues are not changed. By applying Lemma 1, \( |\mu_i| < 1 \) holds for the eigenvalues \( \mu_i = 1 - \tau \lambda_i \) of the matrix \( \mathbf{P} = I_{N-1} - \tau DLU \) with \( \tau \in (0, \frac{1}{\max(\lambda_i)}) \). This proves the claim.

Similarly, we state the observability property.

**Theorem 2 (Observability of the \( \Delta \)-system):** Given the mobile consensus problem in (1), the \( \Delta \)-system with matrices \( D \) and \( \mathbf{U} \) in (7) and sample time \( \tau \in (0, \frac{1}{\max(\lambda_i)}) \) is observable.

**Proof:** We know that the output matrix \( LU \) in (7b) has \( \text{rank}(LU) = N - 1 \). Thus, the observability matrix
\[
M_O = [L U \mathbf{1}_{N-1}]^T \cdots [L U \mathbf{1}_{N-2}]^T \in \mathbb{R}^{(N-1) \times (N-1)}
\]
has \( \text{rank}(M_O) = N - 1 \). This completes the proof.

**Remark 2:** In [17], [18], a similar but regular transformation has been introduced that includes the position of the reference agent. Hence, the transformed system is not observable.

**B. Characteristics of the \( \Delta \)-System**

For better understanding, we consider a system consisting of \( N = 3 \) agents and topology in Fig. 2(a). We choose the reference agent \( r = 1 \). The matrices \( L, D, U \) read
\[
L = \begin{bmatrix}
1 & -1 & 0 \\
0 & 1 & -1 \\
0 & 0 & 0
\end{bmatrix}, \quad D = \begin{bmatrix}
-1 & 1 & 0 \\
-1 & 0 & 1 \\
0 & 0 & 1
\end{bmatrix}, \quad U = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}.
\]

The dynamics \( x^{k+1} \sim p(x^{k+1}|x^k) \) of the closed loop
consensus problem according to (4) is
\[
\begin{bmatrix}
    x_{k+1}^1 \\
    x_{k+1}^2 \\
    x_{k+1}^3
\end{bmatrix} = \begin{bmatrix}
    x_k^1 \\
    x_k^2 \\
    x_k^3
\end{bmatrix} - \tau \begin{bmatrix}
    1 & -1 & 0 \\
    0 & 1 & -1 \\
    0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
    x_k^1 \\
    x_k^2 \\
    x_k^3
\end{bmatrix} + \xi,
\]
(9)
with output \( y^k = Lx^k + v \). The Dynamic Bayesian Network (DBN) in Fig. 2(b) unfolds the joint probability \( p(x^{0:k}, y^{0:k}) \) of system (9) over time [15] and visualizes its factorization
\[
p(x^{0:k}, y^{0:k}) = p(x^0) \prod_{i=1}^k p(x^i|x^{i-1}) \prod_{i=0}^k p(y^i|x^i).
\]
(10)
Note, that the edges in Fig. 2(a) represent the information flow, while the edges in a DBN characterize conditional probabilities (cf. Fig. 2(b)). Applying the transformation, according to (8) the system of reduced order reads
\[
\begin{bmatrix}
    \Delta_{k+1}^1 \\
    \Delta_{k+1}^2
\end{bmatrix} = \begin{bmatrix}
    \Delta_k^1 \\
    \Delta_k^2
\end{bmatrix} - \tau \begin{bmatrix}
    2 & 1 \\
    -1 & 0
\end{bmatrix} + \bar{\omega}
\]
(11)
with output (7b)
\[
y^k = \begin{bmatrix}
    y_1^k \\
    y_2^k \\
    y_3^k
\end{bmatrix} = \begin{bmatrix}
    -1 & 0 \\
    1 & -1 \\
    0 & 0
\end{bmatrix} \begin{bmatrix}
    \Delta_k^1 \\
    \Delta_k^2
\end{bmatrix} + v.
\]
(12)
Again, Fig. 2(c) shows the corresponding DBN with its joint probability
\[
p(\Delta^{0:k}, y^{0:k}) = p(\Delta^0) \prod_{i=1}^k p(\Delta_i^i|\Delta^{i-1}) \prod_{i=0}^k p(y^i|\Delta^i).
\]
(13)

**Remark 3 (Choice of the reference agent):** The DBN in Fig. 2(c) can be derived from the DBN in Fig. 2(b) by merging the nodes \( x_1 \) with \( x_2 \) into \( \Delta_1 \) and the nodes \( x_1 \) and \( x_3 \) into \( \Delta_2 \). Thus, the factorization properties of the \( \Delta \)-system in the DBN significantly depend on the choice of the reference agent \( r \). For instance, for \( r = 2 \), both measurements \( y_r \) only depend on \( \Delta_i \) and the transition is not fully connected.

IV. OPTIMAL KALMAN FILTER

A. Designing a Central Kalman Filter

In contrast to system (1), the \( \Delta \)-system (7) is observable. As a consequence, it is possible to design a Kalman filter for the \( \Delta \)-system. To this end, we assume that each agent has global knowledge, i.e., each agent has access to all measurements and system states at each time step. With this assumption, \( N \) central Kalman filters all using the same reference agent exist in the network, one Kalman filter estimating the state \( \Delta \) on each agent as follows.

**Central Kalman Filter for the \( \Delta \)-system:** Given the \( \Delta \)-system as derived in (7)
\[
\Delta^{k+1} = \Delta^k + \tau D u^k + \bar{\omega}, \quad N(\bar{\omega}|0, \widetilde{Q}),
\]
(14)
\[
y^k = L U \Delta^k + v, \quad N(v|0, R),
\]
(15)
with \( u^k = -LU\Delta^k \). Then, based on the current estimate \( \Delta^k \in \mathbb{R}^{N-1} \), the predicted estimate \( \Delta^{k+1} \in \mathbb{R}^{N-1} \) and the predicted error covariance matrix \( \widetilde{F}^{k+1} \in \mathbb{R}^{(N-1)\times(N-1)} \) of the central Kalman filter are
\[
\Delta^{k+1} = \hat{P} \Delta^k,
\]
\[
\widetilde{F}^{k+1} = \hat{P} \widetilde{F}^k \hat{P}^\top + \bar{Q}.
\]
(16)
(17)
The Kalman update follows the equations
\[
\hat{y}^{k+1} = L U \Delta^{k+1},
\]
\[
\Delta^{k+1} = \Delta^{k+1} + K^{k+1}(y^{k+1} - \hat{y}^{k+1}),
\]
(18)
(19)
\[
\hat{F}^{k+1} = (I_{N-1} - K^{k+1}LU) \widetilde{F}^{k+1},
\]
\[
K^{k+1} = \hat{F}^{k+1}U^\top L^{-1} (LU \hat{F}^{k+1}U^\top L^{-1} + R)^{-1}
\]
(20)
(21)
with Kalman gain \( K^{k+1} \in \mathbb{R}^{(N-1)\times N} \), updated error covariance \( \hat{F}^{k+1} \in \mathbb{R}^{(N-1)\times(N-1)} \) and initial values \( \Delta^0 \) and \( \hat{F}^{0} \).

B. Designing a Decentral Kalman Filter

The decentral Kalman filter we propose works similar to the central Kalman filter. Again, we assume that \( N \) Kalman filters exist in the network, one Kalman filter estimating the entire state \( \Delta \) on each agent. However, each agent \( i \) only has immediate access to its own measurement \( y_i^k \). All other measurements \( y_j^k \neq y_i^k \) need to be communicated through the network. To this end, we assume bidirectional communication for all agents \( i, j \) as in Fig. 1(d), if \( j \) is able to measure the distance to \( j \) or vice versa, i.e., if \( l_{ij} \neq 0 \) or if \( l_{ji} \neq 0 \) with Laplacian matrix \( \mathbf{L} = [l_{ij}] \).

By using bidirectional communication, all measurements \( y_j \) are broadcasted through the communication network. However, the communication introduces time delays [13]. We assume the time delay for passing a measurement \( y_j \) from node \( j \) to node \( i \) is determined by the hop count \( h(j,i) \).

With (3), the measurement vector \( \tilde{y}_i^k \in \mathbb{R}^N \) of agent \( i \) then reads
\[
\tilde{y}_i^k = \begin{bmatrix}
    y_i^k \\
    y_{i+1}^{k-h(1,i)} \\
    \vdots \\
    y_N^{k-h(N,i)}
\end{bmatrix}.
\]
(22)

Clearly, agent \( i \) only has immediate access to its own measurement such that \( y_i^{k-h(i,i)} = y_i^k \). Due to the communication delays, all measurements \( \tilde{y}_i^k \) differ. Hence, the estimates \( \hat{\Delta} \) of the \( \Delta \)-system differ as well. Therefore, we introduce a local state estimate \( \Delta_i^k \in \mathbb{R}^{N-1} \), \( i = 1, \ldots, N \), and local transformation matrices \( D_i \) and \( U_i \) according to (5) and (6) with reference agent \( r = i \). Then each agent runs a decentralized Kalman filter with itself as reference agent as follows.

**Decentral Kalman Filter for the \( \Delta \)-system:** For the \( \Delta \)-system with transition
\[
\Delta_i^{k+1} = \Delta_i^k + \tau D_i u_i^k + \bar{\omega}_i, \quad N(\bar{\omega}_i|0, \widetilde{Q}_i),
\]
(23)
and output (22), the predicted estimate \( \Delta_i^{k+1} \in \mathbb{R}^{N-1} \) and the predicted error covariance matrix \( \widetilde{F}_i^{k+1} \in \mathbb{R}^{(N-1)\times(N-1)} \) of the decentralized Kalman filter are
\[
\Delta_i^{k+1} = \hat{P}_i \Delta_i^k,
\]
\[
\widetilde{F}_i^{k+1} = \hat{P}_i \widetilde{F}_i^k \hat{P}_i^\top + \bar{Q}_i.
\]
(24)
(25)
\[ \Delta_k \text{ is a system with reference agent } \Delta_1 = \Delta_2 = \Delta_3 = \Delta \text{ holds.} \]

The Kalman update follows the equations

\[ \hat{y}_i^{k+1} = LU_i \Delta_i^{k+1}, \]
\[ \Delta_i^{k+1} = \Delta_i^{k+1} + K_{i}^{k+1} (\hat{y}_i^{k+1} - \hat{y}_i^{k+1}), \]
\[ \hat{F}_i^{k+1} = (I_{N-1} - K_i^{k+1} LU_i) \hat{F}_i^{k+1}, \]
\[ K_i^{k+1} = \hat{F}_i^{k+1} U_i^{\top} L_i^{\top} (LU_i \hat{F}_i^{k+1} U_i^{\top} L_i^{\top} + R)^{-1} \]

with \( u_i^k = -LU_i \hat{\Delta}_i^k \), the Kalman gain \( K_1 \in \mathbb{R}^{(N-1) \times (N-1)} \) and the updated error covariance \( \hat{F}_i \) at time step \( k+1 \) and initial values \( \Delta_i^0, \hat{F}_i^0 \) and \( \hat{y}_i^0 = [\hat{y}_i^0, \cdots, \hat{y}_i^N]^\top \).

It is noteworthy, that the bidirectional communication does not change the dynamics, i.e., the Laplace matrix \( L \) of the central and the decentral Kalman filter are the same.

**C. Applying the Reverse Transformation**

The reverse transformation from the \( \Delta \)-system with reference agent \( r \) back to the original system is accomplished by

\[ x_r^{k+1} = 1_N x_r^{k+1} + U_r \hat{\Delta}_r^{k+1} \]

with

\[ x_r^{k+1} = x_r^k - \tau \hat{y}_r^k + w_r \]

where \( \hat{y}_r^k = [\hat{y}_1^k, \cdots, \hat{y}_r^k, \cdots, \hat{y}_N^k]^\top = LU_i \hat{\Delta}_r^k \). According to (31), the accuracy of the state \( \Delta^{k+1} \) of the original system depends on variance of the transitional noise \( w_r \).

**D. Characteristics of the Decentral Kalman Filter**

In order to gain a better understanding, we again consider the example in Fig. 2. The central Kalman filter for the consensus problem depicted in Fig. 2(a) works according to the DBN in Fig. 3(a). Here, each agent runs a central Kalman filter all using the same reference agent to estimate the state \( \Delta \). Given the local measurements \( y_i \), each state estimate depends on all measurements.

In the presence of time delays introduced by the communication, the factorization properties of the decentral Kalman filter change, as shown in Fig. 3(b). Here, each state estimate depends on past measurements, depending on the hop count.

**Remark 4:** In each time step, the measurements \( y_i \) depend on a different set of states \( \Delta_i \). For instance, at time \( t_{k-1} \) the measurement \( y_1^{k-1} \) only depends on the state \( \Delta_1^{k-1} \), which leads to the observation likelihood \( p^{k-1}(y_1^{k-1} | \Delta_1^{k-1}) \). In the next time step \( t_k \), the same measurement \( y_1^{k-1} \) additionally depends on \( \Delta_2^k \) and it follows \( p^{k-1}(y_1^{k-1} | \Delta_1^{k-1}, \Delta_2^k) = p^{k-1}(y_1^{k-1} | \Delta_1^{k-1}) p^{k}(y_1^{k-1} | \Delta_2^k) \). Thus, the DBN is time variant.

**V. Results**

**A. System Representation**

According to Remark 1 we consider a mobile consensus problem with \( N = 5 \) agents with states \( x_i \in \mathbb{R}^2 \) in the plane \( (n = 2) \). Each dimension follows the consensus topology in Fig. 4, which is equivalent to the Laplacian matrix

\[ \tilde{L} = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \]

The Laplacian for the overall system is augmented with the Kronecker product and reads \( L = \tilde{L} \otimes I_5 \in \mathbb{R}^{10 \times 10} \). Since the Kronecker product duplicates all eigenvalues, the augmented Laplacian matrix \( L \) has \( n = 2 \) eigenvalues in \( 0 \). Thus, the transformed \( \Delta \)-system is of order \( 2(N-1) = 8 \).

The measurement noise \( \nu \sim N(0, R) \) and the process noise \( w \sim N(0, Q) \) are characterized by the covariance matrices

\[ R = \text{diag}(1, 1, 1, 1, 1, 1, 1, 1), \]
\[ Q = \tau \cdot 0.1 \cdot R, \]
implies that agent $k$ uses the communicated measurement $y_i$ to agent $j$, $i \neq j$.

The decentral Kalman filters use the communicated measurements according to (22). The diameter of the topology is $\text{diam} = 3$. This maximum delay can be observed in the topology in Fig. 4 for agents 1, 2 and 5. The impact of the delay depends on the sample time $\tau$. Larger $\tau$ imply more outdated communicated measurements and less accurate filtering.

Fig. 5(c) shows the difference between the central filter and the $N$ decentral filters. Again, the state $x_i$ originates from the $i$-th of the $N$ decentral Kalman filters. The error covariance matrix $\Sigma_0 = I_5$.

In the decentral filter case, we assume that all measurements are initially communicated to all agents, meaning that $y_i^{k-h(i,j)} = y_i^k$ if $k - h(j,i) < 0$ in (22). For instance, the hop count from agent 5 to agent 1 is $h(5,1) = 3$. This implies that agent 5 uses the communicated measurement $y_i^0$ in the first $3 + 1 = 4$ filter steps multiple times.

C. Reference Frame

As mentioned in Section III-A, the $\Delta$-system only describes the dynamics of the differences independent of the absolute reference frame. Thus, applying the reverse transformation always interprets the agent states in a local reference frame. Since every agent has its own reference frame (cf. Section IV-C), the states cannot directly be compared.

However, in the following simulation the measurements $y_i$ are build (analog to the scalar case in (3)) by

$$y_i^k = \sum_{j=1}^{N} a_{ij}(x_i^k - x_j^k) + v_i,$$

where $x_i^k$ is taken locally from each agent $j$. Hence, the measurements lead to all agents using the same reference frame and the individual states $x_i$ can be compared in one coordinate system.

D. Simulation Results

Fig. 5(a) shows the unfiltered trajectories of the first dimension of all states $x_i$ for a time frame of 10 seconds. Fig. 5(b) depicts the same trajectories, filtered with the $N$ decentral Kalman filters. Note, that the trajectory of agent $i$ originates from the $i$-th decentral Kalman filter. Obviously, the filters significantly reduce the disturbances introduced by the noise.

Fig. 6(a) shows a snapshot of the agents in the two-dimensional space approximately at the time where consensus is reached. Fig. 6(b) shows the same snapshot but without a filter. The filtered system much better estimates the real trajectories of the agents.

E. Impact of Delayed Measurements

The decentral Kalman filters use the communicated measurements according to (22). The diameter of the topology is $\text{diam} = 3$. This maximum delay can be observed in the topology in Fig. 4 for agents 1, 2 and 5. The impact of the delay depends on the sample time $\tau$. Larger $\tau$ imply more outdated communicated measurements and less accurate filtering.

Fig. 5(c) shows the difference between the central filter and the $N$ decentral filters. Again, the state $x_i$ originates from the $i$-th of the $N$ decentral Kalman filters. The error of $\pm 0.25$ is rather limited due to the short diameter. With increasing diameter, the divergence increases as well.

Further, the average hop count $\bar{h}(i) = \frac{1}{N-1} \sum_{j \neq i} h(i,j)$ of agent $i$ significantly influences the accuracy of the delayed Kalman filter. Clearly, agents $i$ with a minimal average hop count $\bar{h}(i)$ suffer lower time delays compared to agents with large average hop count and, thus, have more accurate filter results.
measurements. The Kalman filter is slightly lower than the sample variance of the filtered system, which is due to the sampling time. The sample variances of the filtered systems are very similar. It is required to first apply a transformation to obtain an estimation for the first dimension, depending on the neighbors. The dashed lines show the ideal trajectories without noise.

**Fig. 6. Trajectories of the mobile consensus problem in the plane, a) filtered with \( N \) decentral Kalman filters, and b) no filter.**

**F. Statistical Evaluation**

Fig. 5(d) shows the sample variance

\[
s^2 = \frac{1}{N-1} \sum_{i=1}^{N} (x_{i,1} - \bar{x})^2
\]

with sample mean \( \bar{x} = \frac{1}{N} \sum_{i=1}^{N} x_{i,1} \) for the first dimension, averaged over 1000 simulation runs. Clearly, the unfiltered consensus problem shows the largest sample variance. The sample variances of the filtered systems are very similar. The reason for this is the rather small diameter and the low sampling time.

In the first 5 seconds, the sample variance of the decentral Kalman filter is slightly lower than the sample variance of the central Kalman filter due to the initialization of the measurements \( y_i^0 \) as described in Section V-B.

**VI. CONCLUSION AND OUTLOOK**

The proposed filters for the mobile consensus problem reduce the effects of noise. Due to relative measurements, it is required to first apply a transformation to obtain an observable system. Then, a central Kalman filter was globally designed to estimate the whole system state. Based on the global knowledge of the consensus topology and assuming bidirectional communication, \( N \) decentral filters were derived. These filters estimate the whole state locally, as communication and computation are completely distributed. Further, Dynamic Bayesian Networks were used to show the factorization properties of the system.

A limitation of the proposed filters is the estimation of all states. It would be of advantage to find distributed filters, where each agent only estimates its own state \( x_i \), or a subset of the full state \( x \) depending on the neighbors. Based on the representation as Dynamic Bayesian Networks, it would be interesting to apply approximate inference algorithms.

Also, it is desirable to evaluate arbitrary formations [4] instead of consensus in the sense of rendezvous.

**REFERENCES**


