Self testing quantum apparatus

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We study, in the context of quantum information and quantum communication, a configuration of devices that includes (1) a source of some unknown bipartite quantum state that is claimed to be the Bell state $\Phi^+$ and (2) two spatially separated but otherwise unknown measurement apparatus, one on each side, that are each claimed to execute an orthogonal measurement at an angle $\theta \in \{-\pi/8, 0, \pi/8\}$ that is chosen by the user. We show that, if the nine distinct probability distributions that are generated by the self checking configuration, one for each pair of angles, are consistent with the specifications, the source and the two measurement apparatus are guaranteed to be identical to the claimed specifications up to a local change of basis on each side. We discuss the connection with quantum cryptography.

I. INTRODUCTION

Typically, when one considers the task of testing a quantum system, for example using quantum state tomography, one makes the assumption that the measurement apparatus are perfect or reasonably close to perfect. Moreover, it is typically assumed that every measured system has the correct dimension. In contrast, here we consider the problem of testing a quantum system without trusting the measuring apparatus that are used in the test, except the fact that two measurements that are space like separated in the ideal specification can be modeled in the real setting by two simultaneous quantum operations on distinct systems. In particular, we assume no apriori information about the dimension of the measured systems or on the rank of the measurement operators. The intuition that applies to ordinary tests where the dimensions are correct does not apply here. The problem that we have is more difficult. In fact, it will already be an interesting challenge to consider the case where the probability distribution for measurement outcomes in the real setting are identical to the corresponding probability distribution for measurement outcomes in the ideal setting.

To be more specific, we describe the setting that we will consider. The source has specification to emit two systems (say photons) $A$ and $B$ in the Bell state $\Phi^+ = (|00\rangle^A^B + |11\rangle^A^B)/\sqrt{2}$. The photons $A$ and $B$ are sent to two measurement apparatus, one for each photon, that respectively receive the classical inputs $\alpha, \beta \in \{-\pi/8, 0, \pi/8\}$ that represent the measurement bases $\{|\alpha+0\rangle, |\alpha+\pi/2\rangle\}$ and $\{|\beta+0\rangle, |\beta+\pi/2\rangle\}$. The photons $A$ and $B$ are respectively measured in the bases $\alpha$ and $\beta$ and the respective classical outcomes $x$ and $y$ of these measurements are noted on each side.

Let $p(\alpha,x),(\beta,y)$ be the probability of the pair of outcomes $\langle x,y \rangle$ given the pair of measurements $\langle \alpha, \beta \rangle$ when the system respects perfectly the original specifications. Let $\tilde{p}(\alpha,x),(\beta,y)$ be the corresponding probabilities for the actual system which not might not be built to the original specification. Let $\Theta = \{-\pi/8, 0, \pi/8\} \times \{0, 1\}$, the space of pairs $\langle \alpha, x \rangle$ where $\alpha$ is a basis and $x$ an associated outcome. Our main result states that if, $\forall (a,b) \in \Theta^2$, we have $\tilde{p}(a,b) = p(a,b)$, the setting is necessarily identical modulo some local isomorphisms (see section III) to the original specification. This result would not be surprising at all if we assumed that the two measured systems are two dimensional systems or that the measurement operators are executed on two dimensional systems, but we do not use any assumption of this kind here. No assumption on the measuring apparatus or the source are required in our proof of the theorem, except that before a measuring apparatus receives its choice of basis and until after the measurement is executed, the apparatus (and whatever has selected the basis) is isolated from the other measuring apparatus. In the specific context of quantum key distribution, this separation assumption is also needed after the measurement to guarantee the privacy of the generated key, but this is another issue. If we only worry about testing the source, we only need the separation assumption until after the test is executed. This separation assumption is only an assumption on the device or mechanism that is used to isolate the measuring apparatus; it is not a direct assumption on the measuring apparatus.

If the actual system is not built to the original specifications, we might have $\tilde{p}(a,b) \neq p(a,b)$. A test could eventually be executed to check how close the probabilities $\tilde{p}(a,b)$ are to the ideal case $p(a,b)$. A robust variation on the theorem should consider the case $|\tilde{p}(a,b) − p(a,b)| \leq \epsilon$, but we do not do this analysis. The problem and the main theorem are described in section III. A specific connection with the BB84 protocol [2] in quantum cryptography is discussed in section IV. Finally, the proof of the theorem is provided in section V.

II. RELATED RESULTS

The result is interesting from both a purely theoretical point of view and a practical point of view. We hope that it will have application in different areas of quantum information processing. The result was obtained in a specific context, the unconditional security [1] of a variation on a protocol proposed by Bennett and Brassard in 1984 [2]. For concreteness, we first explain this (variation on
the) protocol. In this protocol, Alice sends many photons to Bob in one of the four polarisation states $|\alpha + x\pi/2\rangle$, $\alpha \in \{-\pi/8, \pi/8\}$, $x \in \{0,1\}$. Usually, in the literature, the bases are at angles are 0 and $\pi/4$ instead of $-\pi/8$ and $\pi/8$ as we have here, but this is symmetrical. Bob measures each received photon in one of these two bases chosen uniformly at random and notes the outcome $y$. Let \( \Omega \) be the set of positions where Alice and Bob used the same basis. For each position in \( \Omega \), because Alice and Bob used the same basis, in principle they should share the same bit $x = y$. After the quantum transmission, Alice and Bob announce their bases, and therefore they learn the set \( \Omega \). Eve also learns the bases used by Alice, but it is too late for Eve because the photons are already on Bob’s side. Alice and Bob execute a test on a set \( T \subseteq \Omega \) and count the number of errors in \( T \). If too many errors are detected the test fails and the protocol aborts. Otherwise, the set \( T \) is thrown away and the protocol continues. Alice announces redundant bits about $E = \Omega - T$ and Bob uses this extra information to correct the errors in $E$. At this point, Alice and Bob should share a string of bits on $E$ which we call the raw key. Eve has obtained some information about the raw key from the redundant information or directly because she eavesdropped on the quantum channel. To address this problem, Alice and Bob generate a final key $k$ in which each bit $k_i$ is the parity of some subset $K_i \subseteq E$ of the raw key bits, a well known technique to extract a final key from the raw key [3]. As proven in [1], if the number of parity bits is not too large, the final key will be almost perfectly secret, that is, with almost certainty either the test fails or Eve has almost no information about the final key. The connection between this proof and our result is given in section IV.

The novelty of the problem that we consider is that the apparatus that are used in the protocol can be defective. There is no perfect solution to this problem. Therefore, every tool that can be used to improve the situation is normally welcome and our theorem is such a tool. Before we explain in which way our theorem helps, let us describe the basic steps that should be taken even before we use this theorem. The first basic step is simply to propose assumptions that directly say that the source is close to the ideal specification. The second basic step is to justify each of these assumptions (as much as an assumption can be) by considering the specific technology used. An example of such a direct assumption is an upper-bound on the ratio of multi-photon signal that is emitted by a source. In this case, the required technology could be a strong signal of light followed by an attenuator. The intensity of the strong signal is measured and then the attenuator is used to reduce the intensity of the signal to the desired level. It turns out that this specific technology is reasonably trusted. In the same way, a direct assumption must be proposed and justified for every degree of freedom which can encode information. In addition to the photon number space, one has to consider the frequency space, the polarization space and also we must consider the possibility that the information gets encoded into a different system. For example, if any mechanical system is used, the information could get encoded into some vibrational modes. Moreover, one should explain (in the general model of quantum mechanics) why all aspects are covered by the proposed assumptions. So the task is not easy and it is not a perfect solution.

Some people might say that these assumptions should not be trusted, and a test should be conducted to verify them instead. For example, one might propose to use a photon detector to test the intensity of the signal after the attenuator. Fine, but this leads us to our complementary approach. The immediate problem that we must address is that validity of the test depends on the testing apparatus, the photon detector. Let us consider what can be done. In 1991, Artur Ekert proposed a protocol that was based on EPR pairs and violation of a Bell inequality [4]. Though the security analysis of his protocol was incomplete, Ekert’s salient idea of using EPR pairs and violation locality in a protocol is far reaching. First, if the parties on both sides have trusted measuring apparatus, the idea of using an untrusted source of EPR pairs takes care of the defective EPR source issue by itself (without any use of violation of locality). Ekert’s analysis considered the case in which we have perfect measuring apparatus. A little bit later, it was strongly suggested in [6] that violation of a Bell inequality (which is more than just using EPR pairs) does not help in quantum cryptography unless the purpose is to verify the quantum apparatus. In the context of trusted quantum apparatus, violation of Bell like inequalities might not provide better security. Moreover, to our knowledge, it didn’t help as a theoretical tool to obtain a better bound on Eve’s information. However, with violation of locality, one can hope to test both a defective source and defective measuring apparatus. Ekert did not explicitly propose that we should use violation of locality to test both the EPR source and the measuring apparatus, but the idea was implicitly there. It was there, but remained unused until the work of [7] in which violation of locality was used to test both the source and the measuring apparatus (see also [11]). Two papers reported experimental work that was based upon Ekert’s protocol (or a similar protocol) [9, 10]. The purpose of some these papers was to address the imperfect apparatus issue. They cite Ekert for his protocol. However, nobody knows if Ekert’s protocol meets the objective. Even Ekert did not claim that. His main focus was not to test defective apparatus. There are other connections between quantum cryptography and violation of locality or Bell inequalities that focus less on testing defective apparatus than the current paper does, but are not less interesting [12].

This paper, following [7], shows that, except for the location of each apparatus and the time at which they receive their input, we can assume that Eve has designed both the EPR source and the detectors. In practice, to conduct a test on the source, we also need the assumption that the different executions of the overall configuration
in the test are identical and independent. Our main tool is violation of classical locality. In the context of untrusted source and untrusted measuring apparatus, it is not clear whether or not a violation of a Bell inequality, especially if it is only slightly violated, implies that a private key can be generated. Therefore, in this context, it is not sufficient to detect a violation of a Bell inequality to claim that a protocol is secure. So, our statement is not that violation of locality implies security. We use violation of locality indirectly by considering an ideal setting that is known to violate a Bell inequality and by restricting ourselves to attacks that generate the same probability distribution of classical outcomes as in this ideal setting. Again, if the attack modifies the probability distribution of these classical outcomes, this can be detected by Alice and Bob, but we do not do this part of the analysis. Of course, this test will require its own set of assumptions. This approach should not be taken as a way to avoid the basic steps that are described above. Instead, in view of the fact that no solution is perfect, we think that the existence of two complementary approaches is very much welcome.

III. THE PROBLEM AND THE MAIN RESULT

An apparatus emits an unknown system  ˜A ⊗ ˜B in some unknown state  ˜Ψ  ˜AB which will be measured by some unknown measurement operators  ˜Π  ˜a,  ˜Π  ˜b associated with  a,b ∈ Θ, respectively. Without loss of generality, we assume that the source emits a pure state  ˜Ψ  ˜AB. There is no loss of generality because an extra system can be added to  ˜A or  ˜B to purify the state. Without loss of generality, we can further assume that the defective measurement operators  ˜Π  ˜a and  ˜Π  ˜b respectively associated with  a and  b are orthogonal. An auxiliary system in some known state can be added to the measured system to replace a general POVM on this system by an orthogonal measurement on the extended system [15]. The ideal projection on the system  ˜A ( ˜B) associated with an element  a ∈ Θ (  b ∈ Θ) is denoted  ˜P  ˜a (  ˜P  ˜b). We define  p(a,b) =  ∥  ˜P  ˜a  ˜P  ˜b |Φ  ˜AB+∥  2 and  ˜p(a,b) =  ∥  ˜Π  ˜a  ˜Π  ˜b  ˜Ψ  ˜AB ||  2.

As we explained in the Introduction, the hypothesis in our main result (theorem 1) are  ˜p(a,b) = p(a,b),  ˜Π  ˜a =  ˜P  ˜a  ˜I  ˜B and  ˜Π  ˜b =  ˜I  ˜A  ˜P  ˜b. Different systems (  ˜P  ˜a,  ˜P  ˜b,  ˜Ψ  ˜AB) can generate the same probabilities  ˜p(a,b). Let  ˜A (  ˜B) be the support of the residual density matrix of  ˜Ψ  ˜AB after a partial trace over  ˜B ( ˜A). The definition of  ˜P  ˜a and  ˜P  ˜b outside  ˜A and  ˜B respectively cannot affect the probabilities  ˜p(a,b). Therefore the only constraint that we can hope to obtain on  ˜P  ˜a (  ˜P  ˜b) will have to be a constraint on  ˜P  ˜a  ˜P  ˜b =  ˜P  ˜a  ˜P  ˜b (  ˜P  ˜a  ˜P  ˜b) = (  ˜P  ˜a  ˜P  ˜b), where  ˜P  ˜a (  ˜P  ˜b) is the projection on the subspace  ˜A ( ˜B) of  ˜A ( ˜B). Our result takes this fact into account.

Another point is that the defective system  ˜A (the same for  ˜B) might not contain any qubit, that is, there might not exist any qubit  ˜A such that, for some other component  ˜E  ˜A, we have  ˜A =  ˜A ⊗  ˜E  ˜A. A unitary transformation on  ˜A can correspond to a change of basis in  ˜A, but it will not create a qubit because of the trivial fact that the final space is still  ˜A. If the space  ˜A already had a tensor product structure, a unitary transformation on this space could change a pure state into an entangled state, and this could be interpreted as a modification of the tensor product structure of the state but not of the space. If the space  ˜A has no tensor product structure, a unitary transformation on  ˜A will not create one. In opposition, we would like to conclude that somehow the subspace  ˜A ( ˜B) of the defective systems  ˜A ( ˜B) contains the correct qubit  ˜A ( ˜B). One solution is simply to add an extra qubit  ˜A ( ˜B) initially in the state |0  ˜A⟩ ( |0  ˜B⟩) and then consider a local unitary transformation defined on  ˜A ⊗  ˜A ( ˜B ⊗  ˜B) that will extract the information about the correct qubits in  ˜A ( ˜B) and swap it into  ˜A ( ˜B). Another formal solution is to consider an isometry from  ˜A to  ˜A ⊗  ˜E  ˜A. An isometry is the same as an unitary transformation except that it can change the tensor product structure because the final space is not the same as the initial space.

Definition 1 A linear transformation  U from a space  V to a space  W is an isometry if and only if  V and  W have the same dimension and  U preserves the inner product between any pair of states in  V.

Theorem 1 For every setting (  ˜P  ˜a,  ˜P  ˜b,  ˜Ψ  ˜AB) such that  ˜p(a,b) = p(a,b),  ˜P  ˜a acts on  ˜A only and  ˜P  ˜b acts on  ˜B only, there exists two “garbage” spaces  ˜E  ˜A and  ˜E  ˜B, a state  ˜Ψ  ˜E  ˜A  ˜E  ˜B ∈  ˜A ⊗  ˜E  ˜B, an isometry  U  ˜A from  ˜A to  ˜A ⊗  ˜E  ˜A and an isometry  U  ˜B from  ˜B to  ˜B ⊗  ˜E  ˜B such that,

• for every  a ∈ Θ,  U  ˜A  ˜P  ˜a  ˜U†  ˜A = (  ˜P  ˜a ⊗  ˜I  ˜E  ˜A),
• for every  b ∈ Θ,  U  ˜B  ˜P  ˜b  ˜U†  ˜B = (  ˜P  ˜b ⊗  ˜I  ˜E  ˜B),
• (  U  ˜A ⊗  U  ˜B)  ˜Ψ  ˜AB = (  ˜Φ  ˜AB ⊗  ˜Ψ  ˜E  ˜A  ˜E  ˜B).

This theorem essentially states that the real system is identical to the ideal system ((  ˜P  ˜a ⊗  ˜I  ˜E  ˜A), (  ˜P  ˜b ⊗  ˜I  ˜E  ˜A),  ˜Φ  ˜AB ⊗  ˜Ψ  ˜E  ˜A  ˜E  ˜B) up to a local change of basis (that can modify the tensor product structure) on each side. The system ((  ˜P  ˜a ⊗  ˜I  ˜E  ˜A), (  ˜P  ˜a ⊗  ˜I  ˜E  ˜A),  ˜Φ  ˜AB ⊗  ˜Ψ  ˜E  ˜A  ˜E  ˜B) follows exactly the original specification except for an additional system  ˜E  ˜A ⊗  ˜E  ˜B that is in some pure state  ˜Ψ  ˜E  ˜A  ˜E  ˜B which doesn’t interfere at all with this specification. One cannot hope to prove more than theorem 1 using only  ˜p(a,x) = p(a,x). The proof will be given in section V.

IV. CONNECTION WITH THE BB84 PROTOCOL

There are different ways in which our main result could be connected to a security proof. In particular, our self-
checking apparatus can be used in the BB84 protocol in two different ways. In one way, the two measuring apparatus in our self-checking apparatus are on Alice’s side. This is the approach that we will describe here. It corresponds to the original idea of a self-checking source entirely located on Alice’s side. The other approach is that the two measuring apparatus are respectively located on Alice’s side and on Bob’s side. We will not discuss this other option here.

We recall that we consider a variation on the BB84 protocol where every state is rotated of an angle $-\pi/8$ so that the bases used are at angle $-\pi/8$ and $\pi/8$ instead of 0 and $\pi/4$ as is usually the case in the literature. When Alice picks $\alpha = \{ -\pi/8, \pi/8 \}$ for the measurement on the first photon and obtains the outcome $x$, the second photon collapses into the state $(\alpha + x\pi/2)$, as requested in the BB84 protocol. However, our test requires that Alice uses the three angles $-\pi/8$, 0, and $\pi/8$ for the two photons. Therefore, to test the apparatus Alice will pick a random set of positions $R$ and the bases $\alpha, \beta \in \{ -\pi/8, 0, \pi/8 \}$ for the positions in this set. For the non tested positions (i.e., the positions not in $R$), Alice will use $\alpha \in \{ -\pi/8, \pi/8 \}$ for the first photon and send the other photon to Bob. The basic intuition is that, if the test really works, it should be unlikely that this test succeeds on $R$ and would fail if it was executed outside $R$.

Let us prove that, whenever the source respect the conclusion of theorem 1, the protocol is as secure as if an ordinary BB84 source was used. It is convenient to consider $U_{\hat{A}} \otimes U_{\hat{B}}$ as a change of bases which provides an alternative representation for the states of the subsystem $A \otimes B$. In this alternative representation, it is not hard to see that if $(\alpha, x)$ is used/obtained on $A$’s side, the system $B \otimes E_A \otimes E_B$ must be left in the collapsed state $(\alpha + x\pi/2) \otimes \Psi_{E_A E_B}$. If we return to the original representation, the collapsed state is $(U_{\hat{A}} \otimes U_{\hat{B}})(\alpha + x\pi/2)^A \otimes (\alpha + x\pi/2)^B \otimes \Psi_{E_A E_B}$. The transformation $U_{\hat{A}}$ has no effect on $B$’s side, so it’s the same thing as if Eve received the part $B \otimes U_{\hat{B}}(\alpha_1 + x_1\pi/2)^B \otimes \Psi_{E_A E_B}$. An important fact is that $U_{\hat{B}}$ and $\Psi_{E_A E_B}$ are independent of $\alpha$ and $x$. Therefore, with the state $U_{\hat{B}}(\alpha + x\pi/2)^B$, and the part $E_B$ of the state $\Psi_{E_A E_B}$ Eve could herself create the state $U_{\hat{B}}(\alpha + x\pi/2)^B \otimes \Psi_{E_A E_B}$. Therefore, Eve has nothing more than what she could obtain if the ordinary BB84 source was used together with a completely uncorrelated state $\Psi_{E_A E_B}$ that is initially shared between Alice and Bob.

V. THE PROOF

Here we prove the main result (theorem 1). Theorem 1 is given in terms of two local isometries $U_{\hat{A}}$ and $U_{\hat{B}}$ which preserve the tensor product structure of the subspace $A \otimes B$ of $A \otimes B$. We will also need a simpler notion of isomorphism which ignores the tensor product structure of the space $A \otimes B$. If we do not care about the tensor product structure, there is a smaller space which contains $\Psi_{\hat{A} \hat{B}}$ and this space is sufficient to describe the essential of the projections $P_a^A$ and $P_b^B$. This space is the span $S$ of $(P_a^A \otimes P_b^B)\Psi_{\hat{A} \hat{B}} \mid a, b \in \Theta$.

Definition 2 Consider any setting $(P_a^B, P_a^A, \Psi_{\hat{A} \hat{B}})$. The setting $(P_b^B, P_a^A, \Psi_{\hat{A} \hat{B}})$ is inner product isomorph to $(P_a^A, P_b^B, \Phi_{\hat{A} \hat{B}}^\dagger)$ if there exists an isometry $U$ from the span $S$ of $(P_a^A \otimes P_b^B)\Psi_{\hat{A} \hat{B}} \mid a, b \in \Theta$ to $A \otimes B$ such that, for every $a, b \in \Theta$, for every $|\phi_S\rangle \in S$, we have

$$A1: (P_a^A \otimes P_b^B)|\phi_S\rangle = U^\dagger(P_a^A \otimes P_b^B)U|\phi_S\rangle \quad \text{and}$$

$$A2: U\Psi_{\hat{A} \hat{B}} = \Phi_{\hat{A} \hat{B}}^\dagger.$$ 

The proof of theorem 1 proceeds in two main steps. First, we prove that the equality $\tilde{p}(a, b) = p(a, b)$ implies that $(P_b^B, P_a^A, \Psi_{\hat{A} \hat{B}})$ is inner product isomorph to $(P_a^A, P_b^B, \Phi_{\hat{A} \hat{B}}^\dagger)$. Second, we show that this isomorphism implies the conclusion of theorem 1. The reader might find the second step a little bit surprising because $S$, the span of $(P_a^A \otimes P_b^B)\Psi_{\hat{A} \hat{B}} \mid a, b \in \Theta$, is not necessarily identical to $A \otimes B$. In fact, $S$ is not in general the tensor product of two Hilbert spaces. As we will see, the trick is that the inner product structure $A \otimes B$ can be reconstructed because the projections $P_a^A$ and $P_b^B$ are defined on $\hat{A}$ and $\hat{B}$ separately.

A. The inner product isomorphism.

Throughout this subsection we assume that the equality $\tilde{p}(a, b) = p(a, b)$ hold and we try to show that $(P_b^B, P_a^A, \Psi_{\hat{A} \hat{B}})$ is inner product isomorph to $(P_a^A, P_b^B, \Phi_{\hat{A} \hat{B}}^\dagger)$. We have 6 possible values for $a$ (3 bases with 2 outcomes each) and 6 possible values for $b$, so a total of 36 possible (non normalised) states $(P_a^A \otimes P_b^B)\Psi_{\hat{A} \hat{B}}$. If our goal can be achieved, these 36 vectors should lie in a 4 dimensional space and be linearly related as in the ideal specification. For every $\alpha \in \{ -\pi/8, 0, \pi/8 \}$, let $\Theta_\alpha = \{ (\alpha, 0), (\alpha, 1) \}$. To achieve our goal we have that the length of these vectors are uniquely determined by the probabilities $p(a, b)$. We also have that, for every $\alpha \in \{ -\pi/8, 0, \pi/8 \}$, $\sum_{a \in \Theta_\alpha} P_a^A = 1$, $\sum_{b \in \Theta_\alpha} P_b^B = 1$, and also the commutativity of $P_a^A$ and $P_b^B$. The proof of the inner product isomorphism proceeds in three steps. In the first step, we show the following simple proposition.

Proposition 1 For every $a \in \Theta$,

$$P_a^A P_b^B \Psi_{\hat{A} \hat{B}} = P_a^A P_b^B \Psi_{\hat{A} \hat{B}} = P_a^A \Psi_{\hat{A} \hat{B}} = P_a^B \Psi_{\hat{A} \hat{B}}.$$ 

In the second step, we show this other simple proposition.

Proposition 2 For every $(\alpha, \beta) \in \{ -\pi/8, 0, \pi/8 \}$ with $\alpha \neq \beta$, the four non normalised vectors in $B(\alpha, \beta)$ def
Each of these 6 different sets $\mathcal{B}_{(\alpha,\beta)}$ of 4 vectors span a 4 dimensional space. The third and crucial step is to show that, for every $(\alpha, \beta)$ and $(\alpha', \beta')$ with $\alpha \neq \beta$ and $\alpha' \neq \beta'$, the four vectors in $\mathcal{B}_{(\alpha,\beta)}$ are linearly related to the four vectors in $\mathcal{B}_{(\alpha',\beta')}$ with the same coefficients as for the corresponding vectors in the ideal specification. More precisely, we must show the following crucial proposition.

**Proposition 3** For every $(\alpha, \beta)$ and $(\alpha', \beta')$ with $\alpha \neq \beta$ and $\alpha' \neq \beta'$, for every $(x, y) \in \{0, 1\}^2$,

$$P_{(\alpha, x)}^{\alpha} P_{(\beta, y)}^{\beta} \Psi_{\tilde{A}\tilde{B}} = \sum_{(x', y') \in \{0, 1\}} [T^{(\alpha', \beta')}(x', y')] P_{(\alpha', x')}^{\alpha} P_{(\beta', y')}^{\beta} \Psi_{\tilde{A}\tilde{B}}$$

where $[T^{(\alpha, \beta)}]$ is the unique 4 x 4 coefficient matrix such that

$$P_{(\alpha, x)}^{\alpha} P_{(\beta, y)}^{\beta} \Phi^+ = \sum_{(x', y') \in \{0, 1\}} [T^{(\alpha', \beta')}(x', y')] P_{(\alpha', x')}^{\alpha} P_{(\beta', y')}^{\beta} \Phi^+_{\tilde{A}\tilde{B}}.$$

It is not hard to see that proposition 3 implies that the mapping that maps $P_{(\alpha, x)}^{\alpha} P_{(\beta, y)}^{\beta} \Psi_{\tilde{A}\tilde{B}}$ into $P_{(\alpha', x')}^{\alpha} P_{(\beta', y')}^{\beta} \Phi^+$ for $a, b \in \Theta_0 = \{(0, 0), (0, 1)\}$, is an inner product isomorphism (see definition 2). Indeed, proposition 3 implies that any of the 6 sets $\mathcal{B}_{(\alpha,\beta)}$ is a (non normalised) basis for $S$ that is as good as any other basis to check if a projection $P_{(\alpha, x)}^{\alpha} P_{(\beta, y)}^{\beta}$ has the correct matrix representation. So, to check $P_{(\alpha, x)}^{\alpha} P_{(\beta, y)}^{\beta}$, one simply picks the basis $\mathcal{B}_{(\alpha,\beta)}$ that contains $P_{(\alpha, x)}^{\alpha} P_{(\beta, y)}^{\beta} \Psi_{\tilde{A}\tilde{B}}$.

Note that proposition 2 consider 24 vectors, and we said that there are 36 outcomes. This can be explained by two facts. First, the probabilities are the same in the real setting as in the ideal setting and, thus, 6 of these vectors vanish because their associated outcome occurs with probability zero: for every $a \in \{-\pi/8, 0, \pi/8\}$

$$P_{(\alpha, 0)}^{\alpha} P_{(\beta, 0)}^{\beta} \Psi_{\tilde{A}\tilde{B}} = P_{(\alpha, 1)}^{\alpha} P_{(\beta, 0)}^{\beta} \Psi_{\tilde{A}\tilde{B}} = 0.$$  \hspace{1cm} (1)

This brings us back to 30 vectors. Second, proposition 1 says that the 6 non vanishing vectors $P_{(\alpha, x)}^{\alpha} P_{(\beta, y)}^{\beta} \Psi_{\tilde{A}\tilde{B}} = P_{(\beta, y)}^{\beta} P_{(\alpha, x)}^{\alpha} \Psi_{\tilde{A}\tilde{B}}$, $a \in \Theta$, can be written as $P_{(\alpha, x)}^{\alpha} \Psi_{\tilde{A}\tilde{B}}$ or $P_{(\beta, y)}^{\beta} \Psi_{\tilde{A}\tilde{B}}$ and thus are known linear combinations of the 24 vectors considered in proposition 2. Now, we prove proposition 1.

**Proof of proposition 1.** We have $P_{(\alpha, x)}^{\alpha} P_{(\beta, y)}^{\beta} \Psi_{\tilde{A}\tilde{B}}$ because the collapse associated with the projection $P_{(\alpha, x)}^{\alpha}$ on $P_{(\beta, y)}^{\beta} \Psi_{\tilde{A}\tilde{B}}$ occurs with probability 1. Similarly, $P_{(\alpha', x')}^{\alpha'} P_{(\beta', y')}^{\beta'} \Psi_{\tilde{A}\tilde{B}} = P_{(\beta', y')}^{\beta'} P_{(\alpha', x')}^{\alpha'} \Psi_{\tilde{A}\tilde{B}}$. By commutativity, we obtain $P_{(\alpha', x')}^{\alpha'} P_{(\beta', y')}^{\beta'} \Psi_{\tilde{A}\tilde{B}} = P_{(\beta', y')}^{\beta'} P_{(\alpha', x')}^{\alpha'} \Psi_{\tilde{A}\tilde{B}}$. By transitivity, we obtain proposition 1b. This concludes the proof. □

**Proof of proposition 2.** The orthogonality of the four vectors $P_{(\alpha, 0)}^{\alpha} P_{(\beta, 0)}^{\beta} \Psi_{\tilde{A}\tilde{B}}$, $a \in \Theta_0$, and $b \in \Theta_0$, is immediate from the fact that $P_{(\alpha, 0)}^{\alpha}$ is orthogonal to $P_{(\alpha, 1)}^{\alpha}$, $P_{(\beta, 0)}^{\beta}$ is orthogonal to $P_{(\beta, 1)}^{\beta}$, and the commutativity of $P_{(\alpha, x)}^{\alpha}$ and $P_{(\beta, y)}^{\beta}$. The length of the vector $P_{(\alpha, x)}^{\alpha} P_{(\beta, y)}^{\beta} \Psi_{\tilde{A}\tilde{B}}$ is the same as in the ideal specification because it is uniquely determined by the probability $\tilde{p}(a, b) = p(a, b)$.

Now, we proceed with the third step which we feel is the most important step of the entire proof.

**Proof of proposition 3.** Here is a crucial observation for the proof. Let $(\gamma, \beta)$ be either $(-\pi/8, 0)$, $(0, \pi/8)$, $(\pi/8, 0)$, $\pi/8, -\pi/8$ or $(\pi/8, -\pi/8)$, a cyclic permutation of $(-\pi/8, 0, \pi/8)$. In this way, the value of $\beta \in \{-\pi/8, 0, \pi/8\}$ uniquely determines $\alpha$ and $\gamma$. In the specified setting, if we use a fixed basis $\beta$ and a fixed outcome $z$ on one side, say the side $\tilde{B}$, and look at the different final states associated with the two different bases $\alpha$ and $\gamma$ and the two different outcomes 0 and 1 on the other side, we see that the 4 different final states lie in the same real two dimensional plane. This fact is easily understood because the system on the non fixed side is a two dimensional system and the states in the measurement bases are all in the same real plane (no complex numbers). Two of these four states belong to $\mathcal{B}_{(\alpha,\beta)}$ whereas the other two belong to $\mathcal{B}_{(\beta,\gamma)}$.

Note that states that have different value of $z$ are orthogonal. So, the linear relationship between $\mathcal{B}_{(\alpha,\beta)}$ and $\mathcal{B}_{(\beta,\gamma)}$ does not mix different values of $z$: the coefficient $[T^{(\alpha, \beta)}](x', z')$ vanishes when $z \neq z'$. Therefore, it is sufficient to see, for each value of $z$ individually, how the two states associated with $(\gamma, \beta)$ and $z$ (below the dotted line in figure 1) are linearly related to the two states associated $(\alpha, \beta)$ and $z$ (above the dotted line in figure 1). For $z = 0, 1$, let

$$d_{(\beta, z)} = (P_{(\alpha, 0)}^{\alpha} P_{(\beta, z)}^{\beta} - P_{(\alpha, 0)}^{\alpha} P_{(\beta, z)}^{\beta})(\Psi_{\tilde{A}\tilde{B}}).$$

It is not hard to algebraically check, with the help of proposition 1, that the length of

$$d_{\beta} = d_{(\beta, 0)} + d_{(\beta, 1)} = (P_{(\alpha, 0)}^{\alpha} - P_{(\gamma, 0)}^{\alpha}) \Psi_{\tilde{A}\tilde{B}}.$$

is uniquely determined by the probabilities $\tilde{p}(a, b) = p(a, b)$. Indeed, we have

$$\langle \Psi_{\tilde{A}} | (P_{(\alpha, 0)}^{\alpha} - P_{(\gamma, 0)}^{\alpha}) (P_{(\alpha, 0)}^{\alpha} - P_{(\gamma, 0)}^{\alpha}) | \Psi_{\tilde{A}} \rangle = \| P_{(\alpha, 0)}^{\alpha} \Psi_{\tilde{A}} \|^2 - \langle \Psi_{\tilde{A}} | P_{(\alpha, 0)}^{\alpha} (P_{(\alpha, 0)}^{\alpha} | \Psi_{\tilde{A}} \rangle \rangle^2$$

and

$$\langle \Psi_{\tilde{A}} | P_{(\gamma, 0)}^{\gamma} P_{(\alpha, 0)}^{\alpha} | \Psi_{\tilde{A}} \rangle = \langle \Psi_{\tilde{A}} | P_{(\gamma, 0)}^{\gamma} P_{(\alpha, 0)}^{\alpha} | \Psi_{\tilde{A}} \rangle \rangle^2.$$
and similarly for the term $\langle \Psi_{AB} | P^A_{(a,0)} P^A_{(a,0)} | \Psi_{AB} \rangle$. We also have that the two states $d_{(\beta,0)}$ and $d_{(\beta,1)}$ are orthogonal because they have different value of $z$. So, we have obtained

$$
\|d_{(\beta,0)}\|^2 + \|d_{(\beta,1)}\|^2 = \|d_{\beta}\|^2 = \|d_{\beta}^{ideal}\|^2 = \|d_{(\beta,0)}^{ideal}\|^2 + \|d_{(\beta,1)}^{ideal}\|^2.
$$

(2)

After an exhaustive consideration of all cases, one can check that in the ideal specification we have that either, for both $z = 0$ and $z = 1$, $P^A_{(\alpha,0)} P^B_{(z,0)} \Phi^+$ is on the same side of the dotted line (see figure 1) as $P^A_{(\gamma,0)} P^B_{(z,0)} \Phi^+$ or else, for both $z = 0$ and $z = 1$, $P^A_{(\alpha,0)} P^B_{(z,0)} \Phi^+$ is on a different side of the dotted line than $P^A_{(\gamma,0)} P^B_{(z,0)} \Phi^+$. This means that either both $\|d_{(\beta,0)}^{ideal}\|^2$ and $\|d_{(\beta,1)}^{ideal}\|^2$ reach their maximum value or else they both reach their minimum value. So, we have the following.

$$
\|d_{(\beta,0)}\|^2 \leq \|d_{(\beta,0)}^{ideal}\|^2 \quad (\forall z \in \{0,1\})
$$

or

$$
\|d_{(\beta,0)}\|^2 \geq \|d_{(\beta,0)}^{ideal}\|^2 \quad (\forall z \in \{0,1\})
$$

(3)

We combine (2) and (3), we obtain $\|d_{(\beta,0)}\|^2 = \|d_{(\beta,0)}^{ideal}\|^2$ and $\|d_{(\beta,1)}\|^2 = \|d_{(\beta,1)}^{ideal}\|^2$. So, the real setting reaches the same extreme situation as in the ideal setting where the coefficients $\{T_{(\beta,0)}(x,z)\}$ are uniquely determined. This shows that the transformation $\{T_{(\beta)}(x,z)\}$ is the same as in the ideal case.

Now, we must consider arbitrary transformation from $B_{(\alpha,\beta)}$ to $B_{(\alpha',\beta')}$, not just from $B_{(\alpha,\beta)}$ to $B_{(\gamma,\beta)}$. By symmetry, we also have the transformations from $B_{(\alpha,\beta)}$ to $B_{(\alpha',\gamma)}$, that is, we can also change the axes on the side $B$ while keeping the same basis on the side $A$. For any $(\alpha,\beta)$, $(\alpha',\beta')$ and $(\alpha'',\beta'')$, the transformation from $B_{(\alpha,\beta)}$ to $B_{(\alpha',\beta')}$ is the product of the transformation from $B_{(\alpha,\beta)}$ to $B_{(\alpha',\beta'')}$ with the transformation from $B_{(\alpha',\beta'')}$ to $B_{(\alpha'',\beta'')}$. Using this fact, it is easy to obtain all transformations from arbitrary set $B_{(\alpha,\beta)}$ to an arbitrary set $A_{(\alpha',\beta')}$. This concludes the proof. □

**B. The tensor product structure**

Here, we use the result of the previous subsection to prove theorem 1. We want to construct two local isometries $U_A$ and $U_B$ that respects the three conditions of theorem 1. Intuitively, the isometry $U_A$ will extract the information about the correct qubit that is hidden inside $A$, and leave $A$ (without this information) as the garbage space. It will swap the state of a correct qubit that is somehow hidden in $A$ with the state of an additional qubit $A$ that is initially in state $|0\rangle$. This is just an intuition.

Formally, we will only show how to obtain $U_A$ because $U_B$ can be obtained in the same way with an additional qubit $B$ that is initially in the state $|0\rangle$. For two qubits $A'$ and $A$, let us denote $\overline{N}_{AA'}$ the control not operation where $A'$ is the source qubit and $A$ the target qubit. Similarly, let us denote $\overline{N}_{AA'}$ the control not operation where $A'$ is the source and $A$ is the target. Let $N_A$ and $N_A'$ be the not operation on $A$ and $A'$, respectively. One can easily check that

$$
\overline{N}_{AA'} = P^A_{(0,0)} \otimes I_{A'} + P^A_{(0,1)} \otimes N_{A'}
$$

(4)

$$
\overline{N}_{AA'} = I_A \otimes P^A_{(0,0)} + N_A \otimes P^A_{(1,0)},
$$

(5)

and

$$
N_{A'/A'} = \sqrt{2}(P^A_{(\pi/8,0)} - P^A_{(-\pi/8,0)}).
$$

(6)

Note that $\overline{N}_{AA'}$ is the standard swap operation on $A \otimes A'$ given that $A$ is initially in the state $|0\rangle^A$. This suggests that we define the swap operation on $A \otimes A'$ as

$$
U_{AA} \overset{df}{=} \overline{N}_{AA'} \overline{N}_{AA'}
$$

(7)

where the extended control not operations $\overline{N}_{AA'}$ and $\overline{N}_{AA'}$, by analogy with (4), (5) and (6), are defined as

$$
\overline{N}_{AA} \overset{df}{=} P^A_{(0,0)} \otimes I_A + P^A_{(0,1)} \otimes N_A
$$

(8)

and

$$
\overline{N}_{AA} \overset{df}{=} I_A \otimes P^A_{(0,0)} + N_A \otimes P^A_{(1,0)}
$$

(9)

where

$$
N_A \overset{df}{=} \sqrt{2}(P^A_{(\pi/8,0)} - P^A_{(-\pi/8,0)}).
$$

The swap operation $U_{BB}$ can be defined in a similar way.

We recall that the span of $\{(P^B_{a} \otimes P^B_{b}) \Psi_{MN} | a, b \in \Theta\}$ is denoted as $S$. Let us show that that, for every $\Phi_S \in S$, for every $|a\rangle^A \in A$, we have

$$
(U_{AA} \otimes I_B)(|a\rangle^A \otimes \Phi_S) = (I_A \otimes U^\dagger)(N_{AA} \overline{N}_{AA'} \otimes I_{B'})(I_A \otimes U)(|a\rangle^A \otimes \Phi_S)
$$

(10)

where $U$ is the inner product isomorphism from $S$ to $A' \otimes B'$. First, we use the definitions of $U_{AA}$, $N_{AA}$ and $\overline{N}_{AA}$ respectively given in (7), (8) and (9) to expand $U_{AA}$ in the LHS in terms of the projections $P^A_{a}$ and $P_a^A$. Second, use the condition A1 of definition 2 for an inner product isomorphism (summing over $b \in \Theta_a$ to get rid of the projections $P^B_{b}$) to replace every projection $P_a^A$ by its corresponding projection $P_a^{A'}$. This also adds $(I_A \otimes U^\dagger)$ to the left and $(I_A \otimes U)$ to the right. To finally obtain
the RHS, we use the identities (4), (5) and (6) to recover the expression \( (\tilde{N}_{AA} \tilde{N}_{AA'} \otimes I_B') \), but this time on \( A \otimes A' \otimes B' \). We can obtain a similar result for \( U_{BB} \).

With the help of (10) and condition A2 of the inner product isomorphism, we can obtain

\[
(U_{AA} \otimes U_{BB})|0\rangle^A \otimes |0\rangle^B \otimes \Psi_{\bar{A}B} = \Phi_{\bar{A}B}^+ \otimes \Psi_{E_A E_B} (11)
\]

with

\[
\Psi_{E_A E_B} = U^\dagger(|0\rangle^A' \otimes |0\rangle^{B'}).
\]

This is essentially the third condition of theorem 1. It is not exactly the third condition because it is expressed in terms of the swap operation \( U_{AA} \) and \( U_{BB} \), not in terms of isometries. The isometry \( U_{\bar{A}} \) is simply the transformation \( U_{\bar{A}} \) where the space \( \bar{A} \) is always in the fixed state \( |0\rangle^A \) so that the fixed component \( |0\rangle^A \) does not need to appear explicitly in the initial state. The image of \( U_{\bar{A}} \) on \( \bar{A} \) is the image of \( U_{\bar{A}} \) on \( \{ |0\rangle^A \} \otimes \bar{A} \). It is not hard to see using (11) and proposition 4 (provided later) that the image of \( U_{\bar{A}} \) is \( \bar{A} \otimes E_A \) where \( E_A \) is the support of the residual density matrix of \( \Psi_{E_A E_B} \) on \( \bar{A} \). The isometry \( U_{\bar{B}} \) can be defined in a similar way. With these definitions, (11) becomes

\[
(U_{\bar{A}} \otimes U_{\bar{B}})\Psi_{\bar{A}B} = \Phi_{\bar{A}B}^+ \otimes \Psi_{E_A E_B}
\]

which is exactly the third condition. With the help of (10), as a first step toward the first condition, we can obtain that, for every \( \Phi_S \in S \subseteq \bar{A} \otimes \bar{B} \),

\[
(P_a \otimes I_B)\Phi_S = U_{\bar{A}}^\dagger(P_a \otimes I_{\bar{A}B})U_{\bar{A}}\Phi_S
\]

and similarly for the second condition for \( P_b \).

To conclude the proof, it only remains to show that \( U_{\bar{A}} \otimes I_B \) does the right job on \( \bar{A} \otimes \bar{B} \), not only on \( S \) (i.e., \( U_{\bar{A}} \) is an isometry and fully satisfied the first condition of the theorem), and similarly with \( U_{\bar{B}} \) (for the second condition). The following easy to prove proposition is a key ingredient.

**Proposition 4** Let \( \Psi_{XY} \) be any entangled pure state of a bipartite system \( X \otimes Y \). Let \( X \) be the support of the residual density matrix of \( \Psi_{XY} \) on \( X \). Let \( A_X \) and \( A'_X \) be any two linear operators from \( X \) to another space \( Z \). We have that \( A_X \Psi_{XY} = A'_X \Psi_{XY} \) if and only if \( A_X = A'_X \). Moreover, the image of \( A_X \) on \( X \) is the support of the residual density matrix of \( A_X \Psi_{XY} \) on \( Z \).

**Proof.** Consider a Schmidt decomposition

\[
\Psi_{XY} = \sum_{i \in \mathcal{X}} \lambda_i |i\rangle^X \otimes |i\rangle^Y
\]

of \( \Psi_{XY} \) where \( \lambda_i > 0 \) for every \( i \in \mathcal{X} \). Consider the projection \( P^Y_i = |i\rangle^Y \langle i| \) on \( Y \). We have that \( A_X \Psi_{XY} = A'_X \Psi_{XY} \), if and only if, for every \( i \in \mathcal{X} \), \( P^Y_i A_X \Psi_{XY} = P^Y_i A'_X \Psi_{XY} \) which implies that, for every \( i \in \mathcal{X} \), \( A_X |i\rangle^X \otimes |i\rangle^Y = A'_X |i\rangle^X \otimes |i\rangle^Y \), and thus \( A_X |i\rangle^X = A'_X |i\rangle^X \). The image of \( A_X \) on \( X \) is the span of \( \{ A_X |i\rangle^X | i \in \mathcal{X} \} \), but this is also the support of the residual density matrix of \( A_X \Psi_{XY} \) on \( Z \).

We must show that \( U_{\bar{A}} \) is an isometry from \( \bar{A} \) to its image (which we do not need to know here), and then check that this isometry respects the first condition of theorem 1. To obtain that \( U_{\bar{A}} \) is an isometry, it is sufficient to obtain that \( U_{\bar{A}}^\dagger U_{\bar{A}} \) is the identity on \( \bar{A} \). We can use the definition of \( U_{\bar{A}} \) in terms of the projections \( P_a \) and the inner product isomorphism on \( S \) (as we did to obtain (10)), but this time to obtain that \( U_{\bar{A}}^\dagger U_{\bar{A}} \Psi_{AB} = \Psi_{AB} \) and then apply proposition 4 to conclude that \( U_{\bar{A}}^\dagger U_{\bar{A}} \) is the identity on \( \bar{A} \). Now, we verify the first condition of theorem 1 in a similar way. We want to show that

\[
P_a = U_{\bar{A}}^\dagger(P_a \otimes I_{E_a})U_{\bar{A}}
\]

on \( \bar{A} \). We simply use (12) with \( \Phi_S = \Psi_{AB} \) and proposition 4. This concludes the proof of theorem 1.

**VI. DISCUSSION**

Thus far, this result was not generalised to a large class of settings. Of course, if we only vary the angles a little, it will still hold, but nothing is known for more interesting variations. For example, it is still an open question whether a similar result applies if we used the angles \( \{ 0, \pi/4 \} \) on one side and the angles \( \{ \pi/8, -\pi/8 \} \) on the other side, instead of \( \{ \pi/8, 0, -\pi/8 \} \) on both sides. Also, it is known that the GHZ state can also be self tested [19], but no rule is known for a large class of states.

The theorem was obtained in the context of quantum cryptography. It would be useful to obtain a robust variation on this theorem in which the probabilities do not have to exactly respect the ideal specification. The idea is that a statistical test could then be used to obtain enough constraints on the source for the purpose of quantum cryptography. One may also ask what are the implications of this result in the foundation of quantum mechanics. Somehow, the essential processus of science is that we try to figure out what specific models describe the observed classical data in different settings. In particular, we make hypothesis and then conduct experiments to verify them. This is essentially the situation that is analysed in our theorem. Therefore, it would be interesting to see if the analysis of self-checking quantum apparatus says anything interesting about the way we build our different quantum models. If there is a link, it is not a trivial link because, typically, experimentalists know what their measuring apparatus are when they verify a model whereas our theorem applies when the measuring apparatus on each side are not known. Finally, in view of
the fact that configurations such as the one we consider in our theorem have been studied for many years in the context of violation of Bell’s inequalities, it is interesting to know that they are uniquely determined (up to a natural isomorphism) by the probability distributions that they generate.

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FIG. 1: The vector $d_{(\beta,z)}$ has maximum length when the real plane spanned by $P_{(\alpha,0)}^A P_{(\beta,z)}^B \Psi_{\bar{A}\bar{B}}$ and $P_{(\alpha,1)}^A P_{(\beta,z)}^B \Psi_{\bar{A}\bar{B}}$ (above the dotted line) and the real plane spanned by $P_{(\gamma,0)}^A P_{(\beta,z)}^B \Psi_{\bar{A}\bar{B}}$ and $P_{(\gamma,1)}^A P_{(\beta,z)}^B \Psi_{\bar{A}\bar{B}}$ (below the dotted line) are one and the same plane.